THE

# *Young Mathematician's Guide:*

Being a PLAIN and EASY

# INTRODUCTION

TO THE

# MATHEMATICKS.

## In FIVE PARTS.

*VIZ.*

I. **Arithmetick,** Vulgar and Decimal, with all the uſeful Rules; And a General Method of Extracting the Roots of all Single Powers.

II. **Algebra,** or Arithmetick in Species; wherein the Method of Raiſing and Reſolving Equations is rendered Eaſy; and illuſtrated with Variety of Examples, and Numerical Queſtions. Alſo the whole Buſineſs of Intereſt and Annuities, *&c.* performed by the Pen.

III. The **Elements** of **Geometry,** contracted, and Analytically demonſtrated; with a new and eaſy Method of finding the Circle's Periphery and Area to any aſſigned Exactneſs, by one Equation only: Alſo a new way of making Sines and Tangents.

IV. **Conick Sections,** wherein the chief Properties, *&c.* of the Ellipſis, Parabola, and Hyperbola, are clearly demonſtrated.

V. The **Arithmetick** of **Infinites** explained, and rendered Eaſy; with it's Application to ſuperficial and ſolid Geometry.

With an APPENDIX of **Practical Gauging.**

By *JOHN WARD.*

The TENTH EDITION, carefully Corrected.

To which is added,

A SUPPLEMENT, containing the Hiſtory of LOGARITHMS, and an INDEX to the whole WORK.

*LONDON:*

Printed for C. HITCH and L. HAWES, E. WICKSTEED, J. BEECROFT, W. JOHNSTON, J. RICHARDSON, T. LONGMAN, S. CROWDER and Co. P. DAVEY and B. LAW, and T. CASLON. 1758.

To the HONOURABLE

# Sir RICHARD GROSVENOR, of *Eaton*, in the County *Palatine* of *Chefter*, Baronet.

*SIR*,

WHEN requefted by fome Bookfellers in *London*, to revife and prepare this Treatife for a New Impreffion, and once refolved to anfwer their Demands; I was not long confidering at whofe Feet to lay it.

My Memory may indeed be impaired by Age, Misfortunes, and Accidents; nay, I am fenfible it is fo: But it muft be entirely loft, when I am forgetful of the great Obligations I lie under to Sir *Richard Grofvenor*.

Your Hofpitality and Generofity make you ftand unenvied in the Abundance of Fortune. Any Upftart may contrive to fpend a Great Eftate; but it is a Felicity almoft peculiar to Great Birth to become One.

Were I now to defcribe Liberality, without Profufenefs; Steadinefs in Principles, without any private View; Candour and Affability, Good Nature joined to found Judgment, and a Serenity of Temper, which your Enemies will always find the Companion of true Courage; and then pronounce that you are poffeffed of all thefe good Qualities in as high a Degree as moft Men living; No Gentleman that knows you well, would think I flattered you.

Sir, Give me Leave to ſay, I honour your Character, and love your Perſon; My Expreſſions are uncourtly, my Stile unpoliſhed, and therefore more proper to be prefixed to a Work wherein the Matters related are indeed clad in a plain and homely Dreſs; but they are true, and deſigned to propagate Mathematical Learning among ſuch as deſire to be introduced into that Sort of Knowledge; and I am extreamly pleaſed they are permitted to be ſent into the World under your Protection.

That you may long live, to promote the Good of your Country, and that City in whoſe Intereſt you have ſo heartily engaged yourſelf; and that you may ever ſucceed in your own private Affairs, and live to enjoy all the Bleſſings that attend a quiet prudent Life, is the earneſt Prayer of,

*Honoured* SIR,

*Your moſt Obliged, Humble,*

*and Obedient Servant,*

J. WARD.

To

# To the READER.

I *Think it needleſs* (and almoſt endleſs) *to run over all the Uſefulneſs, and Advantages of* Mathematicks *in General*; *and ſhall therefore only touch upon thoſe two admirable Sciences*, Arithmetick *and* Geometry; *which are indeed the two grand Pillars* (or rather the Foundations) *upon which all other Parts of Mathematical Learning depend*

*As to the Uſefulneſs of* Arithmetick, *it is well known that no Buſineſs, Commerce, Trade, or Employment whatſoever, even from the Merchant to the Shop-keeper*, &c. *can be managed and carried on, without the Aſſiſtance of Numbers.*

*And as to the Uſefulneſs of* Geometry, *it is as certain, that no curious Art, or Mechanick-Work, can either be invented, improved, or performed, without it's aſſiſting Principles*; *though perhaps the Artiſt, or Workman, has but little* (nay, ſcarce any) *Knowledge in* Geometry.

*Then, as to the Advantages that ariſe from both theſe Noble Sciences, when duly joined together, to aſſiſt each other, and then apply'd to Practice*, (according as Occaſion requires) *they will readily be granted by all who conſider the vaſt Advantages that accrue to Mankind from the Buſineſs of Navigation only. As alſo from that of Surveying and Dividing of Lands betwixt Party and Party. Beſides the great Pleaſure and Uſe there is from Time-keepers, as Dials, Clocks, Watches*, &c. *All theſe, and a great many more very uſeful Arts*, (too many to be enumerated here) *wholly depend upon the aforeſaid Sciences.*

*And therefore it is no Wonder, That in all Ages ſo many Ingenious and Learned Perſons have employed themſelves in writing upon the Subject of* Mathematicks; *but then moſt of thoſe Authors ſeem to preſuppoſe, that their Readers had made ſome Progreſs in that Sort of Learning before they attempted to peruſe thoſe Books, which are generally large Volumes, written in ſuch abſtruſe Terms, that young Learners were really afraid of looking into thoſe Studies.*

*Theſe Conſiderations firſt put me* (many Years ago) *upon the Thoughts of endeavouring to compoſe ſuch a plain and familiar Introduction to the* Mathematicks, *as might encourage thoſe that were willing* (to ſpend ſome Time that Way) *to venture and proceed on with Chearfulneſs*; *though perhaps they were wholly ignorant of it's firſt Rudiments. Therefore I began with their firſt Elements or Principles.*

*That*

*That is, I began with an Unit in Arithmetick, and a Point in Geometry; and from these Foundations proceeded gradually on, leading the young Learner Step by Step with all the Plainness I could,* &c.

*And for that Reason I published this Treatise* (Anno 1707) *by the Title of the* Young Mathematician's Guide; *which has answered the Title so well, that I believe I may truly say* (without Vanity) *this Treatise hath proved a very helpful Guide to near five thousand Persons; and perhaps most of them such as would never have looked into the* Mathematicks *at all but for it.*

*And not only so, but it hath been very well received amongst the Learned, and* (I have been often told) *so well approved on at the Universities, in* England, Scotland, *and* Ireland, *that it is ordered to be publickly read to their Pupils,* &c.

*The Title Page gives a short Account of the several Parts treated of,* with the Corrections and Additions that are made to this Fifth Edition, *which I shall not enlarge upon, but leave the* Book *to speak for itself; and if it be not able to give Satisfaction to the Reader, I am sure all I can say here in it's Behalf will never recommend it: But this may be truly said, That whoever reads it over, will find more in it than the Title doth promise, or perhaps he expects: it is true indeed, the Dress is but Plain and Homely, it being wholly intended to instruct, and not to amuse or puzzle the young Learner with hard Words, and obscure Terms: However, in this I shall always have the Satisfaction; That I have sincerely aimed at what is useful, tho' in one of the meanest ways; it is Honour enough for me to be accounted as one of the Under-Labourers in clearing the Ground a little, and removing some of the Rubbish that lay in the Way to this Sort of Knowledge. How well I have performed That, must be left to proper Judges.*

*To be brief; as I am not sensible of any Fundamental Error in this Treatise, so I will not pretend to say it is without Imperfections,* (Humanum est errare) *which I hope the Reader will excuse, and pass over with the like Candour and Good-Will that it was composed for his Use; by his real Well-wisher,*

J. WARD.

London, October 10th, 1706.

Corrected, *&c.* at Chester,
January 20th, 1722.

THE

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A N

# AN INTRODUCTION TO THE Mathematicks.

## PART I.

## *PRÆCOGNITA.*

THE *Businefs of* Mathematicks, *in all it's Parts, both* Theory *and* Practice, *is only to fearch out and determine the true Quantity*; *either of* Matter, Space, *or* Motion, *according as Occafion requires.*

*By* Quantity *of* Matter *is here meant the* Magnitude *or Bignefs of any vifible thing, whofe* Length, Breadth, *and* Thicknefs, *may either be meafured, or eftimated.*

*By* Quantity *of* Space *is meant the Diftance of one thing from another.*

*And by* Quantity *of* Motion *is meant the Swiftnefs of any thing moving from one Place to another.*

*The Confideration of thefe, according as they may be propofed, are the* Subjects *of the* Mathematicks, *but chiefly that of* Matter.

*Now the Confideration of* Matter, *with refpect to it's* Quantity, Form, *and* Pofition, *which may either be* Natural, Accidental, *or* Defigned, *will admit of infinite Varieties: But all the Varieties that are yet known, or indeed poffible to be conceived, are wholly comprized under the due Confideration of thefe Two,* Magnitude *and* Number, *which are the proper Subjects of* Geometry, Arithmetick, *and* Algebra. *All other Parts of the* Mathematicks *being only the Branches of thefe three* Sciences, *or rather their Application to particular Cafes.*

**Geometry** *is a Science by which we search out, and come to know, either the whole* Magnitude, *or some Part of any proposed* Quantity; *and is to be obtained by comparing it with another known Quantity of the same Kind, which will always be one of these,* viz. *A* **Line** (*or* Length *only*); *A* **Surface** (*that is,* Length *and* Breadth); *or a* **Solid** (*which hath* Length, Breadth, *and* Depth, *or* Thickness); *Nature admitting of no other Dimensions but these Three.*

**Arithmetick** *is a* Science *by which we come to know what* Number *of* Quantities *there are (either real or imaginary) of any Kind, contained in another* Quantity *of the same Kind: Now this Consideration is very different from that of* Geometry, *which is only to find out true and proper Answers to all such Questions as demand, how* Long, *how* Broad, *how* Big, *&c. But when we consider either more* Quantities *than one, or how often one* Quantity *is contained in another, then we have recourse to* Arithmetick, *which is to find out true and proper Answers to all such Questions as demand, how* Many, *what* Number *or* Multitude *of* Quantities *there are. To be brief, the Subject of* Geometry *is that of* Quantity, *with respect to it's* Magnitude *only; and the Subject of* Arithmetick *is* Quantities *with respect to their* Number *only.*

**Algebra** *is a* Science *by which the most abstruse or difficult Problems, either in* Arithmetick *or* Geometry, *are Resolved and Demonstrated; that is, it equally interferes with them both; and therefore it is promiscuously named, being sometimes called* Specious Arithmetick, *as by* Harriot, Vieta, *and Dr* Wallis, *&c. And sometimes it is called* Modern Geometry, *particularly the ingenious and great Mathematician Dr* Edmund Halley, *Savilian Professor of* Geometry *in the University of* Oxford, *and Royal Astronomer at* Greenwich, *giving this following Instance of the Excellence of our* Modern Algebra, *writes thus:*

'*The Excellence of the* Modern Geometry (*saith he*) *is in* '*nothing more evident, than in those full and* Adequate Solutions '*it gives to* Problems; *representing all the possible Cases at one* '*View, and in one general* Theorem *many times comprehending* '*whole* Sciences; *which deduced at length into* Propositions, *and* '*demonstrated after the Manner of the* Ancients, *might well be-* '*come the Subjects of large Treatises: For whatsoever* Theorem '*solves the most complicated* Problem *of the Kind, does with a* '*due Reduction reach all the subordinate* Cases.' *Of which he gives a notable Instance in the* Doctrine *of* Dioptricks *for finding the* Foci *of* Optick Glasses *universally.* (*Vide* Philosophical Transactions, *Numb.* 205.)

*Thus*

*Thus you have a short and general Account of the proper Subjects of those three noble and useful* Sciences, Arithmetick, Geometry, *and* Algebra. *I shall now proceed to give a particular Account of each; and first of* Arithmetick, *which is the* Basis *or* Foundation *of all* Arts, *both* Mathematick *and* Mechanick; *and therefore it ought to be well understood before the rest are meddled withal.*

## CHAP. I.

### *Concerning the several Parts of* Arithmetick, *with the Definition of such Characters as are used in this Treatise.*

ARithmetick, or the *Art* of *Numbering*, is fitly divided into three distinct Parts, two of which are properly called *Natural*, and the third *Artificial.*

The first, being the most plain and easy, is commonly called *Vulgar Arithmetick* in whole *Numbers*; because every *Unit* or *Integer* concerned in it, represents one whole *Quantity* of some *Species* or thing proposed.

The second is that which supposes an *Unit* (and consequently the *Quantity* or thing represented by that *Unit*) to be *Broken* or *Divided* into equal *Parts* (either even or uneven) and considers of them either as pure *Parts*, *viz.* Each less than an *Unit*, or else of *Parts* and *Integers* intermixt. And is usually called the *Doctrine of Vulgar Fractions.*

The third, or *Artificial Part*, is called *Decimal Arithmetick*; being an *Artificial* Invention of managing *Fractions* or *Broken Numbers*, by a much more commodious and easy Way than that of *Vulgar Fractions*: For the several Operations performed in *Decimals*, differ but little from those in *Whole Numbers*: and therefore it is now become of general Use, especially in *Geometrical Computations.*

Arithmetick (in all it's *Parts*) is performed by the various ordering and disposing of Ten *Arabick Characters* or *Numeral Figures* (which by some are called *Digits*).

| viz. | *One,* | *Two,* | *Three,* | *Four,* | *Five,* | *Six,* | *Seven,* | *Eight,* | *Nine,* | *Cypher,* |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |

*The Use of these* Characters *is said to be first introduced into* England *near six hundred Years ago,* viz. *about the Year* 1130, *vide Dr* Wallis's *Algebra,* Page 12.

The first of these *Characters* is called *Unity*, and represents one of any Kind of *Species* or *Quantity*. As one *World*, one *Star*, one *Man*, &c.

Viz. *Unity* is that by which every thing that is, is called one, (*Euclid*. 7. *Def*. 1.) and is the beginning of all *Numbers*. That is to say, *Number* is a *Multitude* of *Units*. Euclid 7. Def. 2.

For, one more one, makes Two; and one, more one, more one makes Three, &c. *Which is the first and chief* Postulate, or *rather* Axiom *to* Arithmetick.

*Viz.* { That $1+1=2$. $1+1+1=3$. $1+1+1+1=4$. $1+1+1+1+1=5$. And so on to 9.

*Nine* of these *Figures* were thus composed of *Units*, and differently form'd to represent so many *Units* put together into one *Sum*, as was intended each should denote: *Nine* being the greatest *Number* of *Units* that was then thought convenient to be expressed by one single *Character*; the last of the *Ten* is only a *Cypher*, or (as some phrase it) a Nothing, because of itself it signifies nothing; for if never so many *Cyphers* be Added to, or Substracted from, any *Number*, they can neither increase nor diminish that *Number*; but yet, as a *Cypher* (or *Cyphers*) may be placed, the other *Figures* will become of different *Values* from what they were before, as will appear further on.

For the more convenient ordering of the aforesaid *Numeral Figures*, according to the several Varieties that happen in *Computations*; I do advise the young Learner to acquaint himself with the Signification of the following *Algebraick Signs* or *Characters*, which he will find of excellent Use, as being a much shorter, better, and more significant Way of denoting what is to be done (in most Operations) than can otherwise be expressed in Words at length.

Significations.

| *Signs Names.* | |
|---|---|
| + { *Plus* or *more.* | The Sign of *Addition*; as $8+7$ is 8 more 7, and signifies that the *Numbers* 8 and 7 are to be added into one *Sum*. The like is to be understood when several *Numbers* are connected together with the Sign +. As $34+22+9+45$, &c. denotes these are all to be added into one *Sum*. |

The

| | | |
|---|---|---|
| — | *Minus or leſs.* | The Sign of *Subſtraction*; as 9—6 is 9 leſs 6, and ſignifies that 6 is to be taken from 9, that ſo their Difference may be found. |
| × | *Into or with.* | The Sign of *Multiplication*; as 9×6, is 9 into 6, and ſignifies that 9 is to be Multiplied into or with 6. |
| ÷ | *By.* | The Sign of *Diviſion*; as 8 ÷ 2, is 8 by 2, and ſignifies that 8 is to be Divided by 2, alſo thus 2) 8 (4 or thus $\frac{2}{8}$ each ſignifying the ſame thing, to wit, 8 Divided by 2. |
| = | *Equal.* | The Sign of *Equality* or *Equation*, viz. whenever this Sign = is placed betwixt *Numbers* (or *Quantities*) it denotes them to be Equal, as 9=9, or 9+6=15, or 9—6=3, *&c.* That is, 9 is Equal to 9, or 9 more 6 is Equal to 15, and 9 leſs 6 is Equal to 3, *&c.* |
| : : | *So is.* | The Sign of *Proportion*, or that commonly called the *Golden Rule*, or *Rule of Three*, and : : is always placed betwixt the Two middle *Terms* or *Numbers* in Proportion. Thus 2 : 8 : : 6 : 24. To be read thus; as 2, is to 8; ſo is 6, to 24. |

Theſe *Signs* and their *Significations*, being perfectly learnt, will help to ſhorten the Work.

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## CHAP. II.

### *Concerning the* Principal Rules *in* Arithmetick, *and how they are performed in Whole Numbers.*

THE *Rules* by which *Numerical Operations* are perform'd in all the Parts of *Arithmetick*, are many and various, ſeveral of them being form'd and raiſed as Occaſion requires, when applied to *Practice*; yet they are all comprehended within the due Conſideration of theſe Six, *viz.* Numeration (or Notation)

tion) **Addition**, **Subſtraction**, **Multiplication**, **Diviſion**, and **Evolution**, or *Extraction of Roots.*

## Sect. 1. *Of* Numeration *or* Notation.

**Numeration** or *Notation,* teacheth to Read or Expreſs the true Value of any *Number* when writ down; and conſequently to write down any propoſed *Number* according to it's true Value when it is named: And this conſiſteth of Two Parts.

1. The due Order of placing down Figures.
2. The true valuing of each Figure in it's Place.

Both which are plainly exhibited in the following Table.

| *&c.* | *Hundreds of Thouſands of Millions,* | *Tens of Thouſands of Millions* | *Thouſands of Millions* | *Hundreds of Millions;* | *Tens of Millions* | *Millions* | *Hundreds of Thouſands* | *Tens of Thouſands* | *Thouſands* | *Hundreds* | *Tens* | *Units* { *And is the firſt Place in Counting.* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 6 | 7 | 8, | 9 | 8 | 7, | 6 | 5 | 4, | 3 | 2 | 1 |
| | *Period of Thouſands of Millions.* | | | *Period of Millions.* | | | *Period of Thou-ſands.* | | | *Period of Units.* | | |

By this **Numeration** *Table* it is apparent, that the Order of Places is reckoned from the Right-hand towards the Left; the firſt Place of any *Number* being always that which is the outmoſt *Figure* to the Right-hand: and whatever *Figure ſtands* in that Place, doth only ſignify it's own ſimple Value, *viz.* ſo many *Units* as that *Figure* repreſents.

The ſecond Place is that of *Tens,* and any *Figure* ſtanding in that Place ſignifieth ſo many *Tens* as that *Figure* repreſents *Units.*

The

The third Place is *Hundreds*, the fourth Place *Thousands*, &c. That is, each Place towards the Left-hand is *Ten* Times the Value of that next it, towards the Right.

For Instance, suppose 759 were proposed to be read or pronounced according to the Value of each *Figure* as they now stand. The first *Figure* in this *Sum* is 9, because it stands in the Place of *Units*, and therefore signifies but it's own simple Value, to wit, 9 *Units*, or 9. The second *Figure* 5 stands in the Place of *Tens*, and therefore signifies Five *Tens* or *Fifty*. The *Figure* 7 stands in the third Place, or Place of *Hundreds*, and therefore it signifies *Seven Hundred*; and the whole Sum is to be read or pronounced thus, *Seven Hundred Fifty Nine*.

Note, Although the *Figure* 7 stands in the third Place (according to the Order of *Numbering*) yet when the whole *Sum* comes to be read, it is first pronounced; the reading of *Numbers* being performed like that of Letters or Words, always beginning with the outmost *Figure* towards the Left-hand, and so many *Figures* as are placed together without any Point, Comma, Line, or other Note of Distinction between them, are all but one *Sum*, and must be read as such.

For Example, 763596 is but one entire *Sum* or *Number*, notwithstanding it consists of six Places of *Figures*, and is thus read; *Seven Hundred Sixty Three Thousand, Five Hundred Ninety Six*.

The like is to be observed in reading or expressing the true Value of any *Sum* or *Rank* of *Numbers* consisting of *Seven*, *Eight*, *Nine*, or more Places of *Figures*, each *Figure* being to be valued according to it's Distance from the Place of *Unity*: As in the foregoing Table.

Now such Values may as well arise by *Cyphers*, as by other *Figures*; for Instance, 6 standing by itself, represents but Six *Units*: But if a *Cypher* be annext to it thus, 60, then it becomes *Sixty*; for the *Cypher* possessing the Place of *Units*, hath thereby removed the 6 into the Place of *Tens*; and another *Cypher* more would make it 600, *Six Hundred*, &c.

Whence it may be noted, that although a *Cypher* of itself signify nothing (as hath been said before) yet being placed on the Right-hand of any *Figure*, it augments the Value of that *Figure* by advancing it into a higher Place than otherwise it would have been, had not the *Cypher* been there.

Take one Example more in *Numeration* (if you please, that in the Table) *viz.* 678987654321, which is, according as is there signified.

*Six*

*Six Hundred Seventy Eight Thousand Millions,*
*Nine Hundred Eighty Seven Millions,*
*Six Hundred Fifty Four Thousand,*
*Three Hundred Twenty One Units.* Of any proposed *Species* or *Quantities* whatsoever.

And here it may be observed, that every third *Figure* from the Place of *Units*, bears the Name of *Hundreds*; which shews that if any great *Sum* be parted, or rather distinguished into Periods, of Three *Figures* in each *Period* (as in the foregoing Table), it will be of good Use to help the young Learner in the easier valuing and expressing that *Sum*.

---

## Sect. 2. *Of* Addition.

*Postulate or Petition.*

*That any given* Number *may be increased or made more, by putting another* Number *to it.*

**Addition** is that *Rule* by which several *Numbers* are collected and put together, that so their *Sum* or *Total* Amount may be known.

In this *Rule* Two things being carefully observed, the Work will be easily performed.

1. The first is the true placing of the *Numbers*, so as that each *Figure* may stand directly underneath those *Figures* of the same Value, *viz.* place *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c.

Then underneath the lowest Rank (always) draw a Line to separate the given *Numbers* from their *Sum* when it is found.

*Example.* If these *Numbers* 54327, and 2651, were given to be added together, they must be placed

Thus, { 54327
 2651
 ———

2. The second thing to be observed is the due Collecting or Adding together each Row of *Figures* that stand over one another of the same Value: And that is thus performed.

*Rule.*

*Always begin your* Addition *at the Place of* Units, *and Add together all the* Figures *that stand in that Place, and if their* Sum *be under* Ten, *set it down below the Line underneath it's own Place; but if their* Sum *be more than* Ten, *you must set down only the overplus, or odd* Figure *above the* Ten (*or* Tens) *and so many* Tens *as the* Sum *of those* Units *amount to, you must carry to* *to the place of* Tens; *Adding them and all the* Figures *that ſtand in the place of* Tens *together, in the ſame manner as thoſe of the* Units *were added*; *then proceed in the ſame order to the place of* Hundreds, *and ſo on to each place until all is done.*

The *Sum* ariſing from thoſe *Additions* will be the *Total* Amount required.

### *EXAMPLE* 1.

Let it be required to find the Sum of the aforeſaid *Numbers*,

*viz.* } 54327
2651

56978 the *Sum* required.

Beginning at the place of *Units*, I ſay 1 and 7 is 8, which being leſs than 10, I ſet it down (according to the *Rule*) underneath its own place of *Units*; and then proceed to the place of *Tens*, ſaying 5 and 2 is 7, which being leſs than 10, I ſet it down underneath its own place of *Tens*, and proceed to do the like at the place of *Hundreds*, and then at *Thouſands*, ſetting each of their *Sums* underneath their own reſpective places: Laſtly, becauſe there is not any *Figure* in the lower Rank to be added to the Figure 5, which ſtands in the place of *Ten Thouſands*, in the upper Rank, I therefore bring down the ſaid 5 to the reſt, placing it underneath its own place, and then I find that $54327 + 2651 = 56978$, the true *Sum* required.

### *EXAMPLE* 2.

Suppoſe it were required to find the *Sum* of theſe *Numbers*, $3578 + 496 + 742 + 184 + 95$. Theſe being placed, as before directed, will ſtand as in the Margin. Then beginning (as before) at the place of *Units*, ſay 5 and 4 is 9, and 2 is 11, and 6 is 17, and 8 is 25; ſet down the 5 *Units* underneath its own place of *Units*, and carry the 20, or two *Tens*, to the place of *Tens* (at which place they are only 2) ſaying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39; ſet down the 9 underneath its own place of *Tens*, and carry the 30, or three *Tens* (which indeed is 300) to the place of *Hundreds*, at which place they are but 3, ſaying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 5 is 20; here becauſe there is no *Figure* overplus (as before) I ſet down a *Cypher* underneath the place of *Hundreds*, and carry the 2 *Tens* (or rather the 2000) to the place of *Thouſands*, ſaying

3578
496
742
184
95
——
5095

(as before) 2 I carry and 3 is 5, which being the laſt, I ſet it down underneath its own place, and all is finiſhed. And find the *Sum* or *Total* amount to be 5095=3578+496+742+184+95.

If this Example be well conſidered, it will be ſufficient to ſhew the uſual Method of *Addition* in whole *Numbers*; but to make all plain and clear, I ſhall ſhew the young Learner the Reaſon of carrying the *Tens* from one Degree or Row of *Figures*, to the next Superior Degree, which is done purely to ſave Trouble, and prevent the uſing of more *Figures* than are really neceſſary, as will appear by the following Method of adding together the ſame *Numbers* of the laſt Example.

Thus, add together each ſingle Row of Figures by itſelf; as if there were no more but that one Row, ſetting down the *Sum* underneath its own place.

| | | | |
|---|---|---|---|
| 3 | 5 | 7 | 8 |
| | 4 | 9 | 6 |
| | 7 | 4 | 2 |
| | 1 | 8 | 4 |
| | | 9 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| The *Sum* of the Row of *Units*, is | | | 2 | 5 | Add |
| The *Sum* of the Row of *Tens*, is | | 3 | 7 | 0 | |
| The *Sum* of the Row of *Hund.* is | 1 | 7 | 0 | 0 | |
| The three *Thouſand* brought down | 3 | 0 | 0 | 0 | |
| The *Sum* or *Total* Amount as before, is | 5 | 0 | 9 | 5 | |

From hence I preſume it will be eaſy to conceive the true Reaſon of carrying the aforeſaid *Tens*; and alſo that *Cyphers* do not augment or increaſe the *Sum* in *Addition*. (See Page 4.)

I might have here inſerted a Lineal Demonſtration of this *Rule* of *Addition*; but I thought it would rather puzzle than improve a young Learner, eſpecially in this place; beſides the Reaſon of it is ſufficiently evident from that Natural Truth of *the Whole being Equal to all its Parts taken together*. Euclid 1. Axiom 19.

That is, the *Numbers* which are propoſed to be added together, are by that *Axiom* underſtood to be the ſeveral Parts, and their *Sum* or *Total* Amount found by *Addition* is underſtood to be the Whole.

And from thence is deduced the Method of proving the Truth of any *Operation* in *Addition*, *viz.* By parting or ſeparating the given *Numbers* into Two Parcels (or more, according to the Largeneſs of it) and then adding up each Parcel by itſelf: For if thoſe particular *Sums* ſo found, be *added* into one *Sum*, and that *Sum* prove Equal or the ſame with the *Total Sum* firſt found,

found, then all is right; if not, care muſt be taken to diſcover and correct the Error.

*EXAMPLE.*

| | | | |
|---|---|---|---|
| Add | 5647<br>3289<br>4016 | The Sum of theſe Parts is, | 12952 |
| | 2900<br>5007<br>1606 | The *Sum* of theſe, is | 9513 |
| The *Total Sum* of all theſe Parts | 22465 | The *Sum* of each Parcel put together | 22465 |

## Sect. 3. *Of* Subſtraction.

*Poſtulate or Petition.*

*That any* Number *may be diminiſhed, or made leſs, by taking another* Number *from it.*

Subſtraction is that *Rule* by which one *Number* is *deducted* or taken out of another, that ſo the *Remainder*, *Difference*, or *Exceſs* may be known.

As 6 taken out of 9, there *remains* 3. This 3 is alſo the *Difference* betwixt 6 and 9, or it is the *Exceſs* of 9 above 6.

Therefore the *Number* (or *Sum*) out of which *Subſtraction* is required to be made, muſt be greater than (or at leaſt equal to) the *Subtrahend* or *Number to be ſubſtracted.*

Note, *This Rule is the Converſe or Direct contrary to* Addition.

And here the ſame Caution that was given in *Addition*, of placing *Figures* directly under thoſe of the ſame Value, *viz. Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c. muſt be carefully obſerved; alſo underneath the loweſt Rank there muſt be drawn a Line (as before in *Addition*) to ſeparate the given *Numbers* from their *Difference* when it is found.

Then having placed the leſſer *Number* under the greater, the *Operation* may be thus performed.

RULE.

*Begin at the Right Hand* Figure *or place of* Units *(as in* Addition) *and take or ſubſtract the lower* Figure *in that place*

 *from*

*from the* Figure *that ſtands over it, ſetting down the* Remainder *or* Difference *underneath its own place. If the Two* Figures *chance to be Equal, ſet down a* Cypher: *But if the upper* Figure *be leſs than the lower* Figure, *then you muſt* add 10 *to the upper* Figure, *or mentally call it* 10 *more than it is, and from that* Sum ſubſtract *the lower* Figure, *ſetting down the* Remainder (*as* before directed). *Now becauſe the* 10 *thus added, was ſuppos'd to be borrowed from the next ſuperior place* (viz. *of* Tens) *in the upper* Figures, *therefore you muſt either call the upper* Figure *in that place from whence the* 10 *was borrowed, one leſs than really it is, or elſe* (*which is all one, and moſt uſual*) *you muſt call the lower* Figure *in that place one more than it really is, and then proceed to* Subſtraction *in that place, as in the former; and ſo gradually on from one Row of* Figures *to another until all be done.*

*EXAMPLE* 1.

Let it be required to find the *Difference* between 6785, and 4572. That is, let 4572 be *ſubſtracted* from 6785.

Theſe *Numbers* being placed down, as before directed, will ſtand

Thus { 6785
4572
————
2213

Beginning at the place of *Units*, take 2 from 5 and there will *remain* 3 which muſt be ſet down underneath its own place, and then proceed to the place of *Tens*, taking 7 from 8, and there will *remain* 1, to be ſet down underneath its own place; again, at the place of *Hundreds*, take 5 from 7, and there *remains* 2, which ſet down, as before; laſtly, take 4 from 6 and there will *remain* 2, which being ſet down underneath its own place, the Work is finiſhed, and the *Difference* ſo found will be 2213=6785−4572, as was required.

*EXAMPLE* 2.

The *Difference* between 5849, and 7496 is required.

Having placed the *Numbers* as in the Margin, begin at the place of *Units* (as before) and ſay 9 from 6 cannot be, but 9 from 16 and there *remains* 7, to be ſet down under its own place; next proceed to the place of *Tens*, where you muſt now pay the 10 that was borrowed to make the 6, 16, by counting the upper *Figure* 9 in that place one leſs than it is, ſaying 4 from 8 and there *remains* 4, or elſe (which is the moſt practiſed) ſay 1 I borrowed and 4 is 5

7496
5849
————
1647

from

from 9 and there *remains* 4, to be set down under its own place (as before); again, at the place of *Hundreds*, say 8 from 4 that cannot be, but 8 from 14 there will *remain* 6 to be set down; and here I have borrowed 10 (as before) which must be paid in the same manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, saying 5 from 6 there *remains* 1, or else by saying 1 borrowed and 5 is 6 from 7 and there *remains* 1, which being set down under its own place all is done, and the *Difference* required will be 1647 = 7496 — 5849.

*EXAMPLE* 3.

From 830476
Take 741068

*Remains* 89408

By this Example you may perceive that *Cyphers* in the *Subtrahend*, viz. in the *Numbers* to be *substracted*, do not diminish the *Number* from whence *Substraction* is made. See *Page* 4.

These Three Examples, I presume, may be sufficient to shew the young Learner the Method of *Substracting* whole *Numbers*; as for the Reason thereof it is the same with that of *Addition*, Page 10, *viz.* of *the Whole being Equal to all its Parts taken together.*

That is, in this *Rule* the *Number* from which *Substraction* is required to be made, is understood to be the Whole, and the *Subtrahend* or *Number* to be *substracted*, is supposed to be a part of that Whole; consequently if that Part be taken from the Whole, the *Remainder* will be the other part.

From hence is deduced the common Method of proving *Substraction*, by adding together the *Subtrahend* and the *Remainder*. For if the *Sum* of those Two (which are here called Parts) be equal to the *Number* from whence *Substraction* was made (which is here called the Whole) then the Work is right; if not, care must be taken to discover and correct the Error.

*EXAMPLE.*

From 59435
Take 47608 } Add
———
Proof { 11827 }
———
59435 } The *Sum* which is equal to the *Number* from whence *Substraction* was made.

Or

Or from the abovesaid Reason, it will be easy to conceive how to prove the Truth of *Substraction* by *Substraction*.

| | | |
|---|---|---|
| For if from | 59435 | being here the whole, |
| there be taken | 47608 | as part of that whole ; |
| there will *remain* | 11827 | the other part (as before) |
| And if from | 59435 | the whole, there be *substracted* the |
| last part, *viz.* | 11827 | |
| there will *remain* | 47608 | the first part, or *Number* which was required to be first *Substracted*. |

| | | | |
|---|---|---|---|
| From | 75643 | From | 7000000 |
| Take | 9000 | Take | 986432 |
| Remains | 66643 | Remains | 6013568 |

## Sect. 4. *Of* Multiplication.

Multiplication is a *Rule* by which any given *Number* may be speedily increased, according to any proposed *Number* of *Times*.

*That is, One* Number *is said to* Multiply *another, when the* Number multiplied *is so often* added *to itself, as there are* Units *in the* Number multiplying; *and another* Number *is produced.* (Euclid 7. Def. 15.)

To perform *Multiplication*, there is required two given *Numbers*, called *Factors*.

The First is the *Number* to be *multiplied*, which is generally put the greater of the Two *Numbers*, and is commonly called the *Multiplicand*.

The other is that *Number* by which the First is to be *multiplied*, and is usually called the *Multiplicator* or *Multiplier* ; and this denotes the *Number* of *Times* that the *Multiplicand* is required to be *added* to itself. For so many *Units* as are contained in the *Multiplier*, so many times will the *Multiplicand* be really *added* to itself *(as per* Euclid *above).* And from thence will arise a Third *Number*, called the *Product*. *But in* Geometrical Operations *it is called the* Rectangle *or* Plain.

For instance; suppose it were required to increase 6 four times, that is, to *multiply* 6 into or with 4. These two *Numbers* are to be set (or placed) down as in *Addition* or *Substraction*.

Thus

Thus { 6 *Multiplicand,* / 4 *Multiplier,* } or *Factors.*

*Product* 24 *viz.* 4 times 6 is 24, as plainly appears by *Addition, viz.* By ſetting down 6 four times, and then adding them together into one *Sum,* Thus

| | | |
|---|---|---|
| 1 | 6 | Add |
| 2 | 6 | |
| 3 | 6 | |
| 4 | 6 | |
| | 24 | |

*From hence it is evident, that* Multiplication *is only a* Conciſe *or* Compendious *Way* of adding *any given* Number *to itſelf, ſo often as any* Number *of Times may be propoſed.*

Before any Operation can be readily performed in *Multiplication,* the ſeveral *Products* of the ſingle *Figures* one into another muſt be perfectly learn'd by Heart, *viz.* That 2 times 2 is 4, that 3 times 3 is 9, and 3 times 6 is 18, *&c.* According as they are expreſſed in the following *Table;* wherein I have omitted *multiplying* with 2, it being ſo very eaſy that any one may do it.

*Multiplication Table.*

| | | | | | |
|---|---|---|---|---|---|
| 3×3= 9 | 4×4=16 | 5×5=25 | 6×6=36 | 7×7=49 | 8×8=64 |
| 3×4=12 | 4×5=20 | 5×6=30 | 6×7=42 | 7×8=56 | 8×9=72 |
| 3×5=15 | 4×6=24 | 5×7=35 | 6×8=48 | 7×9=63 | 9×9=81 |
| 3×6=18 | 4×7=28 | 5×8=40 | 6×9=54 | | |
| 3×7=21 | 4×8=32 | 5×9=45 | | | |
| 3×8=24 | 4×9=36 | | | | |
| 3×9=27 | | | | | |

I think it needleſs to give any *Explanation* of this *Table;* for if the *Signs* and their *Significations* be well underſtood, (*vide page* 5) it muſt needs be eaſy. Only this may be noted, that $4 \times 3 = 3 \times 4$, or $7 \times 5 = 5 \times 7$, *&c.*

That is, 3 times 4 is the ſame with 4 times 3, or 5 times 7 is the ſame with 7 times 5, *&c.* The like muſt be underſtood of all the reſt in the *Table.*

And when all theſe ſingle *Products* are ſo perfectly learn'd by Heart, as to be ſaid without pauſing; you may then proceed (but not 'till then) to the Buſineſs of *Multiplication;* which will be found very eaſy, if the following *Rule* (and *Examples*) be carefully obſerved.

## RULE.

*Always begin with that* Figure *which ſtands in the* Units *place with the* Multiplier, *and with it* multiply *the* Figure *which ſtands in* *in the* Units *place of the* Multiplicand; *if their* Product *be less than Ten, set it down underneath its own place of* Units, *and proceed to the next* Figure *of the* Multiplicand. *But if their* Product *be above* Ten (*or* Tens) *then set down the Overplus only (or odd* Figure, *as in* Addition) *and bear (or carry) the said* Ten *or* Tens *in mind until you have* multiplied *the next* Figure *of the* Multiplicand, *with the same* Figure *of the* Multiplier; *then to their* Product *add the* Ten *or* Tens *carried in mind, setting down the Overplus of their* Sum *above the* Tens, *as before: and so proceed on in the very same manner, until all the* Figures *of the* Multiplicand *are* multiplied *with that* Figure *of the* Multiplier.

*EXAMPLE* 1.

Suppose it were required to *multiply* 3213 into or with 3.

3213 *Multiplicand* }
3 *Multiplier*, } or *Factors*.

*Product* 9639

Beginning at the *Units* place, say 3 times 3 is 9, which, because it is less than *Ten*, set down underneath its own place; and proceed to the next place of *Tens*, saying 3 times 1 is 3, which set down underneath its own place; then to the next place, *viz.* of *Hundreds*, saying 3 times 2 is 6, which set down, as before; lastly, at the place of *Thousands*, say 3 times 3 is 9, which being set down underneath its own place, the Operation is finish'd; and the true *Product* is $9639 = 3213 \times 3$, as was required.

*EXAMPLE* 2.

Let it be required to *multiply* 8569 into 8. Set down these *Numbers* as before,

Thus { 8569
8

68552

Beginning at the *Units* place, say 8 times 9 is 72, set down the 2 underneath its own place of *Units*, and bear the 70, or 7 *Tens* in mind, and proceed to the next *Figure* of the *Multiplicand* (at which place the 7 *Tens* are only 7) saying 8 times 6 is 48, and the 7 carried in mind is 55; set down the odd 5 underneath its own place of *Tens*, and carry the 50 (which is really 500) to the next place (*viz.* of *Hundreds*) at which place it is only 5, where say, 8 times 5 is 40, and the 5 carried in mind is 45; set down the 5 underneath its own place, and carry the 40 or 4 *Tens* (which is really 4000) to the

next

next place, *viz.* of *Thouſands*, ſaying, 8 times 8 is 64, and 4 carried in mind is 68. (Now this being the laſt Place or *Figure* to be *multiplied)* Set down the whole *Product* 68, and the Work is done.

So that, $8569 \times 8 = 68552$, the *Product* required.

Now the Reaſon of this and all other the like Operations, may be eaſily conceived from this which follows.

| | | | | | |
|---|---|---|---|---|---|
| 8 | 5 | 6 | 9 | | The ſame *Factors* as before. |
| | | | 8 | | |
| | | | 7 | 2 | Here 8 times 9 is 72, as before, becauſe the 9 ſtands in the *Units* place. |
| | | 4 | 8 | 0 | Now here it is not really 8 times 6 = 48, but it is 8 times 60 = 480, becauſe the 6 ſtands in the place of *Tens.* |
| | 4 | 0 | 0 | 0 | And here it is not 8 times 5 = 40, but it is really 8 times 500 = 4000, becauſe the 5 ſtands in the place of *Hundreds.* |
| 6 | 4 | 0 | 0 | 0 | Laſtly, becauſe the 8 in the *Multiplicand* ſtands in the place of the *Thouſands*, it is therefore 8 times 8000 = 64000, and not 8 times 8 = 64. |
| 6 | 8 | 5 | 5 | 2 | The *Sum* of the particular *Products*, which gives the true *Product*, as before. |

By what hath been already ſaid, with a little Conſideration had to the *Examples*, I preſume the Learner may eaſily underſtand how to *multiply* whole *Numbers* with any ſingle *Figure.* And when it is required to *multiply* with more than one; then ſo many Figures as there are in the *Multiplier*, ſo many particular *Products* there muſt be.

That is, all the *Figures* of the *Multiplicand* muſt be *multiplied* with every ſingle *Figure* of the *Multiplier*, as if there were but one ſingle *Figure:* and the *Sum* of all thoſe particular *Products*, will be the true *Product* required. But in thoſe Operations, great Care muſt be taken in ſetting down the particular *Products* (which ariſe by each *multiplying Figure)* in their proper places. Which will be eaſily done, if the following Directions be carefully obſerved.

*Viz.* { Always place the firſt *Figure* (or *Cypher)* of every particular *Product*, directly underneath the *multiplying Figure.* Or thus:

" *The Firſt* Figure *(or* Cypher) *of the ſecond particular* Product *muſt ſtand directly under the ſecond* Figure *(or place) of the Firſt* Product; *and the Firſt* Figure *(or* Cypher) *of the Third particular*

*particular* Product, *must stand directly underneath the Third* Figure *of the First* Product: *And so on until all is done.*

Now the Reason of placing the first *Figure* of every particular *Product* in their Order, will be very obvious to any one that considers the last *Example*; wherein the *Cyphers* are only set down to shew the true Distance of the first *Figure* in each particular *Product* from the *Units* place. And altho' it is not usual to set down *Cyphers* in this manner; yet they are always supposed to be there: That is, their Places are always left void, as in the two following *Examples*; wherein I have placed *Points* instead of *Cyphers*.

### *EXAMPLE* 3.

Let it be required to *multiply* 78094, into or with 7563.

| | | |
|---|---|---|
| 78094 <br> 7563 | } *Factors.* | |
| 234282 | The First particular *Product* with | 3 |
| 468564. | The Second particular *Product* with | 60 |
| 390470.. | The Third particular *Product* with | 500 |
| 546658... | The Fourth particular *Product* with | 7000 |
| 590624922 | The *Total*, or true *Product* required. | |

### *EXAMPLE* 4.

Suppose it be required to *multiply* 57498 into 60008.

| | | |
|---|---|---|
| 57498 <br> 60008 | | |
| 459984 | The *Product* with | 8 |
| 344988.... | The *Product* with | 6000 |

3450339984 = 57498 × 60008, as was required.

Here you may observe, that I pass over the *Cyphers*, and only take care of placing the first *Product* of the last *Figure*, viz. of 60000 according to the foregoing Directions.

When there is a *Cypher* or *Cyphers*, to the Right-hand either of the *Multiplicand* or *Multiplicator*, or to both; in that case *multiply* the *Figures* as before; neglecting the *Cyphers* until the particular *Products* are added together; Then to their *Sum* annex so many *Cyphers* as are in either or both the *Factors*. As in these:

*EXAMPLE*

| *EXAMPLE* 5. | *EXAMPLE* 6. | *EXAMPLE* 7. |
|---|---|---|
| 9538 | 87600 | 785000 |
| 4600 | 79 | 56900 |
| 57228 | 7884 | 7065 |
| 38152 | 6132 | 4710 |
| | | 3925 |
| 43874800 | 6920400 | 44666500000 |

*Take a few Examples without their Work at large.*

75649×579=43800771
687000×356=244572000
530674×45007=23884044718
7901375×30000=237041250000
537084000×590700=317255518800000
102030405×504030201=51426405540261405
987654321×123456789=121932641112635269

*Note*, If it be required to *multiply* any *Number* with 10, 100, 1000, 10000, *&c.* it is only annexing the *Cyphers* of the *Multiplier* to the *Figures* of the *Multiplicand*, and the Work is done.

Thus { 578×10 =5780. 578×1000 =578000
578×100=57800. 578×10000=5780000, &c.

These *Examples* (being well underſtood) are ſufficient to inſtruct the Learner all the Varieties that can happen in *multiplying* of whole *Numbers*, according to the Method generally practiſed: However it may not be amiſs to ſhew here how *Multiplication* may be performed (with many *Figures*) by *Addition* only.

## *EXAMPLE.*

Let it be required to *multiply* 879654 into 79863.

In order to perform this (or any Operation of this kind) by *Addition* only; you muſt make a Tariffa or ſmall Table of the given *Multiplicand*, in this manner:

*Firſt*, Make a ſmall Column, and in it place gradually downward the Nine ſingle *Figures*; viz. 1, 2, 3, 4, 5, *&c.*

Then againſt the *Figure* 1, ſet down the *Multiplicand* (which in this *Example* is 879654) and againſt the *Figure* 2, ſet down the double of the *Multiplicand*, found by *adding* it to itſelf; To this double *add* the *Multiplicand*, ſetting down their *Sum* againſt the *Figure* 3. And ſo proceed on by a continued *Addition* until there be Ten times the *Multiplicand* in the *Table*; which if the Work is true, will be the *Multiplicand* itſelf with a *Cypher* to the Right-hand of it, as in the annexed Table. This being done, it will be eaſy to conceive, that the *Figures* in the ſmall Column of the *Table*, do reſpectively repreſent thoſe of the *Multiplier*: And that the *Numbers* againſt any of thoſe *Figures* in the ſmall Column, will be the true *Product* of the *Multiplicand* agreeing to any *Figure* of the *Multiplier*; as plainly appears by the Work of this *Example*

| | |
|---|---|
| 1 | 879654 |
| 2 | 1759308 |
| 3 | 2638962 |
| 4 | 3518616 |
| 5 | 4398270 |
| 6 | 5277924 |
| 7 | 6157578 |
| 8 | 7037232 |
| 9 | 7916886 |
| 10 | 8796540 |

Then 879654 } The *Factors* as before.
79863 }

| | | | |
|---|---|---|---|
| Againſt | 3, in the *Table* is | 2638962 | =879654×3 |
| | 6, is | 5277924 | =879654×60 |
| | 8, is | 7037232 | =879654×800 |
| | 9, is | 7916886 | =879654×9000 |
| | 7, is | 6157578 | =879654×70000 |
| *The Product required* | | 70251807402 | =879654×79863 |

*Note*, This Method of Tabulating the *Multiplicand*, is both eaſy and certain; being neither ſubject to Errors, nor burdenſome to the Memory, and therefore in large Calculations it may be found very uſeful. But for common Practice the uſeful Method (as in *Page* 18, &c.) is beſt, and to be preferred before this.

Moſt *Maſters* that teach (and ſeveral *Authors* that write of) *Arithmetick*, do teach to prove the Truth of *Multiplication*, by caſting away all the *Nines* that are contained in both the *Factors*, and their *Product*; but becauſe that Method is very erroneous, as might be eaſily ſhewed; I ſhall therefore omit inſerting it, and leave the Proof of *Multiplication* to the next *Section*, wherein (I preſume) the Reaſon and Proof, both of it, and *Diviſion*, will plainly appear.

Sect.

# Sect. 5. *Of* Division.

Division is a *Rule* by which one *Number* may be ſpeedily *ſubſtracted* from another, ſo many times as it is contained therein.

That is, It ſpeedily diſcovers how often one *Number* is contained (or may be found) in another: And to perform that there are required Two Numbers to be given.

1. The one of them is that *Number* which is propoſed to be *divided*, and is called the *Dividend.*

2. The other is that *Number* by which the ſaid *Dividend* is to be *divided*, and is called the *Diviſor.*

And by comparing theſe Two, *viz.* the *Dividend* and the *Diviſor* together, there will ariſe a Third *Number*, called the *Quotient*; which ſhews how often the *Diviſor* is containod in the *Dividend*, or into what *Number* of Equal Parts the *Dividend* is then divided. Therefore,

Diviſion *is by* Euclid *fitly termed the meaſuring of one* Number *by another, viz.* one *Number* is ſaid to meaſure another by that *Number*, which when it *multiplies*, or is *multiplied* by it, it produceth. *Euclid* 7. *Def.* 23.

*And, if a* Number *meaſuring another*, multiply *that* Number *by which it meaſureth, or be* multiplied *by it, it produceth the* Number *which it meaſureth.* Euclic 7. Axiom 9.

That is to ſay, If that *Number* which *divides* another (called the *Diviſor*) be *multiplied* with the *Number* which is produced by *Diviſion* (called the *Quotient*) their *Product* will be the *Number divided* or *Dividend.* Whence it follows, that *Diviſion* and *Multiplication* are the Converſe and Direct Contrary one to another (as *Subſtraction* is to *Addition*) and do mutually prove the Truth of each other's *Operations.*

I ſhall therefore make choice of the foregoing *Examples* in *Multiplication*, in order (as I preſume) to render the Buſineſs of *Diviſion* more plain and eaſy.

Firſt, Let it be required to find how often 6 is contained in 24. That is, to *divide* 24 by 6.

*N.B.* Always place down the given *Numbers* in this Order; Firſt ſet down the *Diviſor*, and to the Right-hand of it draw a crooked Line; then ſet down the *Dividend*, and to the Right of it draw another crooked Line, in which muſt be placed the *Quotient Figure*, or *Figures* as they become found.

Thus

*Dividend.*

Thus *Diviſor* 6) 24 (4 the *Quotient.*

Here I conſider how many times 6 there is in 24, and find it 4, *viz.* 4 times 6 is 24, therefore 4 is the true *Quotient* or *Anſwer* required.

This is apparent by *Subſtraction*, as in the Margin; where 24 the *Dividend* is ſet down, and from it 6 the *Diviſor* continually *ſubſtracted* ſo often as it can be, which is juſt 4 times. Therefore 4 is the true *Quotient* or *Anſwer* required.

Compare this with the *Example, Page* 15.

| | |
|---|---|
| | 24 |
| 1 | 6 |
| | 18 |
| 2 | 6 |
| | 12 |
| 3 | 6 |
| | 6 |
| 4 | 6 |
| | 0 |

*Corollary.*

From hence it is evident; that *Diviſion* is but a *conciſe* or *compendious Method* of *ſubſtracting* one *Number* from another, ſo often as it can be found therein; for if the *Diviſor* be continually *ſubſtracted* from the *Dividend*, accounting an *Unit* (or 1) for each time it is *ſubſtracted* (as above) the *Sum* of thoſe *Units* will be the *Quotient.*

All Operations in *Diviſion* do begin contrary to thoſe of *Multiplication*, *viz.* at the Firſt *Figure* to the Left-hand, or that of the higheſt Value, and decreaſe the *Dividend* by a repeated *Subſtraction* of each *Product* ariſing from the *Diviſor* when *multiplied* into the *Quotient Figure.* And the only difficulty in *Diviſion* of whole *Numbers* (or indeed of any *Numbers*) lies in making choice of ſuch a *Quotient Figure*, as is neither too big, nor too little; and that may be eaſily obtained by obſerving the following *Rule*, which hath two *Caſes.*

RULE.

Caſe 1. *As often as the Firſt* Figure *of the* Diviſor *is taken from the Firſt* Figure *of the* Dividend: *So often muſt the Second* Figure *of the* Diviſor *be taken from the Second* Figure *of the* Dividend, *when it is joined with what* Remains *of the Firſt. And as often muſt the Third* Figure *of the* Diviſor *be taken from the Third* Figure *of the* Dividend, *&c.*

But if the Firſt *Figure* of the *Diviſor* cannot be taken from the Firſt *Figure* of the *Dividend.* Then;

Caſe

Case 2. *So often as the First* Figure *of the* Divisor *is taken from the Two First* Figures *of the* Dividend, *so often must the Second* Figure *of the* Divisor *be taken from the Third* Figure *of the* Dividend, *when it is joined with what* remained *of the Second: And so often must the Third* Figure *of the* Divisor *be taken from the Fourth* Figure *of the* Dividend, *&c.*

That is, the *Quotient Figure* must be such, as being *multiplied* into the *Divisor*, will *produce a Product* equal to such a part of the *Dividend* as is then taken for that *Operation*: But if such a *Product* cannot be exactly found, then the next less must be taken, and ordered, as in the following *Examples*: of which let that in *Page* 16 be the first, wherein there was given 8569 the *Multiplicand*, and 8 the *Multiplier*. To find the *Product* 68552. Let us here suppose the said *Product* 68552, and 8 the *Multiplier*, both given; thence to find the *Multiplicand*. That is, Let it be required to *divide* 68552 by 8.

*Dividend*
*Divisor* 8) 68552 ( *Quotient* when found.

According to the *Rule*, *Case* 1. I compare 8 the *Divisor* with 6 the First *Figure* of the *Dividend*, and finding I cannot take it from that, I then consider (by *Case* 2.) how often 8 can be taken from 68, the two first *Figures* of the *Dividend*, and find it may be taken 8 times; for 8 times 8 is 64, being the greatest *Product* of 8 (into any *Figure*) that can be taken from 68. I therefore place 8 in the *Quotient*, and with it *multiply* 8 the *Divisor*, setting down their *Product* underneath the said Two First *Figures* of the *Dividend*, *subtracting* it from them, and then the Work will stand

Thus 8) 68552 (8
64
———
4

In order to a Second *Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down underneath in its own place to the *Remainder* 4, which will by that means become 45. Then I consider how many times 8 can be taken from 45, and find it may be 5 times; for 5 times 8 is 40, I therefore place 5 in the *Quotient*, and with it *multiply* 8 the *Divisor*, setting down and *subtracting* their *Product*, as before. Then the Work will stand

Thus

```
Thus 8) 68552 (85
        64.
        ---
         45
         40
         --
          5
```

For a Third *Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down, as before, proceeding in all respects, as before; and then the Work will stand

```
Thus 8) 68552 (856
        64..
        ----
         45
         40
         ---
          55
          48
          ---
           7
```

Lastly, I point and bring down the 2, *viz.* the last *Figure* of the *Dividend* to the *Remainder* 7, which will then become 72, and proceeding as in the other *Operations*, I find that 8 the *Divisor* can be taken just 9 times from 72, and the Work is finished, and will stand

```
Thus 8) 68552 (8569
        64...
        ----
         45
         40
         ---
          55
          48
          ---
           72
           72
           ---
           (0)
```

The true *Quotient* is found to be 8569, being exactly the *Eighth* part of 68552, or the *Multiplicand* of the proposed *Example* of *Multiplication*. As was required.

The Reason of the *Operations* will be very plain to any one that will a little consider of it, as follows:

*Divisor*

*Divisor* 8) 6 8 5 5 2 (8000. The First *Quotient Figure.*

*Substract* 6 4 0 0 0 { This *Product* of the *Divisor* into the *Quotient* is 64000, *viz.* 8 times 8000; the *Quotient Figure* being always of the same *Value* or *Degree* with that *Figure* under which the *Unit*'s place of its *Product* stands.

*Divisor* 8) 4 5 5 2 (500. The Second *Quotient Figure.*

*Substract* 4 0 0 0 { And here the *Product* is 4000, *viz.* 8 times 500, not 8 times 5.

*Divisor* 8) 5 5 2 (60. The Third *Quotient Figure.*

*Substract* 4 8 0 { Also here the *Product* is 480, *viz.* 8 times 60, for the *Reasons* abovesaid.

*Divisor* 8) 7 2 (9. The Fourth *Quotient Figure.*

*Substract* 7 2 { Now here the *Product* is but 72, *viz.* 9 times 8, because the 9 stands in the place of *Units.*

*Remains* (0 0) Now the *Sum* of all the several *Quotients*, *viz.* 8000 + 500 + 60 + 9 = 8569, as before.

If the *Process* of this *Example* be well considered and compared with that of *Multiplication*, *Page* 17, it will evidently appear to be only the *Converse* of that; for the particular *Products* are alike in both, only that which is *last* there, is *first* here; there they are *added*, here they are *substracted.* So that whoever understands the true *Reason* of the one, must needs understand the *Reason* of the other, and then *Division* will become very *easy*, although the *Divisor* consists of several places of *Figures.*

*EXAMPLE.*

Let it be required to *divide* 590624922 by 7563.

*Dividend.*

*Divisor* 7563) 590624922 (

'Tis plain at the first sight, that 7563 the *Divisor*, cannot be taken from 5906, the like *Number* of *Figures* in the *Dividend.*

Therefore, by the *Second Case* of the *Rule* (*Page* 23.) there must be allowed Five *Figures* of the *Dividend*, *viz.* 59062 for the *First Operation* or *Quotient*; that so the *First Figure* 7 of the *Divisor* may be taken out of the two *First Figures*, *viz.* 59 of the *Dividend*, &c.

E Then

Then I proceed (*per Case* 2.) and consider how often 7 may be taken from 59, and find it may be taken 8 times, for 8 times 7 is but 56, which I mentally *substract* from 59, and there *remains* 3; to this 3 I mentally adjoin the *Third Figure* of the *Dividend*, *viz.* 0, which makes it 30, out of which I must take the *Second Figure* of the *Divisor*, *viz.* 5, so often as I took the 7 from 59, which was 8 times: But that cannot be, for 8 times 5 is 40, which is more than 30, therefore 8 is too big a *Figure* to be placed in the *Quotient*; yet, hence I conclude, that the next less, *viz.* 7 may be taken without any further *Trial*. I therefore place 7 in the *Quotient*, and with it *multiply* the *Divisor*, setting down their *Product* under the *Dividend*, and *substract* it from thence, as in the other *Example*, and then the *Work* will stand

```
Thus 7563) 590624922 (7
           52941
           -----
            6121
```

In order to a *Second Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 4, and bring it down to the *Remainder* 6121, which will then become 61214, with which I proceed in all respects as I did before with the 59062, and find the next *Quotient Figure* will be 8, with which I *multiply* the *Divisor*, &c. and *substract* their *Product* from the said 61214. Then the *Work* will stand

```
7563) 590624922 (78
      52941·
      -----
       61214
       60504
       -----
         710
```

To this *Remainder* 710, I point and bring down the next *Figure* of the *Dividend*, *viz.* 9, which makes it 7109; now because the *Divisor* 7563 cannot be taken from 7109, I therefore place a *Cypher* in the *Quotient*.

*And this must always be carefully observed*, viz. *That for every* Figure *or* Cypher, *which is brought down from the* Dividend, *in order to a new* Operation, *there must always be either a* Figure *or* Cypher, *set down in the* Quotient. Then the Work will stand

Thus

Thus 7563) 590624922 (780
52941..

61214
60504

7109

To this 7109, I bring down another *Figure* of the *Dividend*, *viz.* 2, and then it will become 71092; then I consider how often 7 can be taken from 71, *&c.* (just as at the first Operation,) and find it may be taken 9 times, therefore I set down 9 in the *Quotient*, and with it *multiply* the *Divisor*, setting down and *substracting* their *Product*, as before; Then the Work will stand

Thus 7563) 590624922 (7809
52941...

61214
60504

71092
68067

3025

To this *Remainder* 3025, I point and bring down the last *Figure* 2 of the *Dividend*, which makes it 30252; then proceeding in all respects as before, I find the *Quotient Figure* to be 4, with it I *multiply* the *Divisor*, setting down and *substracting* their *Product* as before, and then the Work will stand

Thus 7563) 590624922 (78094
52941....

61214
60504

71092
68067

30252
30252

(00000)

Here the Work is ended, and I find the *Quotient* to be 78094, being the true *Multiplicand* of the proposed *Example* of *Multiplication*, Page 18.

That is, 7563 is contained in 590624922 just 78094 times, *&c.*

If the Work of this *Example* be conſidered and compared with the *Rule* (*Page* 22.) the whole Buſineſs of *Diviſion* will be eaſy; for indeed the only Difficulty (as I ſaid before) lies in making choice of a true *Quotient Figure*, which cannot well be done according to the Common Method of *Diviſion*, without Trials, yet thoſe Trials need not be made with the whole *Diviſor*, (as appears by this laſt *Example*) for by the two Firſt *Figures* of the *Diviſor* all the reſt are generally regulated; except the Second *Figure* chance to be 2, 3, or 4, and at the ſame time the Third *Figure* be 7, 8, or 9, then indeed reſpect muſt be had to the Third *Figure*, according as the *Rule* directs.

However, if thoſe Trials are thought too troubleſome, they may be avoided, and the ſame *Quotient Figure* may both eaſily and certainly be found by help of ſuch a ſmall *Table* made of the *Diviſor*, as was of the *Multiplicand* in *Page* 20.

## *EXAMPLE* 4.

Let it be required to *divide* 70251807402 by 79863. *See the Example of Multiplication, Page* 20, *and as there directed make a Table of the Diviſor* 79863,

Thus,

| | *Diviſor.* | *Dividend.* | *Quotient.* |
|---|---|---|---|
| 1 | 79863) | 70251807402 | (879654 |
| 2 | 159726 | 638904..... | |
| 3 | 239589 | 636140 | |
| 4 | 319452 | 559041 | |
| 5 | 399315 | 770997 | |
| 6 | 479178 | 718767 | |
| 7 | 559041 | 522304 | |
| 8 | 638904 | 479178 | |
| 9 | 718767 | 431260 | |
| 10 | 798630 | 399315 | |
| | | 319452 | |
| | | 319452 | |
| | | (000000) | |

The Work of this Operation I preſume may be eaſily underſtood. For thoſe *Figures* in the *Table* are the *Product* of the *Diviſor* into all the 9 *Figures*; conſequently thoſe *Figures* in the ſmall Column do ſhew what *Figure* is to be placed in the *Quotient*; without any doubtful Trials of the *Diviſor*, with the *Dividend*, as before.

This Method of Tabulating the *Diviſor* may be of good Uſe to a Learner; eſpecially until he is well practiſed in *Diviſion*; yea, and even then if the *Diviſor* be large, and a *Quotient* of many *Figures* be required; as in reſolving of high *Æquations*, and calculating of *Aſtronomical Tables*, or thoſe of Intereſt, &c.

Hitherto

Hitherto I have made choice of *Examples* wherein the *Dividend* is truly meafured or *divided* off by the *Divifor*, without leaving any *Remainder*, being exactly compofed of the *Divifor* and *Quotient*. But it moft ufually falls out, that the *Divifor* will not exactly meafure the *Dividend*; in which cafe the *Remainder* (after *Divifion* is ended) muft be fet over the *Divifor* with a fmall Line betwixt them adjoining to the *Quotient*.

*EXAMPLE* 5.

Suppofe it were required to *divide* 379 by 5.

```
5) 379 (75 4/5  4 the Remainder.
                5 the Divifor.
   35 .
   ----
    29
    25
   ----
Remains (4)
```

*EXAMPLE* 6.

Again, Let it be required to *divide* 43789 by 67.

```
67) 43789 (653 38/67 the true Quotient required,
    402 . .
    -----
     358
     335
     ---
      239
      201
      ---
Remains (38)
```

How fuch *Remainders* thus placed over their *Divifors* (which are indeed *Vulgar Fractions*) may be otherwife managed, fhall be fhewed farther on.

*N. B.* When the *Divifor* happens to be an *Unit*, viz. 1, with a *Cypher* or *Cyphers* annexed to it, as 10, 100, 1000, &c. *Divifion* is truly performed by cutting off with a Point or Comma, fo many *Figures* of the *Dividend* as there are *Cyphers* in the *Divifor*; then are thofe *Figures* fo cut off to be accounted a *Remainder*, and the reft of the *Figures* in the *Dividend* will be the true Quotient required, becaufe an *Unit* or 1 doth neither *multiply* nor *divide*.

*EXAMPLE* 7.

Let it be required to *divide* 57842 by 100. The Work may ftand thus, 100) 578,42 the *Quotient* required; or thus 100) 57842 (578 42/100 the fame as before,

Hence it follows, that if any *Divifor* have *Cyphers* to the Right-hand of it, you may cut off fo many of the laft *Figures* in in the *Dividend*, and *divide* the other *Figures* of the *Dividend*, by thofe *Figures* of the *Divifor* that are left when the *Cyphers* are omitted. But when *Divifion* is ended, thofe *Cyphers* fo omitted in the *Divifor*, and the *Figures* cut off in the *Dividend*, are both to be reftored to their own places.

### *EXAMPLE* 8.

Suppofe it were required to *divide* 675469 by 5400.

```
5400) 675469 (125
 ——    ——
       54..
       ——
       135
       108
       ——
        274
       560
       ——
```

*Remains* (4) But the true *Remainder* is 469. Confequently the true *Quotient* is $125\frac{469}{5400}$.

As to the manner of proving the Truth of any Operation, either in *Multiplication* or *Divifion*, I prefume it may be eafily underftood, by what is delivered in *Page* 21, compared with the three firft *Examples* of *Divifion*; for from thence it will be eafy to conceive, that if the *Divifor* and *Quotient* be *multiplied* together, their *Product* (with what *Remains* after *Divifion* being added to that *Product*) will be equal to the *Dividend*. As in the Fifth *Example*, where the *Dividend* is 379, the *Divifor* is 5, the *Quotient* is 75, and the *Remainder* is 4.

I fay, $75\times5=375$, to which *add* the *Remainder* 4, it will be 379.

Again, in the Sixth *Example*, the *Divifor* is 67, the *Quotient* is 653, and the *Remainder* is 38.

Then $653\times67=43751$, and $43751+38=43789$ the *Dividend*, &c.

*There are feveral ufeful Contractions, both in* Divifion *and* Multiplication, *which I have purpofely omitted until I come to treat of* Decimal Arithmetick. Alfo I have omitted the Bufinefs of Evolution or Extracting of *Roots*, until further on; and fo fhall conclude this *Chapter* with a few *Examples* of *Divifion* unwrought at large, leaving them for the Learner's Practice.

579) 43800771 (75649.
Or 75649) 43800771 ( 579.

45007)

45007) 23884044718 (530674.
Or 530674) 23884044718 ( 45007.
356) 244572000 (687000.
59000) 57659066400 (967434.
10000) 679543820000 (67954382.
79) 282016 ( $3569\frac{65}{79}$.

## CHAP. III.

*Concerning* **Addition** *and* **Substraction** *of* Numbers *of different* Denominations, *and how to* reduce *them from one* Denomination *to another.*

## SECT. 1.

### 1. *Of* English *Coin.*

THE leaſt Piece of *Money* uſed in *England* is a *Farthing*, and from thence ariſeth the reſt, as in this *Table*.

| | | | |
|---|---|---|---|
| *Farth.* | | | |
| 4 = | 1 *d. Pen.* | | |
| 48 = | 12 = | 1 *s. Shill.* | |
| 960 = | 240 = | 20 = | 1 *l. Pound Sterling.* |

| | |
|---|---|
| And | 5 *s.* is a *Crown.* |
| | 10 *s.* is an *Angel.* |
| | 6 *s.* 8 *d.* a *Noble.* |
| | 13 *s.* 4 *d.* a *Mark.* |

*Note*, When *l. s. d. q.* are placed over (*or to the Right-hand of*) *Numbers*, they denote thoſe *Numbers* to ſignify *Pounds*, *Shillings*, *Pence*, and *Farthings*.

| | *l.* | *s.* | *d.* | *q.* |
|---|---|---|---|---|
| As | 35 | 10 | 6 | 2. |

Or 35 *l.* 10 *s.* $6\frac{1}{2}$ *d.* Either of theſe do ſignify 35 *Pounds*, 10 *Shillings*, 6 *Pence*, 2 *Farthings*.

The ſame muſt be underſtood of all the following *Characters*, belonging to their reſpective *Tables*, *viz.* Of *Weights*, *Meaſures*, &c.

### 2. *Troy Weight.*

The Original of all *Weights* uſed in *England*, was a *Corn* of *Wheat* gathered out of the *middle* of the *Ear*, and being well dried, 32 of them were to make one *Penny Weight*, 20 *Penny Weights* one *Ounce*, and 12 *Ounces* one *Pound Troy*. *Vide Statutes* of 51 *Hen.* III. 31 *Edw.* I. 12 *Hen.* VII.

But

But in later Times it was thought ſufficient to *divide* the aforeſaid *Penny Weight* into 24 equal *Parts*, called *Grains*, being the leaſt *Weight* now in common Uſe; and from thence the reſt are computed as in this *Table*.

| *Gr. Grain.* | | | |
|---|---|---|---|
| 24 = | 1 *P. W. Penny Weight.* | | |
| 480 = | 20 = | 1 *Oz. Ounce.* | |
| 7560 = | 240 = | 12 = | 1 ℔ *Pound.* |

*Note*, By *Troy Weight* are weighed *Jewels*, *Gold*, *Silver*, *Corn*, *Bread*, and all *Liquors*.

Beſides the common *Diviſions* of *Troy Weight*, I find in *Angliæ Notitia*, or, *The Preſent State of England*, Printed in the Year 1699, that the *Moneyers* (*as that Author calls them*) do *ſubdivide* the *Grain*.

Thus:
- 24 *Blanks* = 1 *Periot.*
- 20 *Periots* = 1 *Droite.*
- 24 *Droites* = 1 *Mite.*
- 20 *Mites* = 1 *Grain*, &c. as before.

## 3. *Apothecaries Weights.*

The *Apothecaries divide* a *Pound Troy*, as in this *Table*.

| *Gr. Grain.* | | | | |
|---|---|---|---|---|
| 20 = | 1 ℈ *Scruple* | | | |
| 60 = | 3 = | 1 ʒ *Dram* | | |
| 480 = | 24 = | 8 = | 1 ℥ *Ounce* | |
| 5760 = | 288 = | 96 = | 12 = | 1 ℔ *Troy*, the ſame as before. |

By theſe *Weights* the *Apothecaries* compound their Medicines: but buy and ſell their Drugs by *Averdupois Weight*.

## 4. *Averdupois Weight.*

When *Averdupois Weight* became firſt in *Uſe*, or by what *Law* it was firſt ſettled, I cannot find out in the *Statute Books*; but on the contrary, I find that there ſhould be but one *Weight* (*and one Meaſure*) uſed throughout this *Realm*, *viz.* that of *Troy*, (*Vide* 14 *Ed.* III. and 17 *Ed.* III.) So that it ſeems (*to me*) to be firſt introduced by *Chance*, and ſettled by *Cuſtom*, *viz.* from giving good or large *Weight* to thoſe *Commodities* uſually weighed by it, which are ſuch as are either very *Coarſe* and *Droſſy*, or very

very subject to waste; as all kind of Grocery Wares. And *Pitch*, *Tar*, *Rosin*, *Wax*, *Tallow*, *Flax*, *Hemp*, &c. *Copper*, *Tin*, *Steel*, *Iron*, *Lead*, &c. Also *Flesh*, *Butter*, *Cheese*, *Salt*, &c. To these and the like (I presume, it was thought convenient to allow a greater *Weight* than the Laws had provided, which happen'd to be about a Sixth part more: For I found by a very nice Experiment, that one *Pound Averdupois* is equal to 14 *Ounces*, 11 *Penny Weights*, and 15½ *Grains Troy*. And it is now computed as in the following Table:

| *Drams.* | | | | |
|---|---|---|---|---|
| 16 = | 1 *Oz.* *Ounces.* | | | |
| 256 = | 16 = | 1 lb *Pounds.* | | |
| 28672 = | 1792 = | 112 = | 1 *C.* *Hundred.* | |
| 573440 = | 35840 = | 2240 = | 20 = | 1 *Tun.* |

And { lb 14 = *a Stone*; 28 = ¼ *of C.*; 56 = ½ *of C.*; 84 = ¾ *of C.*

## 5. *Long Measure.*

As the least part of *Weight* came at first from a *Wheat Corn*, so (it is generally said) the least part of a *Long Measure* was at first a *Barley Corn*, taken out of the middle of the Ear, and being well dried, three of them in length were to make one *Inch*; and thence the rest, as in this Table.

| *Barley Corns.* | | | | | | |
|---|---|---|---|---|---|---|
| 3 = | 1 *In.* *Inches.* | | | | | |
| 36 = | 12 = | 1 *F.* *Feet.* | | | | |
| 108 = | 36 = | 3 = | 1 *Y.* *Yards.* | | | |
| 594 = | 198 = | 16½ = | 5½ = | 1 *P.* *Poles.* | | |
| 23760 = | 7920 = | 660 = | 220 = | 40 = | 1 *Furlong.* | |
| 190080 = | 63360 = | 5280 = | 1760 = | 320 = | 8 = | 1 *Mile.* |

And { 4 *Nails* = ¼ of a *Yard.*; 1¼ *Yard* = 1 *Ell.*; 2 *Yards* = 1 *Fathom.*

*Note*, That forty *Poles* (or *Perches*) in Length, and four in Breadth, do make a Statute *Acre* of *Land*.

That is, 220 *Yards*, *multiplied* into 22 *Yards* = 4840 Square *Yards* are a Statute *Acre*.

And according to the Transactions of the *French Academy*, *Anno* 1687, a *Paris Foot Royal* is = 12,8 *Inches English*; Six of those Feet make a *Toise*; and 57060 *Toises* = [illegible]5184 *English Feet*, are the *Measure* of one Degree of a great Circle upon the Surface of the Earth. So that one Degree is 69 *Miles* and 288 *Yards*, which is very near to our Country-man Mr *Norwood*'s Experiment made betwixt *London* and *York*, *Anno* 1635; who found that 367196 *Feet* = 69 *Miles*, and 958 *Yards* do make a

F Degree.

Degree. And not 60 *Miles*, according to the common received Opinion and Practice of the *Navigators* or *Seamen*.

Hence, according to the *French* Account, the Circumference of the Earth (supposing it to be a true *Spherical Figure*) is 24899 *English Miles*.

## 6. *Of Liquid Measures.*

All *Measures* of Capacity, both Liquid and Dry, were at first made from *Troy Weight*, *Vide Statutes* 9 *H.* III. 51 *H.* III. 12 *H.* VII. *&c.* wherein it is enacted, that eight *Pound Troy Weight*, of *Wheat*, gathered out of the middle of the Ear, and well dried, should make one *Gallon* of *Wine Measure:* And that there should be but one *Measure* for *Wine*, *Ale*, and *Corn*, throughout this Realm. (*Vid. Stat.* 14 *Ed.* III. 15 *Rich.* II.) But Time and Custom hath altered *Measures*, as they have done *Weights* (and perhaps for one and the same Reason) for now we have three different *Measures*, *viz.* one for *Wine*, one for *Ale* or *Beer*, and one for *Corn*.

I have inserted *Tables* of each, as they are now computed by *Cubick Inches*, and practised in the Art of *Gauging*, &c.

The common *Wine Gallon* sealed at *Guild-Hall* in *London*; by which all *Wines*, *Brandies*, *Spirits*, *Strong-waters*, *Mead*, *Perry*, *Cyder*, *Vinegar*, *Oil*, and *Honey*, &c. are measured and sold; is supposed to contain 231 *Cubick Inches*, and from thence the rest are computed, as in this Table.

| *Cubick Inches.* | | | | | | |
|---|---|---|---|---|---|---|
| 231 = | 1 *G. Gallons.* | | | | | |
| 9702 = | 42 = | 1 *Terce.* | | | | |
| 14553 = | 63 = | 1½ = | 1 *Hogshead.* | | | |
| 19404 = | 84 = | 2 = | 1⅓ = | 1 *Puncion.* | | |
| 29106 = | 126 = | 3 = | 2 = | 1½ = | 1 *Butt or Pipe.* | |
| 58212 = | 252 = | 6 = | 4 = | 3 = | 2 = | 1 *Tun.* |

*Note*, *Gallons.* 18 = 1 *Rundlet*, and 31½ makes a *Wine* or *Vinegar Barrel.* (*Vide* 1 *R.* III.)

But Dr *Wybard* in his Tectometry, *Page* 289, doth suppose the *Wine Gallon* to contain but 224, or 225 *Cubick Inches* at the most, and pursuant to this Account an Experiment was made by Mr *Richard Walker* and Mr *Philip Shales*, two General Officers in the Excise. They caused a Vessel to be very exactly made of Brass, in Form of a *Parallelopipedon*, each Side of its Base was 4 *Inches*, and its Depth 14 *Inches*; so that its just Content was 224 *Cubick Inches.* This Vessel was produced at *Guild-Hall* in *London* (*May* 25, 1688.) before the *Lord-Mayor*, the *Commissioners* of *Excise*, the Reverend Mr *Flamstead*, Astr. Reg. Mr

Mr *Halley*, and ſeveral other ingenious Gentlemen, in whoſe Preſence Mr *Shales* did exactly fill the aforeſaid Brazen Veſſel with clear Water, and very carefully emptied it into the old Standard *Wine Gallon* kept in *Guild-Hall*, which did ſo exactly fill it, that all then preſent were fully ſatisfied the *Wine Gallon* doth contain but 224 *Cubick Inches. (This notable Experiment I ſaw tried.)* However, for ſeveral Reaſons, it was at that time thought convenient to continue the former ſuppoſed Content of 231 *Cubick Inches* to be the *Wine Gallon*, and that all Computations in *Gauging* ſhould be made from thence, as above.

The *Beer* or *Ale Gallon* (which are both one) is much larger than the *Wine Gallon*; it being (as I preſume) made at firſt to correſpond with *Averdupois Weight*, as the *Wine Gallon* did with *Troy Weight*: For (as I ſaid before, *Page* 33.) one *Pound Averdupois* is equal to 14 *Ounces* 12 *Penny Weights Troy*, very near.

And, as one *Pound Troy* is in proportion to the *Cubick Inches* in a *Wine Gallon*, ſo is one *Pound Averdupois* to the *Cubick Inches* in an *Ale Gallon*. That is, 12 : 231 :: $14\frac{1}{2}\frac{2}{0}$ : $281\frac{1}{2}$, very near the *Cubick Inches* contained in an *Ale Gallon*, as appears from an Experiment made by one *Nicholas Gunton*, General *Gauger* in the *Exciſe*, about 41 Years ago, who, by ſuch a Veſſel mentioned before in the laſt *Page*, did find the Standard *Ale-Quart* (kept in the *Exchequer*, *Vid.* 12 *Car.* II.) to contain juſt $70\frac{1}{2}$ *Cubick Inches*, conſequently the *Ale Gallon* muſt contain 282 *Cubick Inches*, and from thence the following *Tables* are computed.

*Ale-Meaſure.*

*Cubick Inches.*
282 = 1 *Gallon.*
2256 = 8 = 1 *Firkin.*
4512 = 16 = 2 = 1 *Kilderkin.*
9024 = 32 = 4 = 2 = 1 *Barrel.*
13536 = 48 = 6 = 3 = $1\frac{1}{2}$ = 1 *Hogſhead.*

Note, *A Firkin of Soap and of Herrings are the ſame with that of Ale.*

*Beer Meaſure.*

*Cub. Inches.*
282 = 1 *Gallon.*
2538 = 9 = 1 *Firkin.*
5076 = 18 = 2 = 1 *Kilderkin.*
10152 = 36 = 4 = 2 = 1 *Barrel.*
15228 = 54 = 6 = 3 = $1\frac{1}{2}$ = 1 *Hogſhead.*

*N. B.* This Diftinction or Difference betwixt *Ale* and *Beer-Meafure*, is now only ufed in *London*. But in all other Places of *England* the following *Table* of *Beer* or *Ale*, whether it be ftrong or fmall, is to be obferved, according to a Statute of *Excife* made in the Year 1689

| *Cub. Inches.* | | | | | |
|---|---|---|---|---|---|
| 282 = 1 *Gallon.* | | | | | |
| 2397 = | $8\frac{1}{2}$ = 1 *Firkin.* | | | | |
| 4794 = | 17 = | 2 = 1 *Kilderkin.* | | | |
| 9588 = | 34 = | 4 = | 2 = 1 *Barrel.* | | |
| 14382 = | 51 = | 6 = | 3 = | $1\frac{1}{2}$ = 1 *Hogfhead.* | |

7. Of *Dry Meafure.*

*Dry Meafure* is different both from *Wine* and *Ale Meafure*, being as it were a Mean betwixt both, tho' not exactly fo; which upon Examination I find to be in proportion to the aforefaid old Standard *Wine Gallon*, as *Averdupois Weight* is to *Troy Weight*; That is, As one *Pound Troy* is to one *Pound Averdupois*, fo is the *Cubick Inches* contained in the old *Wine Gallon*, to the *Cubick Inches* contained in the *Dry* or *Corn Gallon.*

*Viz.* 12 : $14\frac{12}{20}$ : : 224 : $272\frac{1}{2}$, which is very near to $272\frac{1}{4}$, the common received Content of a *Corn Gallon:* Altho' now it is otherwife fettled by an Act of Parliament made in *April* 1697, the Words of that Act are thefe:

*Every round* Bufhel *with a plain and even Bottom, being made* eighteen Inches *and a half wide throughout, and eight* Inches *deep, fhould be efteemed a Legal* Winchefter Bufhel, *according to the Standard in his* Majefty's Exchequer.

Now a Veffel being thus made will contain 2150,42 *Cubick Inches*, confequently the *Corn Gallon* doth contain but $268\frac{4}{5}$ *Cubick Inches.*

| *Cub. Inches.* | | | | |
|---|---|---|---|---|
| 268,8 = | 1 *Gallon.* | | | |
| 537,6 = | 2 = | 1 *Peck.* | | |
| 2150,4 = | 8 = | 4 = | 1 *Bufhel.* | |
| 17203,2 = | 64 = | 32 = | 8 = | 1 *Quarter.* |

Note, { 4 *Bufhels* = a *Comb.* 10 *Quarters* = a *Wey*, and 12 *Weys* = a *Laft of Corn.* }

I obferved amongft the Lead-Mines in *Derbyfhire*, (*Anno* 1692) that the *Miners* bought and fold their Lead Ore, by a *Meafure* which they call'd an Ore Difh; whofe Dimenfions I carefully took, and found it

| | | | |
|---|---|---|---|
| Thus | Length | 21.3. | *Inches.* |
| | Breadth | 6. | |
| | Depth | 8.4. | |

Confe-

Consequently its Content is 1073,52 *Cubick Inches*, which is very near equal to 4 *Corn Gallons*, according to the above-mentioned Settlement.

Nine of those Dishes they call a Load of Ore, which if it be pretty good, will produce about 3 hundred Weight of Lead.

## 8. *Of* Time.

It is not an easy Thing to give a true *Definition* of *Time*; for (according to the *Philosophick Poet*)

> Time *of itself is nothing, but from Thought*
> *Receives its Rise, by labouring Fancy wrought*
> *From Things consider'd, whilst we think on some*
> *As present, some as past, or yet to come.*
> *No Thought can think on* Time, *that's still confest,*
> *But thinks on Things in Motion or at Rest.*

And so on, Vide *Lucretius*, Book I.

That is, *Time* only shews the *Duration* or *Mutation* of Things, a Year being the *Standard* or *Integer*, by which such Continuance or Change is computed. And a *Year* is that *Space* of *Time* in which the *Sun* (apparently) compleats its *Revolution* from any one *Point* in the *Ecliptick* (an imaginary *Circle* in the *Heavens*) to the same *Point* again, which, according to modern *Observations*, is performed in 365 *Days*, 5 *Hours*, 48 *Minutes*, 57 *Seconds*, 21 *Thirds*, &c. But a *Second* being the least part of *Time* that can be truly *measured* by the *Motion* of any *Mechanical Engine*, as a *Clock*, &c. (a *Third* being less than the Twinkling of an Eye) I begin the following Table with *Seconds*.

| *Seconds*.'' | | | | |
|---|---|---|---|---|
| 60=1' | *Minute*. | | | |
| 3600= | 60=1 | ° *Hour*. | | |
| 86400= | 1440= | 24=1 | *Day*. ° ' '' | |
| 31556937= | 525949= | 8765= | 365+5+48+57= | 1 *Year*, called *a Solar Year*. |

But the common *Year*, usually called the *Julian Year*, doth consist of 365 *Days* and 6 *Hours*, and is divided into twelve unequal *Months*, called *Calendar Months*, whose Names and *Number* of *Days* are the Subject of every *Almanack*.

To

To these *Tables* it may not be amiss to give a brief Account of such *Coins*, *Weights*, and *Measures*, as are frequently mentioned in the Scriptures. As I have deduced them from those which seem to be the most Correct, inserted in the *Index* to the large *Bible*, Printed *Anno* 1702, and compared with those used in *England*, by the Lord Bishop of *Peterborough* [*Cumberland*].

The *Hebrew Weights*, compared with *Troy Weight*.

| | Oz. | Pw. | Gr. |
|---|---|---|---|
| *A Gerah*= | 0 | 0 | $10\frac{12}{20}$ |
| 10 *Gerahs*=*a Bekah*= | 0 | 4 | $13\frac{1}{2}$ |
| 2 *Bekahs*=*a Shekel*= | 0 | 9 | 3 |
| 100 *Shekels*=*a Menah*= | 45 | 12 | 12 |

*Note*, A *Shekel* is said to be their Original *Weight*.

Their *Coin*, *English Coin*.

| | l. | s. | d. | |
|---|---|---|---|---|
| A *Silver Menah*= | 7 | 1 | $5\frac{1}{4}$ | Weight 60 *Shekels*. |
| *Talent* of *Silver*= | 357 | 11 | $10\frac{1}{2}$ | Weight is 300 *Shekels*. |
| *Talent* of *Gold*= | 5075 | 15 | $7\frac{1}{2}$ | The same Weight men- |
| The *Gold Dram*= | 1 | 0 | 4 | tioned *Ez.* ii. 19. |

The *Roman Money* mentioned in the *New Testament*.

A *Denarius*, or *Silver Penny*=7 *d.* 3 *Farthings*.
*Asses* of *Copper*=0 . 3 *Farthings*.
*Assarium*=0 . $1\frac{1}{2}$ *Farthing*.
*Quadrans*=0 . $\frac{3}{4}$ of a *Farthing*.
*A Mite*=0 . $\frac{1}{3}$ of a *Farthing*.

Their *Long Measure*, compared with *English Measure*.

| | Yar. | Feet. | In. Pts. |
|---|---|---|---|
| *A Finger's Breadth*= | 0 | 0 | 0,912 |
| 4 *Fingers*=*a Hand's Breadth*= | 0 | 0 | 3,648 |
| 2 *Hands*=*the least Span*= | 0 | 0 | 7,296 |
| 3 *Hand's Breadth*=*the longest Span*= | 0 | 0 | 10,944 |
| 2 *Spans*=*the longest Cubit*= | 0 | 1 | 9,888 |
| 4 *Cubits*=*a Fathom*= | 2 | 1 | 3,552 |
| 6 *Cubits*=*Ezekiel's Reed*= | 3 | 1 | 11,328 |
| 400 *Cubits*=*a Stadium*= | 243 | 0 | 7,2 |
| 10 *Stadiums*=*a Mile*= | 2432 | 0 | 0 |
| 3 *Miles*=*a Parasang*= | 7296 | 0 | 0 |
| Which is 4 *English Miles* and | 256 | | |

Their

| Their *Measures* of *Capacity*, compared with | *English Wine.* *Gal.* | *Pints.* | *Inch.* |
|---|---|---|---|
| *A Cotyla* = | 0 | . $0\frac{1}{2}$ | 3,037 |
| *A Log* = | 0 | . $0\frac{1}{2}$ | 9,83 |
| 4 *Logs* = *a Cab* = | 0 | . 3 . | 10,458 |
| 10 *Catyla's* = *an Omer* = | 0 | . 6 . | 1,5 |
| 3 *Cabs* = *a Hin* = | 1 | . 2 . | 2,5 |
| 2 *Hins* = *a Seah* = | 2 | . 4 . | 5, |
| 3 *Seahs* = *an Epha* = | 7 | . 4 . | 15, |
| 10 *Epha's* = *a Chomer* = | 75 | . 5 . | 5,625 |

## Sect. 2. Addition of *Weights*, &c.

The foregoing *Tables* being so well understood, as that you can readily tell (without pausing) how many *Units* of any one *Denomination*, do make one of the next *Superior Denomination (especially in those* Tables *which are most useful for your Business)* it will then be as easy to *add* or *substract* them, as to *add* or *substract whole Numbers*, due Care being taken in placing all *Numbers* that are of one *Denomination* exactly underneath each other. That is to say, in *Money*, place *Pounds* under *Pounds*, *Shillings* under *Shillings*, *Pence* under *Pence*, &c. Understand the like in *Weights* and *Measures*, &c. according to their several *Denominations:* Then in *Addition* observe this *Rule.*

RULE.

*Always begin with those* Figures *of the lowest or least* Denomination, *and* add *them all together into one* Sum, *then consider how many of the next* Superior Denomination *are contained in that* Sum, *so many Units you must carry to the said next* Superior Denomination *to be* added *together with those* Figures *that stand there*; *and if any thing remain over or above those* Units *so carried, that* Overplus *must be set down underneath its own* Denomination: *And so proceed on from one* Denomination *to another until all be finished.*

*Example in* Coin.

Let it be required to *add* 35 *l.* 14 *s.* 06 *d.* and 27 *l.* 02 *s.* 10 *d.* and 54 *l.* 13 *s.* 04 *d.* and 10 *l.* 17 *s.* 09 *d.* into one *Sum.*

The particular *Sums* being placed, as before directed, will stand as in the Margin following.

Then according to the *Rule*, I begin with the *Pence* (being here the lowest or least *Denomination*) and *adding* them all together, I find their *Sum* to be 29 *d.* that is 2 *s.* and 5 *d.* (for

24 *d.*

24 = 2 *s.* and 29 — 24 = 5) the 5 *d.* I ſet down underneath its own *Denomination*, and carry the 2 *s.* to the Place of *Shillings*, *adding* them and all the *Shillings* together, I find the *Sum* to be 48 *s.* *viz.* 2 *l.* 8 *s.* I ſet down the 8 *s.* underneath its own place of *Shillings*, and carry the 2 *l.* to the Place of *Pounds*, *adding* them and all the *Pounds* together, I find their *Sum* is 128 *l.* conſequently the *Total Sum* required is 128 *l.* 08 *s.* 05 *d.*

| *l.* | *s.* | *d.* |
|---|---|---|
| 35 | 14 | 06 |
| 27 | 02 | 10 |
| 54 | 13 | 04 |
| 10 | 17 | 09 |
| 128 | 08 | 05 |

Now, for as much as it often happens in keeping Books of *Accounts*, (and in other Buſineſs) that it is required to *add* up large *Sums* of Money, conſiſting of 30, 40, or more ſeveral particular *Sums*, nay, perhaps, filling up the whole length of a Sheet of Paper, I humbly conceive in thoſe Caſes the beſt and eaſieſt way will be to part them into Parcels, not exceeding above 10 or 12 particular *Sums* in each Parcel; that done, add together all the *Sums* of thoſe Parcels into one *Sum*, and that will be the *Total Sum* required.

Alſo to avoid the making of *Points*, or other *Marks* amongſt your *Figures*, it will be convenient to get the following *Tables* by heart,

| *The* Pence Table. | | *The* Shillings Table. | |
|---|---|---|---|
| *d.* *s.* | *d.* *s.* | *s.* *l.* | *s.* *l.* |
| 12=1 | 72= 6 | 20=1 | 120= 6 |
| 24=2 | 84= 7 | 40=2 | 140= 7 |
| 36=3 | 96= 8 | 60=3 | 160= 8 |
| 48=4 | 108= 9 | 80=4 | 180= 9 |
| 60=5 | 120=10 | 100=5 | 200=10 |

The Uſe of theſe *Tables* is ſo obvious, that I preſume it is needleſs to explain them.

*Examples in* Addition *of* Weights.

*Troy Weight.*

| | lb. | *Oz.* | *Pw.* | *Gr.* |
|---|---|---|---|---|
| | 3 | 09 | 00 | 10 |
| | 5 | 08 | 15 | 21 |
| | 10 | 10 | 12 | 22 |
| | 0 | 11 | 19 | 23 |
| *Sum* | 21 | 04 | 09 | 04 |

*Averdupois Weight.*

| | *Tun.* | *C.* | *Q.* | *lb.* | *Oz.* |
|---|---|---|---|---|---|
| | 12 | 15 | 2 | 24 | 12 |
| | 7 | 10 | 3 | 21 | 15 |
| | 0 | 18 | 1 | 14 | 11 |
| | 1 | 19 | 3 | 27 | 15 |
| *Sum* | 23 | 05 | 0 | 05 | 05 |

*Examples*

*Examples* in *Addition* of *Long-Meaſure.*

| *Yards* | *Qrs.* | *Nails* | *Miles* | *Fur.* | *Poles* | *Yards* | *Feet* | *Inch.* |
|---|---|---|---|---|---|---|---|---|
| 35 | 2 | 3 | 2 | 6 | 32 | 4 | 2 | 9 |
| 17 | 3 | 1 | 0 | 7 | 27 | 3 | 1 | 10 |
| 129 | 1 | 2 | 1 | 3 | 39 | 2 | 2 | 11 |
| 182 | 3 | 2 Sum | 5 | 2 | 10 | 0 | 1 | 6 |

I think it needleſs to ſet down more *Examples* of this kind, for if theſe 5 (eſpecially the laſt) be well underſtood, they will be ſufficient to ſhew how any other may be performed.

## Sect. 3. Subſtraction *of Weights*, &c.

SUbſtraction is but the Converſe of the precedent Work, and may be performed by obſerving this *Rule.*

RULE.

*Begin with the Loweſt or Leaſt* Denomination (*as before in* Addition) *and* Take *or* Subſtract *the* Figure (*or* Figures) *in that place of the* Subtrahend, *from the* Figure (*or* Figures) *that ſtand over them of the ſame* Denomination; *ſetting down the* Remainder. (*as in* Page 12.) *But if that cannot be done, then you muſt increaſe the upper* Figure (*or* Figures) *with one of the next Superior* Denomination, *and from that* Sum *make* Subſtraction; *and ſo proceed to the next Superior* Denomination, *where you muſt pay the one borrowed, by adding* Unity *to the* Subtrahend *in that place*, &c. *as in whole* Numbers.

*Examples in Coin.*

| | *l.* | *s.* | *d.* | | *l.* | *s.* | *d.* |
|---|---|---|---|---|---|---|---|
| From | 386 | 09 | 08 | From | 569 | 10 | 06 |
| Take | 173 | 04 | 06 | *Subſt.* | 389 | 15 | 08 |
| *Remains* | 213 | 05 | 02 | | 179 | 14 | 10 |

The Firſt of theſe *Examples* is ſelf-evident. In the Second *Example*, beginning at the place of *Pence* (being here the Leaſt *Denomination*) I am to take 8*d.* from 6*d.* but becauſe that cannot be done, I muſt (according to the *Rule*) borrow one of the next *Denomination*, viz. 1*s.* and *add* it to the 6*d.* which makes it 18*d.* (for 1*s.*=12*d.* and 12*d.*+6*d*=18*d.* then I take 8*d.* from that 18*d.* and there remains 10*d.* to be ſet down underneath the place of *Pence*; that done, I proceed to the place of *Shillings*, where I muſt now pay the 1*s.* ſaying one borrowed and 15 makes 16 from 10 cannot be, but

16 from 30 and there *remains* 14. That is, I borrow one of the next *Denomination*, viz. 1 *l.* and add to it the 10 *s.* which makes it 30 *s.* for 1 *l.*=20 *s.* and 20 *s.*+10=30) having set down the *Remaining* 14 *s.* underneath its own place of *Shillings*, I proceed to the place of *Pounds*, where paying the 1 *l.* borrowed, it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there *Remains* 9, and so on as in whole *Numbers* until all be finished; and the *Remainder* will be 179 *l.* 14 *s.* 10 *d.*

This *Example* being a little considered will render all others in this *Rule* easy.

*Examples in Weights.*

*Troy Weight.*

| | *lb.* | *oz.* | *pwt.* | *gr.* |
|---|---|---|---|---|
| From | 9 | 10 | 16 | 18 |
| Take | 5 | 09 | 18 | 22 |
| *Rests* | 4 | 00 | 17 | 20 |

*Averdupois Weight.*

| *c.* | *qr.* | *lb.* | *oz.* |
|---|---|---|---|
| 17 | 2 | 15 | 10 |
| 14 | 3 | 18 | 12 |
| 2 | 2 | 24 | 14 |

*Examples in Long Measure.*

| | *yds.* | *qrs.* | *nails* |
|---|---|---|---|
| From | 78 | 3 | 2 |
| Take | 29 | 3 | 3 |
| *Rests* | 48 | 3 | 3 |

| *miles* | *fur.* | *pol.* | *yds.* | *feet* | *inches* |
|---|---|---|---|---|---|
| 22 | 3 | 26 | 3½ | 0 | 9 |
| 18 | 6 | 29 | 4 | 2 | 11 |
| 3 | 4 | 36 | 4 | 0 | 10 |

*Example in Time.*

| | *days* | ° | ′ | ″ |
|---|---|---|---|---|
| From | 27 | 18 | 35 | 21 |
| *Substract* | 16 | 21 | 46 | 36 |
| *Remains* | 10 | 20 | 48 | 45 |

The Proof of *Addition* and *Substraction* in these *Numbers* of different *Denominations*, is the very same with that of whole *Numbers* in *Page* 13. I shall therefore refer you to that place, and omit repeating it here.

## Sect. 4. *Of* Reduction.

BY *Reduction*, *Numbers* of different *Denominations* are brought into one *Denomination*.

That is, it alters or changes any Superior *Denomination* proposed, into any Inferior or Lesser *Denomination* Required; still

ſtill keeping them equivalent in value. And by that means they become fitly prepared for *Multiplication* and *Diviſion*; which otherwiſe could not ſo conveniently be performed. Therefore the Buſineſs of *Reduction* is very uſeful in the *Rule of Proportion*, (commonly called the *Golden Rule*, or *Rule of Three*) eſpecially to thoſe who do not underſtand either *Vulgar* or *Decimal Fractions*. And it is thus performed;

## RULE.

*Conſider how many* Units *of the* Denomination *Required, make one of that* Denomination *propoſed to be Reduced (which is eaſily known by its reſpective Table) and with that* Number *of* Units, Multiply *the* Denomination *propoſed, and their* Product *will be the* Number *Required.*

### *Example in Coin.*

Let it be Required to Reduce or Change 357 *l.* into *Shillings*, and thoſe *Shillings* into *Pence*, which ſhall ſtill be equal in value with the 357 *l.*

357
*Multiply* with 20 the *Shillings* in one *Pound*.

7140 = the *Shillings* in 357 *l.*
*Multiply* with 12 the *Pence* in one *Shilling*.

1428
714

85680 = the *Pence* in 357 *l.* as was Required.

Or 357 *l.* may be reduced into *Pence*, at one Operation; Thus,

357 *l.*
*Multiply* with 240 the *Pence* contained In one *Pound*.

1428
714

85680 = the *Pence* in 357 *l.* as before.

But when the *Numbers* propoſed to be Reduced are of ſeveral *Denominations*, and it is required to bring them all to the Loweſt; you muſt Reduce the higheſt or greateſt *Denomination* to the next leſs, Adding the *Numbers* that are of that leſs *Denomination* together, then *Reduce* their *Sum* to the next lower *Denomination*, Adding together all the *Numbers* that are of that *Denomination*, and ſo proceed gradually on 'till all is done.

*EXAMPLE.*

Let it be required to Reduce 375 *l.* 17 *s.* 10 *d.* 3 *q.* into one *Denomination*, viz. into *Farthings.*

375 *l.* 17 *s.* 10 *d.* 3 *q.*
20
―――
7500 = the *Shillings* in 375 *l.*
+ 17 *s.*
―――
7517 = the *Shillings* in 375 *l.* 17 *s.*
12
―――
15034
7517
―――
90204 = the *Pence* in 375 *l.* 17 *s.*
+ 10 *d.*
―――
90214 = the *Pence* in 375 *l.* 17 *s.* 10 *d.*
4
―――
360856 = the *Farthings* in 375 *l.* 17 *s.* 10 *d.*
+ 3 *q.*
―――
360859 *Farth.* = 375 *l.* 17 *s.* 10 *d.* 3 *q.* as was required.

The Work of this *Example*, and all other Operations of this kind, may be somewhat shortened by observing the following Method.

375 *l.* 17 *s.* 10 *d.* 3 *q.*
20 *Multiply* and *Add* in the 17 *s.*
―――
7517
12 *Multiply* and *Add* in the 10 *d.*
―――
15034
7517
―――
90214
4 *Multiply* and *Add* in the 3 *qrs.*
―――
360859 the *Farthings* as before.

*Example in Troy Weight.*

Suppose it be Required to *Reduce* 29 *lb.* 8 *oz.* 18 *pwt.* 21 *gr.* into the Least *Denomination*, viz. into *Grains.*

Thus,

Thus 29 *lb.* 8 *oz.* 18 *pwt.* 21 *gr.*
*Multiply* with 12 the *oz.* in 1 *lb.* and *add* in the 8 *oz.*

66
29

356 = the *oz.* in 29 *lb.* 8 *oz.*
*Multiply* with 20 the *pwts* in 1 *oz.* and *add* in the 18 *pwt.*

7138 = the *pwts* in 29 *lb.* 8 *oz.* 18 *pwt.*
*Multiply* with 24 the *grs* in 1 *pwt.* and *add* in the 21 *grs.*

28553
14278

171333 the *grs* = 29 *lb.* 8 *oz.* 18 *pwts.* 21 *grs.*

These two Examples at large being well underſtood, may ſuffice to ſhew how all Operations of this kind are performed; either in *Weights*, *Meaſures*, or *Time*. I ſhall only inſert a few Examples of each ſort for the Learner's Practice.

1. In 23 *C.* 3 *qrs.* 21 *lb.* 9 *oz.* Averdupois Weight; How many *Ounces?* Anſw. 42905 *Ounces.*

2. In 252 *Eng. Miles*, How many *Yards*, *Feet*, and *Inches?* Anſw. 443520 *yds.* = 1330560 *Feet* = 15966720 *Inches.*

3. In 1692 common *Years*, How many *Days*, *Hours*, and *Minutes?* Anſw. 618003 *Days*, 14832072 *Hours*, 889924320 *Minutes.*

*Note*, a common *Year* = 365 *Days*, 6 *Hours*, ſee *Page* 37.

4. In 5786 *Pounds*, 17 *Shillings*, 9 *Pence*, Sterling; How many *Shillings*, *Pence*, and *Farthings?* Anſw. 115737 *s.* 1388853 *d.* or 5555412 *Farthings.* That is, 5786 *l.* 17 *s.* 9 *d.* = 115737 *s.* 9 *d.* = 1388853 *d.* &c.

The next Thing will be to ſhew how to bring Numbers from a leſſer to a greater Denomination, which by moſt Authors is called (tho' very improperly)

## Reduction *aſcending.*

This Work is the Converſe of the laſt, and is performed by *Diviſion.* Thus,

### RULE.

*Conſider how many of the* Denomination *propoſed make one of the* Denomination *required, and make that* Number *your* Diviſor, *by which divide the* Denomination *propoſed*; *and the* Quotient *will be the* Number *required.*

*EXAMPLE.*

## EXAMPLE.

Let it be required to find how many *Shillings* and *Pounds* are contained in 85680 *Pence*.

The *Pence* in 1*s:* are 12) 85680 (7140*s* = 85680 *d.*

Again the *Shillings* in 1 *l.* are 20) 7140 (357 *l.* the Anſwer required.

### *Another Example in* Coin.

How many *Pence*, *Shillings*, and *Pounds*, are contained in 264859 *Farthings.*

```
           12)         20)
4) 264859  (66214 d.  (5517 s.  (275 l.
    24      62         151
    08      21         117
    05      94         (17) s.
    19      (10) d.
```

*Remains* (3) *q.* { *Note*, the *Remainder* is always of the ſame Denomination with the *Dividend.*

The laſt *Quotient* 275 *l.* together with the ſeveral *Remainders*, give the Anſwer required.

Viz. 275 *l.* 17 *s.* 10 *d.* 3 *q.* = 264859 *Farthings.*

### *Example in* Troy Weight.

Suppoſe it were required to find how many *Pwts. Ozs.* and *lbs.* are contained in 171333 *Grains.*

```
                20)             12)
24) 171333 gr. (7138 pw.      (356  (29 lb.
    168...      113             24
    33          138            116
    24          ———            108
    ——          (18) pws.      ——
    93                         (8) oz.
    72
    213
    192
```

*Remains* (21) *gr.*

Anſw. 29 *lb.* 8 *oz.* 18 *pwt.* 21 *grs.* This and the laſt Example are the Reverſe or Proof of thoſe in Pages 43, 45.

1. In 42905 *Ounces, Averdupois weight*; How many *Pounds*, &c. Thus,

```
                 28)          4)
Thus  16) 42905   (2681 lb.   (95 qrs. (23 C.
          109      252         15
          130      161         (3)
           25      140
           (9)     (21) Answ. 23 C. 3 qrs. 21 lb. 9 oz.
```

2. In 15966720 *Inches*; How many *English Miles*, &c.

Answ. 252 *Miles*, &c. as occasion requires.

There are many useful Questions may be answered by the help of *Reduction* only: As the changing of one sort of Coin for another; and comparing one sort of Measure with another, &c.

For Instance: Suppose one had 347 *Rixdollars*, at 4 *s.* 6 *d. per* Dollar; and desired to know how many *Pounds Sterling* they make,

```
       347
        54 = the Pence in one Dollar, viz. 4 s. 6 d. = 54 d.
      1388
      1735        20)
  12) 18738 d.    (1561 s. (78 l.
       67          161
       73          (1) s.
       18
       (6) d.
```

Answ. 78 *l.* 1 *s.* 6 *d. Sterl.* are = 347 *Rixdollars.*

*Quest.* 2. In 645 *Flemish* Ells; How many Ells *English?*

Note, 3 *Quarters of a Yard English make one Ell Flemish, and* 1 $\frac{1}{4}$, *or* 5 *Quarters of a Yard, is an English Ell.*

```
        Therefore, 645
                     3 = the qrs of a Yard in 1 Ell Flemish.
qrs in 1 Ell = 5) 1935 (387 English Ells for the Answer.
```

*Quest*. 3. Suppose a Bill of Exchange were accepted at *London*, for the Payment of 400 *l. Sterl.* for the Value delivered at *Amsterdam* in *Flemish* Money at 1 *l.* 13 *s.* 6 *d.* for 1 Pound *Sterl.* How much *Flemish* Money was delivered at *Amsterdam?*

*First*, 1 *l.* 13 *s.* 6 *d.* = 402 *d.* the Value of one Pound *Sterl.* at *Amsterdam.*

Then, 402 *d.* x 400 = 160800 *d.* = 670 *l. Flemish*, and so much was delivered at *Amsterdam.*

CHAP.

## CHAP. IV.

# Of Vulgar Fractions.

### Sect. 1. Of Notation.

A *Fraction*, or *Broken Number*, is that which reprefents a *Part* or *Parts* of any thing propofed, (*vide* Page 3.) and is expreffed by two Numbers placed one above the other with a Line drawn betwixt them:

Thus, $\frac{3}{4}$ { 3 *Numerator.* 4 *Denominator.* }

The Denominator, or Number placed underneath the Line, denotes how many equal Parts the thing is fuppofed to be divided into (being only the Divifor in Divifion). And the Numerator, or Number placed above the Line, fhews how many of thofe Parts are contained in the Fraction (it being the Remainder after Divifion). (See Page 29.) And thefe admit of three Diftinctions:

Viz. { *Proper* or *Simple*, *Improper*, *Compound* } *Fractions.*

A proper, pure, or *Simple Fraction*, is that which is lefs than an Unit. That is, it reprefents the immediate Part or Parts of any thing lefs than the whole, and therefore it's Numerator is always lefs than the Denominator.

As { $\frac{1}{4}$ is one *Fourth Part.* $\frac{1}{3}$ is one *Third Part.* } And { $\frac{1}{2}$ is one *Half.* $\frac{2}{3}$ is two *Thirds*, &c. }

An *Improper Fraction* is that which is greater than an Unit. That is, it reprefents fome Number of Parts greater than the whole thing; and it's Numerator is always greater than the Denominator.

As $\frac{4}{3}$ or $\frac{9}{7}$ or $\frac{41}{15}$ &c.

A *Compound Fraction* is a Part of a Part, confifting of feveral Numerators and Denominators connected together with the Word [of].

As $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$, *&c.* and are thus read, The *one Third* of the *three Fourths* of the *two Fifths* of an Unit.

That is, when a Unit (or whole thing) is firft divided into any Number of equal Parts, and each of thofe Parts are fubdivided ſubdivided into other Parts, and ſo on: Then thoſe laſt Parts are called *Compound Fractions*, or *Fractions of Fractions*.

As for inſtance, ſuppoſe a Pound Sterling (or 20 *s.*) be the Unit or Whole; then is 8 *s.* the $\frac{2}{5}$ of it, and 6 *s.* the $\frac{3}{4}$ of thoſe two Fifths, and 2 *s.* is the $\frac{1}{3}$ of thoſe three Fourths; *viz.* 2 *s.* $=\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ of one Pound Sterling.

All *Compound Fractions* are reduced into ſingle ones, Thus,

RULE.

*Multiply all the* Numerators *into one another for a* Numerator, *and all the* Denominators *into one another for the* Denominator.

Thus the $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ will become $\frac{6}{60}$. Or $\frac{1}{10}$.
For $1 \times 3 \times 2 = 6$ the Numerator, and $3 \times 4 \times 5 = 60$ the Denominator, but $\frac{6}{60}$ or $\frac{1}{10}$ of a *l.* Sterl. is 2 *s.* As above.

## Sect. 2. *To* Alter *or* Change *different* Fractions *into one* Denomination *retaining the ſame Value.*

IN order to gain a clear Underſtanding of this Section, it will be convenient to premiſe this Propoſition, *viz.* If a Number multiplying two Numbers produce other Numbers, the Numbers produced of them ſhall be in the ſame proportion that the Numbers multiplied are, 17 *Euclid* 7.

That is to ſay, If both the Numerator and Denominator of any *Fraction* be equally multiplied into any Number, their Products will retain the ſame Value with that Fraction.

As in theſe, $\frac{2\times2}{3\times2}=\frac{4}{6}$. Or $\frac{2\times3}{3\times3}=\frac{6}{9}$. Or $\frac{2\times5}{3\times5}=\frac{10}{15}$, &c.

That is, $\frac{2}{3}$ and $\frac{4}{6}$. Or $\frac{2}{3}$ and $\frac{6}{9}$. Or $\frac{2}{3}$ and $\frac{10}{15}$ are of the ſame Value, in reſpect to the Whole or Unit.

From hence it will be eaſy to conceive, how two or more Fractions that are of different Denominations, may be altered or changed into others that ſhall have one common Denominator, and ſtill retain the ſame Value.

*Example.* Let it be required to change $\frac{2}{3}$ and $\frac{3}{7}$ into two other Fractions that ſhall have one common Denominator, and yet retain the ſame Value.

According to the foregoing Propoſition, if $\frac{2}{3}$ be equally multiplied with 7, it will become $\frac{14}{21}$, *viz.* $\frac{2\times7}{3\times7}=\frac{14}{21}$. Again, if $\frac{3}{7}$ be equally multiplied with 3, it will become $\frac{9}{21}$, *viz.* $\frac{3\times3}{7\times3}=\frac{9}{21}$.

And by this means I have obtained two new Fractions, $\frac{14}{21}$ and $\frac{9}{21}$, that are of one Denomination, and of the same Value with the two first proposed, *viz.* $\frac{14}{21}=\frac{2}{3}$ and $\frac{9}{21}=\frac{3}{7}$.

And from hence doth arise the general Rule for bringing all Fractions into one Denomination.

RULE.

*Multiply all the* Denominators *into each other for a new (and common)* Denominator. *And each* Numerator *into all the* Denominators *but it's own, for new* Numerators.

*Example.* Let the proposed Fractions be $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{4}$, and $\frac{6}{7}$.

Then, by the Rule,

| A new Denominator will be thus found. | And the new Numerators will be thus found. | | | |
|---|---|---|---|---|
| 3 | 1. | 2. | 3. | 6 |
| 5 | 5 | 3 | 3 | 3 |
| 15 | 5 | 6 | 9 | 18 |
| 4 | 4 | 4 | 5 | 5 |
| 60 | 20 | 24 | 45 | 90 |
| 7 | 7 | 7 | 7 | 4 |
| 420 | 140. | 168. | 315. | 360 |

Hence 420 is the common *Denominator*; and 140 . 168 . 315 . 360, are the new *Numerators*, which being placed Fraction-wise are $\frac{140}{420}$ . $\frac{168}{420}$ . $\frac{315}{420}$ . $\frac{360}{420}$, the New Fractions required.

That is, $\frac{140}{420}=\frac{1}{3}$. $\frac{168}{420}=\frac{2}{5}$ $\frac{315}{420}=\frac{3}{4}$. and $\frac{360}{420}=\frac{6}{7}$.

## Sect. 3. *To bring mix'd* Numbers *into* Fractions, *and the contrary.*

MIX'D *Numbers* are brought into improper *Fractions* by the following Rule.

RULE.

*Multiply the* Integers, *or whole Numbers, with the* Denominator *of the given Fraction, and to their Product add the* Numerator, *the Sum will be the Numerator of the Fraction required.*

*Example.* $9\frac{4}{5}$ by the Rule will become $\frac{49}{5}$. For $9\times5=\frac{45}{5}$. And, $\frac{45}{5}+\frac{4}{5}=\frac{49}{5}$, the improper Fraction required.

Again, $13\frac{11}{15}$ will become $\frac{206}{15}$. For $13\times15=\frac{195}{15}$. And $\frac{195}{15}+\frac{11}{15}=\frac{206}{15}$. And so for any other as occasion requires.

To find the true Value of any improper Fraction given is only the Converse of this Rule. For if $\frac{49}{5}=9\frac{4}{5}$, as before is evident:

Then it follows that if 49 be divided by 5, the Quotient will give $9\frac{4}{5}$. And if 206 be divided by 15, it will give $13\frac{11}{15}$, &c. confequently it follows, that

If the Numerator of any improper Fraction be divided by it's Denominator, the Quotient will difcover the true Value of that Fraction.

*EXAMPLES.*

$\frac{35}{7} = 5$. And $\frac{41}{9} = 4\frac{5}{9}$, And $\frac{121}{20} = 6\frac{1}{20}$. Or $\frac{15}{4} = 3\frac{3}{4}$, &c.

When whole Numbers are to be expreffed Fraction-wife, it is but giving them an Unit for a Denominator. Thus 45 is $\frac{45}{1}$, 9 is $\frac{9}{1}$, and 25 is $\frac{25}{1}$, &c.

---

## Sect. 4. *To* Abbreviate *or* Reduce Fractions *into their Loweft or Leaft Denomination.*

THIS is done, not out of any neceffity, but for the more convenient managing of fuch *Fractions* as are either propofed in large terms; or fwell into fuch, either by *Addition* or otherwife: befides it is moft like an Artift to exprefs or fet down all *Fractions* in the loweft terms poffible; and to perform that, it will be neceffary to confider thefe following Propofitions.

### Numbers are either Prime or Compofed.

1. A Prime Number is that which can only be meafured by an Unit. *Euclid* 7. *Defin.* 11.

That is, 3, 5, 7, 11, 13, 17, &c. are faid to be Prime Numbers, becaufe it is not poffible to divide them into equal Parts by any other Number but Unity or 1.

2. Numbers Prime the one to the other, are fuch as only an Unit doth Meafure, being their common Meafure. *Euclid* 7. *Defin.* 12.

For inftance, 7 and 13 are Prime Numbers to each other, becaufe they cannot be divided by any Number but an Unit. And 9 and 14 are alfo Prime Numbers to each other, for altho' 3 will meafure or divide 9 without leaving a Remainder; yet 3 will not meafure 14 without leaving a Remainder: Again, altho' 2 will meafure 14 without any Remainder, yet 2 will not meafure 9 without leaving a Remainder, &c.

3. A compofed Number is that which fome certain Number meafureth. *Euclid* 7. *Defin.* 13.

For inftance, 15 is a compofed Number of 3 and 5, for $5 \times 3 = 15$, confequently 3 or 5 will juftly meafure 15. Alfo 20



is

is compoſed of 5 and 4, *viz.* $5\times4=20$, therefore 5 and 4 will each juſtly meaſure 20.

4. Numbers compoſed the one to the other, are they which ſome Number, being a common Meaſure to them both, doth meaſure. *Euclid* 7. *Defin.* 14.

That is, if two or more Numbers can be divided by one and the ſame Diviſor; then are thoſe Numbers ſaid to be compoſed one to another.

For Inſtance, 14 and 21 are Numbers compoſed the one to the other, becauſe they can both be meaſured or divided by 7. For $7\times2=14$, and $7\times3=21$; therefore 7 is a common Meaſure to 14 and 21. So that if $\frac{14}{21}$ were propoſed to be abbreviated, it will become $\frac{2}{3}$.

$$\text{Thus }\left\{\begin{array}{l}7)\,14=2\\7)\,21=3\end{array}\right.$$

And how thoſe greateſt common Meaſures may be found comes from *Euclid* 7. *Prob.* 1, 2, 3, and is thus:

## RULE.

*Divide the greater Number by the leſſer, and that Diviſor by the Remainder (if there be any), and ſo on continually until there be no Remainder left: Then will that laſt Diviſor be the greateſt common Meaſure (and if it happen to be* 1, *then are thoſe Numbers Prime Numbers, and are already in their loweſt Terms; but if otherwiſe) Divide the Numbers by that laſt Diviſor, and their Quotients will be their leaſt Terms required.*

## *EXAMPLE.*

Let it be required to find the greateſt common Meaſure of 72 and 108, *viz.* of $\frac{72}{108}$.

```
72) 108 (1
     72
   ------
   36) 72 (2
       72
     ----
       (0)
```

{ Here becauſe there is no Remainder; 36 is the greateſt common Meaſure.

$$\text{Therefore, }\left\{\begin{array}{l}36)\,72=2\\36)\,108=3\end{array}\right\}$$ Hence $\frac{72}{108}$ is abbreviated to $\frac{2}{3}$ the loweſt Terms.

Again, to find the greateſt common Meaſure of 744 and 899.

Thus,

Thus, 744) 899 (1
744
155) 744 (4
620
124) 155 (1
124
31) 124 (4
124
(0)

Here 31 is found to be the greateſt common Meaſure by which 744 and 899 may be abbreviated to 24 and 29 their loweſt Terms.

Thus, $\frac{31}{31}$) $\frac{744}{899}$ (= $\frac{24}{29}$, &c.

*Note*, If the propoſed Numbers be even, they may be brought lower by a continued halfing of them, ſo long as they can be halfed, *viz.* divided by 2.

*EXAMPLE.*

It is required to Reduce $\frac{56}{84}$ to it's leaſt Terms.

Firſt, $\frac{2}{2}$) $\frac{56}{84}$ (= $\frac{28}{42}$. Again, $\frac{2}{2}$) $\frac{28}{42}$ (= $\frac{14}{21}$.

This done, you eaſily perceive that 7 will be the common Meaſure to 14 and 21, *viz.* $\frac{7}{7}$) $\frac{14}{21}$ (= $\frac{2}{3}$, &c.

If the Numbers propoſed to be reduced have each a Cypher, or Cyphers annexed to them, they will be abbreviated by cutting off a like Number of Cyphers from both.

Thus, $\frac{150}{300}$ will be $\frac{15}{30}$. And $\frac{200}{300}$ will be $\frac{2}{3}$, &c.

That is, $\frac{150}{300} = \frac{15}{30} = \frac{1}{2}$. And $\frac{200}{300} = \frac{2}{3}$. And $\frac{360}{400} = \frac{36}{40} = \frac{18}{20} = \frac{9}{10}$.

## Sect 5. Addition *of* Fractions.

WHAT hath been done by the Rules in this Chapter, is chiefly to prepare and fit *Fractions* of different Denominations for *Addition* or *Subtraction*, as Occaſion requires, *viz.* If they are *Compound Fractions*, they muſt be reduced to Simple or Pure *Fractions*, *per Rule*, *Sect.* 1.

If they are of different Denominations, they muſt be altered or changed, *per Rule*, *Sect.* 2.

That is, all *Fractions* muſt be brought into one Denomination before they can either be added or ſubtracted; and that being done, *Addition* is thus performed.

RULE.

*Add together all the Numerators, and their Sum will be a New Numerator, under which ſubſcribe the Common Denominator.*

*Examples*

## *Examples in* Simple Fractions.

Let it be propoſed to add $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{4}$ together. Firſt, $\frac{1}{3} = \frac{20}{60}$, $\frac{2}{5} = \frac{24}{60}$, and $\frac{3}{4} = \frac{45}{60}$, *per Sect.* 2.

Then $\frac{20}{60} + \frac{24}{60} + \frac{45}{60} = \frac{89}{60}$, the Sum required, which according to *Section* 3, is $1\frac{29}{60}$, *viz.* $\frac{89}{60} = 1\frac{29}{60}$.

## *Examples in* Compound Fractions.

Let it be required to add $\frac{3}{7}$ and $\frac{2}{3}$ of $\frac{3}{4}$ into one Sum. Firſt $\frac{2}{3}$ of $\frac{3}{4}$ becomes $\frac{6}{12}$ or $\frac{1}{2}$ per Sect. 1. And (per Sect. 2.) $\frac{3}{7}$ and $\frac{1}{2}$ is $\frac{6}{14}$ and $\frac{7}{14}$, *viz.* $\frac{3}{7} = \frac{6}{14}$, and $\frac{1}{2} = \frac{7}{14}$; but $\frac{7}{14} + \frac{6}{14} = \frac{13}{14}$ the Sum required, *viz.* $\frac{3}{7} + \frac{2}{3}$ of $\frac{3}{4} = \frac{13}{14}$.

## *Examples in* mixed Numbers.

It is required to add $5\frac{2}{5}$ to $7\frac{3}{4}$, theſe per Sect. 3. will be $\frac{17}{3}$ and $\frac{31}{4}$. But $\frac{17}{3}$ and $\frac{31}{4}$ will become $\frac{68}{12}$ and $\frac{93}{12}$ per Sect. 2. Then $\frac{68}{12} + \frac{93}{12} = \frac{161}{12}$, and $\frac{161}{12} = 13\frac{5}{12}$ the Sum required.

Or you may bring only the *Fractions* to one Denomination.

Thus, $5\frac{2}{3}$ and $7\frac{3}{4}$ will become $5\frac{8}{12}$ and $7\frac{9}{12}$.

Then $5\frac{8}{12} + 7\frac{9}{12} = 12\frac{17}{12}$. That is, $13\frac{5}{12}$. As before.

---

## Sect. 6. Subtraction *of* Fractions.

### RULE.

*SUBTRACT one Numerator from the other (according as the Queſtion requires) and their Difference will be a new Numerator, under which ſubſcribe the Common Denominator, as in* Addition.

### *EXAMPLE* 1.

Let it be required to take $\frac{2}{9}$ out of $\frac{3}{7}$. Firſt $\frac{2}{9}$ and $\frac{3}{7}$ per Sect. 2. will become $\frac{14}{63}$ and $\frac{27}{63}$; then $\frac{27}{63} - \frac{14}{63} = \frac{13}{63}$, that is, $\frac{3}{7} - \frac{2}{9} = \frac{13}{63}$. As was required.

### *EXAMPLE* 2.

It is required to ſubtract $\frac{2}{3}$ of $\frac{8}{9}$ from $\frac{13}{14}$. Firſt, $\frac{2}{3}$ of $\frac{8}{9} = \frac{16}{27}$ per Sect. 1. Again $\frac{16}{27}$ and $\frac{13}{14}$ will become $\frac{224}{378}$ and $\frac{351}{378}$. per Sect. 2. Then $\frac{351}{378} - \frac{224}{378} = \frac{127}{378}$.

### *EXAMPLE* 3.

From $6\frac{1}{8}$ ſubtract $3\frac{19}{48}$. Firſt, $6\frac{1}{8} = \frac{49}{8}$, and $3\frac{19}{48} = \frac{163}{48}$ per Rule Sect. 3. Again, $\frac{49}{8} = \frac{2352}{384}$, and $\frac{163}{48} = \frac{1304}{384}$, per Rule Sect. 2. Then, $\frac{2352}{384} - \frac{1304}{384} = \frac{1048}{384} = 2\frac{280}{384} = 2\frac{35}{48}$. Or otherwiſe thus:

Firſt,

First, $6\frac{1}{8} = 5\frac{9}{8}$, then bring $\frac{9}{8}$ and $\frac{19}{48}$ into one Denomination, *viz.* $5\frac{9}{8} = 5\frac{432}{384}$ and $3\frac{19}{48} = 3\frac{152}{384}$.

Then $5\frac{432}{384} - 3\frac{152}{384} = 2\frac{280}{384} = 2\frac{35}{48}$. As before.

*EXAMPLE.*

Let it be required to subtract $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$ from 7.
First, $\frac{3}{5}$ of $\frac{5}{9}$ of $\frac{2}{3} = \frac{30}{189}$. And $7 = 6\frac{189}{189}$.
Then $6\frac{189}{189} - \frac{30}{189} = 6\frac{159}{189} = 6\frac{53}{63} = 7 - \frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$. As was required.

If these few *Examples* be well understood, the whole Business of adding and subtracting *Vulgar Fractions* will be easy; which is really much more difficult to perform than either *Multiplication* or *Division*; as will appear in the next *Section*.

## Sect. 7. Multiplication *of* Fractions.

IN order to perform either *Multiplication* or *Division*, you must prepare the Terms to be multiplied (or divided) thus; reduce Compound *Fractions* to Simple ones, *per Sect.* 1. Bring mixed Numbers into improper *Fractions*, and express whole Numbers *Fraction-wise*, *per Sect.* 3. Also it will be convenient to abbreviate them to their smallest Terms, when it can be done. Then *Multiplication* may be thus performed.

Rule. { *Multiply the Numerators one into another for a new Numerator; and the Denominators one into another for a new Denominator. As in these*

*EXAMPLES.*

1. The Product of $\frac{2}{5}$ into $\frac{3}{7} = \frac{6}{35}$. That is, $\frac{2 \times 3}{5 \times 7} = \frac{6}{35}$.
2. And the Products of $\frac{9}{16}$ into $\frac{20}{27} = \frac{180}{432}$. Or $\frac{5}{12}$.
3. Again, the Product of $\frac{7}{11}$ into $\frac{2}{5}$ of $\frac{5}{7} = \frac{70}{385}$. Or $\frac{2}{11}$.
For $\frac{2}{5}$ of $\frac{5}{7} = \frac{10}{35}$. Then $\frac{7}{11} \times \frac{10}{35} = \frac{70}{385} = \frac{2}{11}$.
4. Let it be required to multiply 6 with $3\frac{2}{5}$. These prepared for the Work will stand thus. $\frac{6}{1} \times \frac{17}{5}$.
*viz.* $6 = \frac{6}{1}$ and $3\frac{2}{5} = \frac{17}{5}$. Then $\frac{6}{1} \times \frac{17}{5} = \frac{102}{5}$, or $20\frac{2}{5}$.
Or, otherwise thus $6 \times 3 = 18$. And $\frac{2}{5} \times 6 = \frac{12}{5} = 2\frac{2}{5}$.
Then $18 + 2\frac{2}{5} = 20\frac{2}{5}$. As before.
5. Let it be required to multiply $7\frac{4}{9}$ with $5\frac{3}{7}$.
First $7\frac{4}{9} = \frac{67}{9}$ and $5\frac{3}{7} = \frac{38}{7}$. Then $\frac{67}{9} \times \frac{38}{7} = \frac{2546}{63} = 40\frac{26}{63}$.

Now the Reason of this Rule for multiplying of *Fractions*, and consequently of these Operations, and all others performed by it; will be evident from this following.

*Viz.*

*Viz.* If $\frac{4}{2}$ be multiplied with $\frac{12}{3}$ acccording to the Rule, their Product will be $\frac{48}{6}$. But $\frac{48}{6} = 8$.

Now $\frac{4}{2} = 2$, and $\frac{12}{3} = 4$ per Sect. 3. But $4 \times 2 = 8$. *Ergo*, &c.

---

## Sect. 8. Division *of* Fractions.

THE *Fractions* being first prepared as before directed, *Division* may be thus performed:

Rule. { *Multiply the Numerator of the Dividend into the Denominator of the dividing Fraction for a new Numerator: and multiply the other Numerator and Denominator together for a new denominator.*

### *EXAMPLES.*

1. Let $\frac{6}{35}$ be divided by $\frac{3}{7}$, *viz.* $\frac{3}{7}$) $\frac{6}{35}$ ($\frac{42}{105} = \frac{2}{5}$ the Quotient.

That is, according to the Rule $6 \times 7 = 42$ the new Numerator, and $35 \times 3 = 105$, the new Denominator, *&c.* as above.

2. Let it be required to divide $\frac{20}{27}$ by $\frac{5}{12}$, *viz.* $\frac{5}{12}$) $\frac{20}{27}$ ($\frac{240}{135} = 1\frac{7}{9}$.

For $12 \times 20 = 240$ the new Numerator, and $27 \times 5 = 135$ the new Denominator.

3. Suppose it were required to divide $\frac{2}{11}$ by $\frac{2}{5}$ of $\frac{5}{7}$.

First, $\frac{2}{5}$ of $\frac{5}{7} = \frac{10}{35}$. Then $\frac{10}{35}$) $\frac{2}{11}$ ($\frac{70}{110} = \frac{7}{11}$.

4. Let $20\frac{2}{5}$ be divided by $3\frac{2}{5}$; *viz.* $\frac{102}{5}$ by $\frac{17}{5}$:

For $20\frac{2}{5} = \frac{102}{5}$, and $3\frac{2}{5} = \frac{17}{5}$. Then $\frac{17}{1}$) $\frac{102}{5}$ ( $= 6$ the Quotient.

5. Let it be required to divide $40\frac{26}{63}$ by $5\frac{3}{7}$.

First, $40\frac{26}{63} = \frac{2546}{63}$, and $5\frac{3}{7} = \frac{38}{7}$. Then $\frac{38}{7}$) $\frac{2546}{63}$ ($\frac{17822}{2394}$.

But $\frac{17822}{2394} = 7\frac{4}{9}$ the true Quotient required.

6. Suppose it were required to divide 13 by $\frac{5}{7}$.

First, $13 = \frac{13}{1}$. Then $\frac{5}{7}$) $\frac{13}{1}$ ($\frac{91}{5} = 18\frac{1}{5}$, the Quotient.

7. Again, let it be required to divide $\frac{5}{7}$ by 6.

*Viz.* $\frac{6}{1}$) $\frac{5}{7}$ ($\frac{5}{42}$ for the Quotient required.

*N. B.* From hence you may observe, that when any whole Number is divided by a Fraction less than Unity or 1, the Quotient will be greater than the Number proposed to be divided: But if any Fraction be divided by a whole Number, greater than 1, then the Quotient will be less than the Dividend: As in the two last *Examples.*

As

As to the *Reaſon* (or *Proof)* of this *Rule* for *dividing Fractions:* It is only the *Converſe* to that of *Multiplication*, and will be very evident from this following.

Let $\frac{48}{6}$ be *divided* by $\frac{4}{2}$. Which according to the *Rule* is thus, $\frac{4}{2}$) $\frac{48}{6}$ ($\frac{26}{24} = 4$. The true *Quotient*. Now $\frac{48}{6} = 8$. And $\frac{4}{2} = 2$. *per Sect.* 3. Conſequently $\frac{48}{6}$ *divided* by $\frac{4}{2}$ is but the ſame with 8 *divided* by 2, *viz.* 2) 8 (4. The *Quotient* as before.

I could have inſerted *Geometrical Demonſtrations*, for the *Rules* of *Multiplication* and *Diviſion* of *Fractions*; but ſuppoſing the Learner purely unacquainted with thoſe kind of Demonſtrations, I thought theſe might be more intelligible to him, eſpecially in this place.

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## CHAP. V.

# *Of* Decimal Fractions.

*WHEN, or by whom, this excellent Invention of* Decimal Arithmetick, *was firſt introduced is uncertain*; *but doubtleſs it's Improvements, and the Perfections it is now in, is owing to latter Years.*

### Sect. 1. *Of* Notation.

IN *Decimal Fractions*, the *Integer* or *whole Thing* (whether it be *Coin*, *Weight*, *Meaſure*, or *Time*, &c.) is ſuppoſed to be *divided* into *Ten equal Parts*; and every one of thoſe *Ten Parts* are ſuppoſed to be *ſubdivided* into other *Ten equal Parts*, &c. *ad infinitum.*

The *Integer* being thus *divided* (*by Imagination*) into 10, 100, 1000, 10000, *&c.* *equal Parts*, becomes the *Denominator* to the *Decimal Fractions*.

Thus, $\frac{2}{10}$. $\frac{3}{100}$. $\frac{7}{1000}$. $\frac{53}{10000}$. $\frac{754}{100000}$, *&c.*

Now theſe *Denominators* are ſeldom or never ſet down, but only the *Numerators*; and thoſe are either diſtinguiſhed, or ſeparated from *whole Numbers* by a *Point* or a *Comma.*

Thus, 5,4 is 5 $\frac{4}{10}$. and 0,7 is $\frac{7}{10}$. 35,05 is 35 $\frac{5}{100}$, *&c.*

But before we proceed further in *Notation*, it will be convenient for the Learner to conſider the following *Table*, (taken out of the learned Mr *Oughtred*'s *Clavis Mathematica*) which ſhews the very Foundation of *Decimal Fractions.*

| *Whole Numbers,* | | | | | | *Decimal Parts.* | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 4 | 3 | 2 | 1 | 0, | 1 | 2 | 3 | 4 | 5 | 6 | |
| *&c.* | *Tens of Thousands.* | *Thousands.* | *Hundreds.* | *Tens.* | *Units Place.* | *Parts of Ten, or* $\frac{1}{10}$. | *Parts of a Hundred.* | *Parts of a Thousand.* | *Parts of Ten Thousand.* | *Parts of 100 Thousand.* | *Parts of a Million.* | *&c.* |

By this *Table* it is evident, that as in whole *Numbers* or *Integers*, every *Degree* from the *Units* Place increases towards the left-hand by a Ten-fold *Proportion:* So in *Decimal Parts* every *Degree* is decreased towards the right-hand by the same *Proportion*, viz. by *Tens.*

Therefore these *Decimal Parts* or *Fractions*, are really more *Homogeneal*, or agreeing with *whole Numbers*, than *Vulgar Fractions*; for indeed all plain *Numbers* are in effect but *Decimal Parts* one to another.

That is, suppose any *Series* of *equal Numbers*, as 444, *&c.* The first 4 towards the Left is *Ten* times the *Value* of the 4 in the middle, and that 4 in the middle is *Ten* times the *Value* of the last 4 to the Right of it, and but the *Tenth Part* of that 4 on the left, *&c.*

Therefore all or any of them may be taken either as *Integers*, or *Parts* of an *Integer:* If *Integers*, then they must be set down without any *Comma* or separating *Point* betwixt them thus, 444 But if *Integers*, and one *Part* or *Fraction*, put a *Comma* betwixt them thus, 44,4 which signifies 44 *whole Numbers*, and 4 *Tenths* of an *Unit:* Again, if two Places of *Parts* be required, separate them with a *Comma* thus, 4,44 *viz.* 4 *Units*, and 44 *hundred Parts* of an *Unit*, *&c.*

From hence (duly compared with the *Table*) it will be easy to conceive that *Decimal Parts* take their *Denomination* from the Place of their last *Figure*.

That is, $\left\{\begin{array}{l} ,5 = \frac{5}{10} \\ ,56 = \frac{56}{100} \\ ,056 = \frac{56}{1000} \end{array}\right\}$ *Parts* of an *Unit*, &c.

*Cyphers*

*Cyphers* annexed to *Decimal Parts*, alter not their *Value*. As ,50, and ,500, or ,5000, *&c.* are each but 5 *Tenths* of an *Unit*. For $\frac{50}{100} = \frac{5}{10}$. And $\frac{500}{1000} = \frac{5}{10}$. Or $\frac{5000}{10000} = \frac{5}{10}$; *Per Sect.* 4. of the last *Chapter*.

But *Cyphers* prefixed to *Decimal Parts* decrease their *Value*, by removing them further from the *Comma*.

Thus, { ,5 = 5 *Tenth Parts*.
,05 = 5 *Parts* of a *Hundred*.
,005 = 5 *Parts* of a *Thousand*.
,0005 = 5 *Parts* of *Ten Thousand*, &c.

Consequently the true *Value* of all *Decimal Parts* are known by their Distance from the *Units* Place; the which being once rightly understood, the rest will be easy.

## Sect. 2. Addition *and* Substraction *of* Decimals.

IN setting down the proposed Numbers to be added, or subtracted, great care must be taken in placing every Figure directly underneath those of the same Value, whether they be mix'd Numbers, or pure Decimal Parts, and to perform that you must have a due regard to the Comma's, or separating Points, which ought always to stand in a direct Line one under another; and to the Right-hand of them carefully place the Decimal Parts, according to their respective Values, or Distances from Unity. Then

Rule { *Add or subtract them, as if they were all whole Numbers; and from their Sum, or Difference, cut off so many Decimal Parts as are the most in any of the given Numbers.*

### *EXAMPLES* in Addition.

Let it be required to find the Sum of these following Numbers, *viz.* 34,5 + 65,3 + 128,7 + 95 + 87,8 + 7,9, which being truly placed, will stand

Thus, {
34,5
65,3
128,7
95,0
87,8
7,9

Their Sum required, 419,2

*EXAMPLE* 2.

Let it be required to find the Sum of 25,854 + 34,578 + 9,076 + 13,907.

| |
|---|
| 25,854 |
| 34,578 |
| 9,076 |
| 13,907 |
| 83,415 The Sum required. |

When the Decimal Parts propoſed to be added (or ſubtracted) have not the ſame Number of Places, you may for convenience of Operation ſupply or fill up the void Places, by annexing Cyphers. As in theſe *Examples.*

| *EXAMPLE* 3. | *EXAMPLE* 4. | *EXAMPLE* 5. |
|---|---|---|
| 45,0700 | 574,678953 | 0,975642 |
| 50,7580 | 95,796430 | ,745257 |
| 123,0057 | 78,054600 | ,000598 |
| 74,7020 | 54,789000 | ,800700 |
| 24,8000 | 8,900000 | ,640530 |
| 318,3357 | 812,218983 | 3,162727 |

## *EXAMPLES in* Subſtraction.

Let it be required to find the Difference between 45,375 and 74,284.

| | *EXAMPLE* 1. | *EXAMPLE* 2. | *EXAMPLE* 3. |
|---|---|---|---|
| That is, | From 74,284 | From 437,5 | From 75,0034 |
| | Take 45,375 | Take 89,657 | Take 57,875 |
| | Remains 28,909 | 347,843 | 17,1284 |

*EXAMPLE* 4.

Let it be required to find the Exceſs between 562 and 93,5784.

| | *EXAMPLE* 4. | *EXAMPLE* 5. |
|---|---|---|
| That is, | From 562, | From 345,7578 |
| | Take 93,5784 | Take 157, |
| | The Exceſs 468,4216 | 188,7578 |

*Note*, The two laſt Examples are ſuppoſed to be ſupplied with Cyphers, which if actually done would ſtand thus,

| | | | |
|---|---|---|---|
| | 562,0000 | | 345,7578 |
| | 93,5784 | | 157,0000 |
| Remains | 468,4216 | As before, | 188,7578 |

*EXAMPLE*

| EXAMPLE 6. | | EXAMPLE 7. | |
|---|---|---|---|
| From | 0,547893 | From | 1,000000 |
| Take | 0,439758 | Take | 0,997543 |
| | 0,108135 | | 0,002457 |

The Proof of Addition and Subtraction in Decimals, is the ſame with that of whole Numbers, page 13, *&c.*

## Sect. 3. Multiplication *of* Decimals.

WHETHER the Factors or Numbers to be multiplied are pure Decimals, or mixed. Multiply them as if they were all whole Numbers, and for the true Value of their Product obſerve this

Rule { *Cut off* (viz. *ſeparate with a Comma*) *ſo many Places of Decimal Parts in the Product, as there are in both the Factors accounted together. As in theſe:*

| *EXAMPLE* 1. | *EXAMPLE* 2. |
|---|---|
| 3,024 | 32,12 |
| 2,23 | 24,3 |
| 90 72 | 9 63 6 |
| 604 8 | 128 48 |
| 6 048 | 642 4 |
| 6,743 52 | 780,51 6 |

The Reaſon why ſuch a Number of Decimal Parts muſt be cut off in the Product, may be eaſily deduced from theſe Examples. Thus,

In Example 1. It is evident, that 3, the whole Number in the Multiplicand, being multiplied with 2, the whole Number in the Multiplier; can produce but 6 (*viz.* $3 \times 2 = 6$). So that of neceſſity all the other Figures in the Product muſt be Decimal Parts; according as the Rule directs.

Or, the Rule is evident from the Multiplication of whole Numbers only: Thus, ſuppoſe 3000 were to be multiplied with 200, their Product will be 600000; That is, there will be ſo many Cyphers in the Product, as are in both the Factors. (*Vide page* 18.) Now if, inſtead of thoſe Cyphers in the Factors, we ſuppoſe the like Number of Decimal Parts; then it follows, that there ought to be the ſame Number of Decimal Parts in the Product, as there were Cyphers in the Factors.

Again, the Rule may be otherwiſe made evident from Vulgar Fractions, thus: Let 32,12 be multiplied with 24,3, and

and their Product will be 780,516 as in *Example* 2, above. Now 32,12 = 32 $\frac{12}{100}$. and 24,3 = 24 $\frac{3}{10}$. which being brought into Improper Fractions (*per Sect.* 3. *page* 50.) will become 32 $\frac{12}{100}$ = $\frac{3212}{100}$. and 24 $\frac{3}{10}$ = $\frac{243}{10}$.

Then $\frac{3212}{100} \times \frac{243}{10} = \frac{780516}{1000}$. *per Sect.* 7. *page* 55.
But $\frac{780516}{1000}$ = 780 $\frac{516}{100}$. *viz.* 780,516, as before.

Any of these three Ways do, I presume, sufficiently prove the Truth of the abovesaid Rule, *&c.*

*EXAMPLE* 3.

```
    78,546
       436
   -------
    471276
   235638
  314184
  --------
  34246,056
```

*EXAMPLE* 4.

```
      5745
     ,0675
   -------
     28725
    40215
   34470
  --------
  387,7875
```

N. B. *It sometimes falls out in multiplying Parts with Parts, that there will not be so many Figures in the Product, as there ought to be places of Decimal Parts by the Rule: In that Case you must supply their Defect by prefixing Cyphers to the Product; as in these Examples.*

*EXAMPLE* 5.

```
      ,2365
      ,2435
     ------
      11825
      7095
     9460
    4730
  ---------
  ,05758775
```

*EXAMPLE* 6.

```
      ,0347
      ,0236
     ------
       2082
      1041
      694
  ---------
  ,00081892
```

When any proposed Number of Decimals is to be multiplied with 10 . 100 . 1000 . 10000, *&c.* It is only removing the separating Point in the Multiplicand, so many places towards the Right-hand, as there are Cyphers in the Multiplier.

Thus, ,578 × 10 = 5,78. And, ,578 × 100 = 57,8.
Again, ,578 × 1000 = 578. Or, ,578 × 10000 = 5780.

These

These things being conſidered, it will be eaſy to multiply Decimals, and determine their true Products. As in theſe following Examples.

57,056 multiplied into 0,578 will produce 32,978368
7,6543 into 5,4246 will produce 41,52151578
0,56879 × 0,05674 = 0,0322731446
0,03246 × 0,02364 = 0,0007672544
87649 × 0,03687 = 3231,61863
94,35786 × 6,57869 = 620,7511100034
3,141592 × 52,7438 = 165,6995001296

Now it oftentimes happens, that it will be needleſs to expreſs all the Figures of the Product at large, (eſpecially, when the Factors have each of them many places of Decimal Parts, as in the two laſt Examples) only ſo many of them as may ſuffice for the intended Deſign; and yet the Product may be as true to ſo many Figures as are retained, as if the Factors had been multiplied at large. And ſuch compendious Contractions are not only of Curioſity, but may alſo be found of great Eaſe and Uſe to the ingenious Practitioner; eſpecially in reſolving adfected Equations, or in calculating of Trigonometrical Problems by the Natural Sines and Tangents, &c. All which may be thus performed.

Viz. *Set the Unit's Place of the Multiplier directly underneath that Figure of the Multiplicand, whoſe Place you intend to keep in the Product; and place all the other Figures of the Multiplier in a quite contrary Order to the uſual way. Then in multiplying, always begin at that Figure of the Multiplicand which ſtands over the Figure wherewith you are then a multiplying, ſetting down the firſt Figure of each particular Product, directly underneath one another; yet herein you muſt have a due Regard to the Increaſe which would ariſe out of the two next Figures to the Right-hand of that Figure in the Multiplicand which you then begin with.*

*EXAMPLE.*

Let it be required to multiply 3,141592 with 52,7438 and let there be only four Places of Decimal Parts retained in the Product.

If the propoſed Numbers were to be multiplied at large they muſt ſtand in a direct Order as uſual.

Thus, { 3,141592
52,7438 } And would produce ten Places of Parts, as in the laſt Example.

But

But seeing it is required to have only four Places of those Parts in the Product, set them down as before directed, and they will stand

| Thus | |
|---|---|
| 3,141592 | The Multiplicand placed as before. |
| 8347,25 | The multiplier in a reverse Order. |
| 1570796 | The Product with 5, regard had to 5 times 2. |
| 62832 | The Product with 2, increased with 9 × 2. |
| 21991 | Product with 7, increased with 5 × 7 + 9 × 7. |
| 1257 | Product with 4, increased with 1 × 4 + 5 × 4. |
| 94 | Product with 3, increased with 4 × 3. |
| 25 | Product with 8, increased with 4 × 8 + 1 × 8. |
| 165,6995 | The true Product as was required. |

The Reason of this Contraction is very obvious from the whole Operation wrought at large.

Thus { 3,141592 / 52,7438

| | |
|---|---|
| 25 | 132736 |
| 94 | 24776 |
| 1256 | 6368 |
| 2 1991 | 144 |
| 6 2831 | 84 |
| 157 0796 | 0 |
| 165,6995 | 001296 |

*From hence it is evident, that all the Figures in the Square to the Right-hand, are wholly omitted in the former Contraction; and that the last single* Product *here, is the first there; consequently the Reason of placing the* Multiplier *in a reverse Order, must needs appear very plain.*

## *EXAMPLE* 3.

Suppose it were required to multiply 257,356 with 76,48 and to have only the entire Product of integers.

| |
|---|
| 257,356 |
| 84,67 |
| 18015 |
| 1544 |
| 103 |
| 20 |
| 19682 |

The same at large { 257,356 / 76,48

| | |
|---|---|
| 20 | 58848 |
| 102 | 9424 |
| 1544 | 136 |
| 18014 | 92 |
| 19682, | 58688 |

The chiefest Care and Difficulty that attends these Contractions, is the true setting down of the Units place in the Multiplier underneath the proper Figure of the Multiplicand, according to the designed Product.

*Viz.*

*Viz.* In Example 1. It was required to have four Places of Decimal Parts in the Product; therefore the Unit's Place of the Multiplier, was set under the fourth Place of Decimals in the Multiplicand: And in Example 2, because it was required to have an entire Product of Integers only; therefore the Unit's Place of the Multiplier, was set under the Unit's Place of the Multiplicand. This, I say, being once rightly understood, will render the Method easy in Practice.

## Sect. 4. Division *of* Decimals.

DIVISION is accounted the most difficult Part of Decimal Arithmetick: In order therefore to make it plain and easy, it will be convenient to resume what has been said in *page* 25.

Viz. { *The Quotient Figure is always of the same Value or Degree with that Figure of the Dividend, under which the Unit's Place of it's Product stands.*

As for Instance, Let 294 be divided by 4.

```
4) 294 (7    { This is not 7 but 70, because the Units
   28        { Place of 4 × 7 stands under the Tens Place
             { of the Dividend.
   --
    14 (3 But this is only 3.
    12
    --
Remains (2) Hence 73 2/4 is the Quotient,
```

Now if to the Remainder 2 there be annexed a Cypher (thus, 2,0) and then divided on, it must needs follow that the Unit's Place of the Product arising from the Divisor into the Quotient, will stand under the annexed Cypher; consequently the Quotient Figure will be of the same Value or Degree with the Place of that Cypher: But that is the next below the Unit's Place, therefore the Quotient Figure is of the next Degree or Place below Unity; That is, in the first Place of Decimal Parts.

Thus 4) 2,0 (,5

So that 4) 294,0 (73,5 the true Quotient required.

This being well understood, Division of Decimals may (in all the various Cases) be easily performed. However, that it may be rendered plain and easy even to the meanest Capacity, if possible; Let Division be again defined, as in *page* 21.

Viz. *If that Number which divides another, be multiplied with the Number which is quoted, their Product will be the Number divided.*

This Definition alone (if compared with the Rule, *page* 61.) will afford a general Rule for difcovering the true Value of the Quotient Figure in Divifion of Decimals.

Rule { *The Place of Decimal Parts in the Divifor and Quotient, being counted together, muft always be equal in Number with thofe in the Dividend. And from this general Rule arifeth four Cafes.*

*Cafe* 1. When the Places of Parts in the Divifor and Dividend are equal, the Quotient will be whole Numbers.

As in thefe *Examples.*

```
8,45) 295,75 (35          0,0078) ,4368 (56
      253 5                       390
       42 25                       468
       42 25                       468
        (0)                        (0)
```

*Cafe* 2. When the Places of Parts in the Dividend exceed thofe in the Divifor; cut off the Excefs for Decimal Parts in the Quotient. As in thefe *Examples.*

```
24,3) 780,516 (32,12                      436) 34246,056 (78,546
      729                                      3052
       515                                      3726
       486                                      3488
        291                                      2380
        243                                      2180
        486     ,534) ,30438 (,57                 2005
        486            2670                        1744
        (0)             3738                        2616
                        3738                        2616
                         (0)                         (0)
```

*Cafe* 3. When there are not fo many Places of Parts in the Dividend, as are in the Divifor; annex Cyphers to the Dividend to make them equal. Then will the Quotient be whole Numbers, as in Cafe 1.

EX-

*EXAMPLES.*

Let it be required to divide 192,1 by 7,684, and 441 by ,7875.

| | |
|---|---|
| 7,684) 192,100 (25 | ,7875) 441,0000 (560 |
| 153 68 | 393 75 |
| 38 420 | 47 250 |
| 38 420 | 47 250 |
| (0) | (0) |

*Case* 4. If after Division is finished, there are not so many Figures in the Quotient, as there ought to be Places of Parts by the general Rule; supply their defect by prefixing Cyphers to it.

*EXAMPLES.*

Let it be required to divide 7,25406 by 957.

957) 7,25406 (,00758 the true Quotient required.

| | |
|---|---|
| 6 699 | |
| 5550 | Again ,575) ,0007475 (,0013 |
| 4785 | 575 |
| 7656 | 1725 |
| 7656 | 1725 |
| (0) | (0) |

*Note*, When Decimal Numbers are to be divided by 10. 100. 1000. 10000. *&c.* that is, when the Divisor is an Unit with Cyphers; Division is performed by removing or placing the separating Point in the Dividend, so many Places towards the Left-hand, as there are Cyphers in the Divisor.

*EXAMPLE.*

| | |
|---|---|
| 10) 5784 (578,4 | 100) 578,4 (57,84 |
| 1000) 5784 (5,784 | 100,00) 578,4 (,05784 |

Note, *These Operations are the direct Converse to those in* page 62.

I presume it needless to give more Examples at large; only I shall insert a few Dividends, and Divisors, with their Quotients, wherein are contained all the Varieties that can happen in Division of Decimals.

| | |
|---|---|
| 574) 493,066 ( 859 | 5,74) 49,3066 (8,59 |
| 574) 483,066 (,859 | 5,74) 493066,00 (85900 |
| 574) 49,3066 (,0859 | ,0574) 493,0665 (8590 |
| 5,74) 4930,66 ( 859 | ,0574) ,493066 (8,59 |



There

There is alſo a compendious Way of contracting Diviſion, like that of Multiplication, *page* 64, by which much Labour may be ſaved; eſpecially when the Diviſor hath many Places of Decimal Parts in it: And it is thus performed.

Having determined how many Places of whole Numbers there will be in the Quotient, if any at all; or if none, of what Value or Place the firſt Figure in the Quotient will be: Then omit, or prick off one Figure of the Diviſor at each Operation; *viz.* for every Figure you place in the Quotient, prick off one in the Diviſor; having a due Regard to the Increaſe which would ariſe from the Figure ſo omitted.

### *EXAMPLE.*

Let it be required to divide 70,23 by 7,9863.

```
The Work contracted.              The ſame at Length.
7,9863) 70,2300 (8,7938           7,9863) 70,2300 (8,7938
 ....   63 8904                           63 8904
         6 3396                            6 3396|0
         5 5904                            5 5904|1
           7492                              7491|90
           7187                              7187|67
            305                               304|230
            239                               239|589
             66                                64|6410
             64                                63|8904
            (2)                                 0|7506
```

The Work contracted I preſume is ſo obvious (if compared with the ſame at large) that it is needleſs to give any farther Explanation of it.

## Sect. 5. *To* Reduce Vulgar Fractions *into* Decimals, *and the contrary.*

ANY Vulgar Fraction being given, it may be reduced, or rather changed into Decimal Parts equivalent to it. Thus,

Rule { *Annex Cyphers to the Numerator, and then divide it by the Denominator, the Quotient will be the Decimal Parts equivalent to the given Fraction; or at leaſt ſo near it as may be thought neceſſary to approach.*

*EX-*

*EXAMPLE.*

It is required to change or reduce $\frac{3}{4}$ into Decimals.

4) 3,00 (,75 The Decimal Parts required.

That is, $\frac{3}{4} = \frac{75}{100} = ,75$.

Again $\frac{1}{2} = ,5$; thus 2) 1,0 ( ,5. And $\frac{1}{4} = ,25$; 4) 1,00 (,25.

Suppofe it were required to change $\frac{4}{7}$ into Decimals.

7) 4,0000000000 (,5714285714 &c. $= \frac{4}{7}$.

*Note*, When the laft Figure of the Divifor, (that is, the Denominator of the propofed Fraction) happens to be one of thefe Figures; *viz.* 1 . 3 . 7. or 9. (as in the Example) then the Decimal Parts can never be precifely equal to the given Fraction; yet by continuing the Divifion on, you may bring them to be very near the Truth. As in this Example; Suppofe it was required to change $\frac{1}{13}$ into Decimal Parts.

13) 1,0000 (,07692307692307 &c. *ad infinitum.*

```
 91..
 ----
  90
  78
  ---
  120
  117
  ---
    30
    26
    --
     40
     39
     --
     10
```

That is, 0,07692307692307 $= \frac{1}{13}$ *ferè.*

And from hence it may be farther obferved; that in thefe imperfect Quotients, the Figures do return again and circulate in the fame Order as before: as you may eafily perceive they begin to do in the feventh Place of both thefe laft Examples.

&c. As at firft.

Thefe being underftood, it will be eafy to find the Decimal Parts equivalent to any known Part or Parts of Coin, Weights, Meafures, Time, &c. If you firft reduce the given Parts of Coin, &c. into a Vulgar Fraction, whofe Denominator is the Number of thofe known Parts contained in the Integer, and the given Parts it's Numerator.

*Examples in Coin,* &c.

1. Let it be required to find the Decimals of 16 *s.* 6 *d.* Firft 16 *s.* $= \frac{16}{20}$ of one Pound, and 6 *d.* $= \frac{1}{40}$ of 1 *l.*

But $\frac{16}{20} + \frac{1}{40} = \frac{33}{40}$. Then 40) 33,000 (,825 the Decimal Parts required: That is, 825 $=$ 16 *s.* 6 *d.*

Again, Suppofe it were required to find the Decimals equal to 3 *l.* 13 *s.* 4 *d.*

Here

Here 3*l.* is 3 Integers, and 13*s.* = $\frac{13}{20}$ of 1 *l.* and 4*d.* = $\frac{4}{240}$. But $\frac{13}{20} + \frac{4}{240} = \frac{160}{240}$. Then 240) 160,000 (0,666666 &c. Hence 3*l.* 13*s.* 4*d.* = 3,666666 &c. As was required.

2. What are the Decimals equal to 7 $\frac{3}{4}$ Inches, one Foot being made the Integer.

First, 7 Inches are $\frac{7}{12}$ of 1 Foot, and $\frac{3}{4}$ of 1 Inch are $\frac{3}{48}$. But $\frac{7}{12} + \frac{3}{48} = \frac{31}{48}$. Then 48) 31,000 (,64583 &c. = 7 $\frac{3}{4}$ Inches.

3. Let it be required to change 8 *Oz.* 19 *Pwt.* 8 *Grains* into Decimals; one Pound Troy being the Integer.

These being reduced into the least Terms, and added together, will become $\frac{4304}{5760}$ of 1 Pound.

Then 5760) 4304,000 (,74722 &c. The Decimals required.

And thus may any proposed Parts of Coin, Weights, Measures, &c. be reduced or changed into Decimal Parts; which perhaps may at first seem somewhat tedious in Practice, but being a little acquainted with them it will be found very easy; and the ingenious Practitioner will (with a little Consideration) soon find how to reduce them almost mentally; or with the help of a very few Figures, without the use of such large Tables as are usually inserted in Books of Decimal Arithmetick; or at most they may be contracted into such as these following, which if duly applied to those Tables in Chap. 3. will be found very useful.

*Decimal Tables.*

*In* English Coin.

| | |
|---|---|
| 0,05 . . . . . | = 1 *s.* |
| 0,0046667 | = 1 *d.* |
| 0,00104167 | = 1 *Farthing.* |

1 *l.* being the *Integer.*

*Averdupois Weight.*

| | |
|---|---|
| 0,0625 . . . . | = 1 *Ounce* |
| 0,00390625 | = 1 *Dram.* |

1 lb. being the *Integer.*

*Troy Weight.*

| | |
|---|---|
| 0,05 . . . . . . | = 1 *Pwt.* |
| 0,00208333 | = 1 *Grain.* |

1 *Oz.* being the *Integer.*

*Averdupois Great Weight.*

| | |
|---|---|
| 0,25 . . . . . . | = $\frac{1}{4}$ *C.* |
| 0,00892857 | = 1 lb. |
| 0,00055803 | = 1 *Ounce.* |

1 *C.* being the *Integer.*

*Apothecaries Weight.*

| | |
|---|---|
| 0,125 . . . . . | = 1 *Dram.* |
| 0,04166667 | = 1 *Scruple.* |
| 0,00208333 | = 1 *Grain.* |

1 *Oz.* being the *Integer.*

*Time.*

| | |
|---|---|
| 0,04166667 | = 1 *Hour.* |
| 0,00069444 | = 1 *Minute.* |
| 0,00001157 | = 1 *Second.* |

1 *Day*, or 24 *Hours*, being made the *Integer.*

The Use of these Tables will be evident by the following

*EXAMPLE.*

*EXAMPLE.*

Let it be required to find the Decimal Parts equivalent to 17*s.* 9*d.* 2 *Farthings*

First 0,05 = 1*s.* Therefore 17 × ,05 = ,85 .... = 17*s.*
And ,004166 = 1*d.* Therefore ,004166 × 9 = ,037494 = 9*d.*
Also 2) ,004166 (= ,002083 = ½*d.*
Consequently their Sum, *viz.* 0,889577 = 17*s.* 9½*d.*

Now to find the Value of Decimals in known Parts of Coin or Weights, *&c.* is only the Converse of the former Work, and is thus performed.

Multiply the given Decimals with the Denominator of the Vulgar Fraction required: That is, multiply the Decimals with such a Number of Units as are contained in the next lower Denomination of that Kind or Species which your Decimal is of; and the Product will be the Number required.

*EXAMPLE.*

1. What is the Value of 0,825 Decimals of 1 Pound Sterling. That is, how many Shillings, Pence, *&c.* = ,825. First, the next lower Denomination is 20, because 20*s.* make one Pound.

Therefore 0,825
20
Shillings 16,500 and Parts of 1 Shilling,
12
Pence 6,000 Answer 0,825 = 16*s.* 6*d.*

Again, What are the known Parts of English Coin equal to 3,666666 Decimals.

Here the 3 Integers are 3 Pounds. Then ,666666
20
Shillings 13,333320
12
666640
3 3332
Pence 3,999840 = 4 near.

Answer 3,666666 = 3*l.* 13*s.* 4*d.*

What is the Value of 0,74722 Parts of 1 lb *Troy*.

First, ,74722
12
1 49444
7 4722
*Oz.* 8,96664

Then, ,96664
20
*Pwts.* 19,33280

Again, ,33280
24
1 3312
6 656
7,98720

These collected are *Oz.* 8. *Pwt* 19. *Gr.* 8. very near.

And

And thus any propofed Number of Decimals may be turned or changed into the known Parts of what they reprefent, *viz.* Whether they be Parts of Coin, Weights, Meafures, or Time, *&c.*

I have omitted inferting more Examples of this kind, becaufe I take the Excellency, and indeed the chief Ufe of Decimal Fractions to confift more in Geometrical Computations, than in the common or practical Parts of Arithmetick, as will appear further on; although even in thofe they are very ufeful upon feveral Accounts; efpecially in the Computations of Intereft and Annuities, *&c.* But of that more in it's proper Place. I fhall therefore conclude this Chapter, with a Remark or two upon the Nature and Properties of Fractions in general.

If any given Number (whether it be whole or mixed) be multiplied with a Fraction either Vulgar or Decimal, the Product will be lefs than the Multiplicand, in fuch a Proportion as the multiplying Fraction is lefs than an Unit or 1.

*That is; as the Denominator of the Fraction is to it's Numerator, fo will the given Number be to the Product.*

Therefore, whenever any Number is to be multiplied with a Fraction, whofe Numerator is an Unit: Divide that Number by the Denominator of the Fraction, and the Quotient will be the Product required. Thus $12 \times \frac{1}{4} = 3$. And $12 \div 4 = 3$. Again, $12 \times \frac{1}{2} = 6$. And $12 \div 2 = 6$, *&c.*

From hence it follows, that if any Number be divided by a Fraction, the Quotient will be greater than the Dividend, by fuch a Proportion as Unity is greater than the dividing Fraction.

Thus $12 \div \frac{1}{4} = 48$, *viz.* $\frac{1}{4} : 1 :: 12 : 48$, *&c.* But the Truth of thefe will be beft underftood after the next Chapter.

## CHAP. VI.

### *Of* Continued Proportions, *and how to change or vary the Order of Things.*

#### Sect. 1. *Concerning* Arithmetical Progreffion, *ufually called* Arithmetical Proportion Continued.

WHEN any Rank or Series of Numbers do either increafe or decreafe by an equal Interval or common Difference; thofe Numbers are faid to be in Arithmetical Progreffion.

* As

As $\begin{cases} 1 . 2 . 3 . 4 . 5 . 6 . 7 \text{ \&c.} \\ 7 . 6 . 5 . 4 . 3 . 2 . 1 \end{cases}$ Here the Interval or common Difference is 1.

Or $\begin{cases} 2 . 4 . 6 . 8 . 10 . 12 . 14 \text{ \&c.} \\ 1 . 3 . 5 . 7 . 9 . 11 . 13 \text{ \&c.} \end{cases}$ Here the common Difference is 2.

And ſo of any other Series, whoſe common Difference is 3 . 4 . 5. &c.

*Lemma* 1.

If any three Numbers be in Arithmetical Progreſſion, the Sum of the two Extremes (*viz.* the firſt and laſt) will be equal to the Double of the Mean or middle Number.

As in theſe, 2 . 4 . 6. Or 3 . 6 . 9. Or 3 . 7 . 11.
*Viz.* $2+6=4+4$. Or $3+9=6+6$. And $3+11=7+7$. &c.

*Lemma* 2.

If any four Numbers are in Arithmetical Progreſſion, the Sum of the two Extremes will be equal to the Sum of the two Means.

As in theſe, 2 . 4 . 6 . 8. Or 3 . 6 . 9 . 12.
*Viz.* $2+8=4+6$. And $3+12=6+9$. &c.

*Corollary* 1.

*From theſe two Lemma's it is eaſy to conceive, that if never ſo many Numbers be in Arithmetical Progreſſion, the Sum of the two Extremes will be equal to the Sum of any two Means, that are equally diſtant from thoſe Extremes.*

As in theſe, 2 . 4 . 6 . 8 . 10 . 12 . 14 . 16.
Then $2+16=4+14=6+12=8+10$.

Or if the Number of Terms be odd, as theſe,
2 . 4 . 6 . 8 . 10 . 12 . 14 . 16 . 18. &c.
Then $2+18=4+16=6+14=8+12=10+10$.

*Lemma* 3.

Every Series of Numbers in Arithmetical Progreſſion is compoſed of the Interval or common Difference, ſo often repeated as there are Terms in the Progreſſion, except the firſt.

As in theſe, 1 . 3 . 5 . 7 . 9 . 11 . 13 . 15 . 17. &c.
Here the Interval or common Difference being two, it will be $1+2=3$. $3+2=5$. $5+2=7$. $7+2=9$. $9+2=11$. $11+2=13$. $13+2=15$. $15+2=17$. &c.

*Corollary* 2.

*Hence it is evident, that the Difference betwixt the two Extremes* (viz. 1 *and* 17) *is compoſed of the common Difference, multiplied into the Number of all the Terms, excepting the firſt.*

As in the aforeſaid Progreſſion, 1 . 3 . 5 . 7 . 9 . 11 . 13 . 15 . 17.

L The

The Number of Terms without the firſt is 8 } Multiply
The common Difference is 2 }
The Difference betwixt the two Extremes is 16

*Propoſition* 1.

In any Series of Numbers in Arithmetical Progreſſion, the two Extremes, and the Number of Terms being given, thence to find the Sum of all the Series.

Theorem 1. { *Multiply the Sum of the two Extremes into the Number of all the Terms; and divide the Product by* 2. *The Quotient will be the Sum of all that Series. Per Corol.* 1.

*EXAMPLE* 1.

It is required to find the Number of all the Strokes a Clock ſtrikes in one whole Revolution of the Index, *viz.* twelve Hours.

Here 1 + 12 = 13 the Sum of the two Extremes.
× 12 the Number of all the Terms
26
13
Then 2) 156 (78. The Number of Strokes required

*EXAMPLE* 2.

Suppoſe one Hundred Eggs were placed in a Right Line a Yard diſtant from one another, and the firſt Egg were a Yard from a Baſket; whether or no may a Man gather up theſe 100 Eggs ſingly one after another, ſtill returning with every Egg to the Baſket and putting it in, before another Man can run four Miles. That is, which will run the greater Number of Yards.

In this Queſtion 200 + 2 = 202 Is the Sum of the two Extremes.
And × 100 Is the Number of all the Terms.
Then 2) 20200 (10100 { The Number of Yards he runs that takes up the Eggs.

Now 4 Miles = 7040 Yards { The Yards he runs that takes up
But 10100 — 7040 = 3060 { the Eggs more than the other.

*Propoſition* 2.

In any Series of Numbers in Arithmetical Progreſſion, the two Extremes and Number of Terms being given; thence to find the common Difference of all the Terms in that Series.

Theorem 2. { *The Difference betwixt the two Extremes, being divided by the Number of Terms leſs an Unit or* 1. *The Quotient will be the common Difference of the Series. Per Corol.* 2.

*EXAMPLE*

*EXAMPLE* 1.

One had Twelve Children that differed alike in all their Ages; the youngeſt was Nine Years old, the eldeſt was Thirty-ſix and a half; what was the Difference of their Ages, and the Age of each?

Here 36,5 — 9 = 27,5 The Difference of the two Extremes.
And 12 — 1 = 11. The Numbers of Terms leſs an Unit.
Then 11) 27,5 (2,5 The common Difference required.
Conſequently 9 + 2,5 = 11,5 The Age of the youngeſt but one.
And 11,5 + 2,5 = 14 The Age of the youngeſt but two. And ſo on for the reſt. *Per Corol.* 2.

*EXAMPLE* 2.

A Debt is to be diſcharged at eleven ſeveral Payments to be made in Arithmetical Progreſſion. The firſt Payment to be Twelve Pounds Ten Shillings, and the laſt to be Sixty-three Pounds. What is the whole Debt, and what muſt each Payment be?

*Per Theorem* 1. Find the whole Debt thus:

12,5 + 63 = 75,5 The Sum of the Extremes.
11 The Number of Terms.

755
755

2) 830,5 (415,25 = 415*l.* 5*s.* The whole Debt.

Then, *per Theorem* 2. find the common Difference of each Payment.

Thus 63 — 12,5 = 50,5 The Difference of the Extremes.
And 11 — 1 = 10 The Number of Terms leſs 1.
Then 10) 50,5 (5,05 = 5*l.* 1*s.* The common Difference.

Conſequently 12*l.* 10*s.* + 5*l.* 1*s.* = 17*l.* 11*s.* The ſecond Payment.

And 17*l.* 11*s.* + 5*l.* 1*s.* = 22*l.* 12*s.* The third Payment, &c.

*EXAMPLE* 3.

A Man is to travel from *London* to a certain Place in ten Days, and to go but two Miles the firſt Day, increaſing every Day's Journey by an equal Exceſs; ſo that the laſt Day's Journey may be Twenty-nine Miles; what will each Day's Journey be, and how many Miles is the Place he goes to diſtant from *London*?

First 29 — 2 = 27 The Difference of the Extremes.
And 10 — 1 = 9 The Number of Terms less 1.
Then 9) 27 (3 The common Difference.
Consequently 2 + 3 = 5 The second Day's Journey.
And 5 + 3 = 8 The third Day's Journey, &c.
Again 29 + 2 = 31 The Sum of the Extremes.
10 The Number of Terms.
2) 310 (155 The Distance required.

There are eighteen Theorems more relating to Questions in Arithmetical Progression; but because they would require a great many Words to shew the Reason of them: I therefore refer the Reader to the Second Part, *viz.* That of *Algebra*, where he may find their *Analytical Investigation.*

---

## Sect. 2. Concerning Geometrical Proportion *continued*; sometimes called *Geometrical Progression.*

WHEN a Rank or Series of Numbers do either increase by one common Multiplicator, or decrease by one common Divisor; Those Numbers are said to be in Geometrical Proportion continued.

As { 2 . 4 . 8 . 16 . 32. &c. here 2 is the common Multiplier.
64 . 32 . 16 . 8 . 4. &c. here 2 is the common Divisor.

Or { 2 . 6 . 18 . 54 . 162. &c. here 3 is the common Multiplier.
162 . 54 . 18 . 6 . 2. here 3 is the common Divisor.

*Note*, The common Multiplier (or Divisor) is called the Ratio; and it shews the Habitude or Relation the Numbers have to one another, *viz.* whether they are Double, Triple, Quadruple, &c. which *Euclid* thus defines.

Ratio (*or* Rate) *is the mutual Habitude or Respect of two Magnitudes (consequently two Numbers) of the same kind each to other, according to Quantity*, Eucl. 5. Def. 3.

Proportion (rather Proportionality) is a Similitude of *Ratio*'s. Eucl. 5. Def. 4.

So that there cannot be less than three Terms to form a Proportionality or Similitude of Ratio's; and if but three Terms, the second must supply the Place of two, As in these 2 . 4 . 8. That is, 2 : 4 : : 4 : 8. (of : : see *page* 5.)

Here 4 the middle Term supplies the Place of two Terms, to wit, of the second and third; 8 bearing the same Reason, Likeneſs, or Proportion, to 4, as 4 doth to 2. *viz.* As 2 : is to 4 :: So is 4 : to 8.

*Lemma* 1.

If three Numbers are proportional, the Rectangle, or Product of the two Extremes; *viz.* of the firſt and laſt Terms will be equal to the Square of the Mean or middle Term. (20 *Eucl.* 7.)

As in theſe 2 : 4 :: 4 : 8. Here $8 \times 2 = 16$ the Product of the Extremes.

And $4 \times 4 = 16$ the Square of the Mean. *Ergo* $8 \times 2 = 4 \times 4$.

*Corol.* 1.

Hence it follows, that if the Product of any two Numbers be equal to the Square of a third Number; thoſe three Numbers will be in Proportion.

*Lemma* 2.

If four Numbers are proportional, the Product of the two Extremes will be equal to the Product of the two Means, (19 *Euclid* 7.)

As in theſe, 2 : 4 :: 8 : 16. Here $16 \times 2 = 32$.
And $8 \times 4 = 32$. Conſequently $16 \times 2 = 8 \times 4$.

*Corol.* 2.

*From hence it follows, that if the Product of any two Numbers, be equal to the Product of any other two Numbers, thoſe four Numbers are Proportionals.*

And from theſe two *Lemma*'s it will be eaſy to conceive, that if never ſo many Numbers are in continued Proportion; the Product of the two Extremes, will be equal to the Product of any two Means, that are equally diſtant from the Extremes.

As in theſe 2 . 4 . 8 . 16 . 32 . 64. &c.

Here $64 \times 2 = 32 \times 4 = 16 \times 8$. &c. And if the Number of Terms be odd,

As in theſe 2 . 4 . 8 . 16 . 32 . 64 . 128. &c.
Then $128 \times 2 = 64 \times 4 = 32 \times 8 = 16 \times 16$.

Note, *The Character made Uſe of to ſignify continued Proportionals is* $\div\!\!\div$.

In

In every Series of ∻ (*viz. of continued Proportionals*) that Number which is compared to another, is called the Antecedent of the Ratio; and that Number to which it is compared, is called it's Consequent.

As in these, 2 : 4 :: 4 : 8. Here 2 is the Antecedent, and 4 is the Consequent; and 4 the middle Term is an Antecedent to 8 it's Consequent: whence it follows, that in every Series of ∻ all the middle Terms between the first and last are both Antecedents and Consequents.

As in these, 2 . 4 . 8 . 16 . 32 . 64. &c. Here 4 . 8 . 16 . 32. are both Consequents and Antecedents.

For 2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 32 :: 32 : 64. &c.

So that all the Terms except the last are Antecedents. And all the Terms except the first are Consequents.

*Lemma* 3.

If never so many Numbers are proportional, it will be: As any one of the Antecedents is to it's Consquent: So will the Sum of all the Antecedents be; to the Sum of all the Consequents. (12 *Euclid* 5.)

That is, in the foregoing Series.

$2 : 4 :: 2 + 4 + 8 + 16 + 32 : 4 + 8 + 16 + 32 + 64.$

For it is evident, that $4 + 8 + 16 + 32 + 64$, the Sum of all the Consequents, is double to $2 + 4 + 8 + 16 + 32$ the Sum of all the Antecedents; as 4 is to 2, according to the Ratio, and would have been Triple, or Quadruple, &c. had the Ratio been 3 or 4, &c.

Note, *In every Series of* ∻ *the Ratio is found by dividing any of the Consequents by it's Antecedent.*

As in these 2 : 6 :: 6 : 18 :: 18 : 54 :: 54 : 162.

Here 2) 6 (3 the Ratio. Or 6) 18 (3 &c.

From the second and third Lemma's may be raised two general Theorems or Rules, for finding the Sum of any Series in ∻ without a continued Addition of all the Terms.

Let the Series 2 . 4 . 8 . 16 . 32 . 64 . 128. be given, to find it's Sum.

Suppose $z =$ the Sum of all the Terms.

Then will $z - 128 =$ the Sum of all the Antecedents.

And $z - 2 =$ the Sum of all the Consequents.

But $2 : 4 :: z - 128 : z - 2$. *per Lemma* 3.

*Ergo* $4z - 512 = 2z - 4$. *per Lemma* 2.

Consequently

Consequently $4z - 2z = 512 - 4$.

Theorem. $z = \frac{512 - 4}{4 - 2}$ In Words at length thus,

Theorem 1. *From the Product of the second and last Terms substract the Square of the first Term, and that Remainder being divided by the second Term less the first, will give the Sum of all the Series.*

Or if the first Term, the common Ratio, and the last Term be only given. Then,

Theorem 2. *Multiply the last Term into the Ratio, and from their Product substract the first Term; divide that Remainder by the Ratio less Unity or 1, and it will give the Sum of all the Series.*

For $4z - 2z = 512 - 4$. As above.

Consequently $2z - z = 256 - 2$. *viz.* the last divided by 2.

Then $z = \frac{256 - 2}{2 - 1}$ *Theorem* 2.

*EXAMPLE.*

Let 2 . 6 . 18 . 54 . 162 . 486. be the given Series. Here 2 is the first Term, 3 is the Ratio, and 486 the last Term.

But $486 \times 3 = 1458$. And $1458 - 2 = 1456$.
Then $3 - 1 = 2$) 1456 (728 the Sum required.
That is, $728 = 2 + 6 + 18 + 54 + 162 + 486$.

Since in either of these Theorems it is required to have the last Term known (the which in a long Series of ∺, will be very tedious to come at by a continued Multiplication) it will therefore be convenient to shew how to obtain either the last Term or any other Term, whose Place is assigned, without producing all the Terms.

In order to that, it will be necessary to premise the Coherence or Similitude that is betwixt Numbers in Arithmetical Progression and those in Geometrical Proportion.

If to any Series of Numbers in ∺, when the first Term is not an Unit or 1, there be assigned a Series of Numbers in Arithmetical Progression, beginning with an Unit or 1, and whose common Difference is 1. called Indices or Exponents.

Thus, { 1 . 2 . 3 . 4 . 5 . 6 . 7. Indices
2 . 4 . 8 . 16 . 32 . 64 . 128. &c. ∺ }

Then

Then will the Addition or Subftraction of any two of thofe Indices (or Numbers in Arithmetical Progreffion) directly correfpond with the Product, or Quotient of their refpective Terms in the Series of ∺.

That is, { As $3 + 4 = 7$. So $8 \times 16 = 128$ the feventh Term in ∺

Again, { As $6 + 4 = 10$. So $64 \times 16 = 1024$. the tenth Term in ∺

Or, { As $7 - 3 = 4$. So $128 \div 8 = 16$. Or, { As $6 - 2 = 4$. So $64 \div 4 = 16$. &c.

But if the Series of ∺ begin with an Unit, the Indices muft begin with a Cypher.

As in thefe, { 0 . 1 . 2 . 3 . 4 . 5 . 6. &c. 1 . 2 . 4 . 8 . 16 . 32 . 64.

Now by the help of the Indices, and a few of the firft Terms in any Series of ∺; it is plain that any Term whofe Place or Diftance from the firft Term is affigned, may be fpeedily obtained without producing the whole Series.

## *EXAMPLE* 1.

A Man bought a Horfe, and was to give a Farthing for the firft Nail, two for the fecond, four for the third, &c. in ∺, the Number of Nails was to be 7 in every Shoe, *viz.* 28 Nails in all. What muft he have paid for the Horfe?

Firft, { 0 . 1 . 2 . 3 . 4 . 5. Indices 1 . 2 . 4 . 8 . 16 . 32. Farthings in ∺

Then, { $5 + 5 = 10$ $32 \times 32 = 1024$ And, { $10 + 10 = 20$ $1024 \times 1024 = 1048576$

Again, { $4 + 3 = 7$ $16 \times 8 = 128$ Laftly, { $20 + 7 = 27$ $1048576 \times 128 = 134217728$

Which is here to be accounted the 28 and laft Term. Becaufe the firft Term in the Series is 1, which doth neither multiply nor divide.

Now this 134217728 being the Number of Farthings to be paid for the laft Nail, by it the common Ratio which is 2, and the firft Term which is 1, may be found the Sum of all the Series, *per Theorem* 2.

$$\begin{array}{r} 134217728 \\ 2 \\ \hline 268435456 \end{array}$$ From this Product ſubſtract 1.

*Viz.* 268435456 — 1 = 268435455. Then 2 — 1 = 1 the Diviſor.

Conſequently 268435455 is the Sum of all the Series, or Price of the Horſe in Farthings, which being brought into Pounds, (See *page* 46) will be 279620 *l.* 5 *s.* 3 *d.* 3 *qrs.*

## *EXAMPLE* 2.

A cunning Servant agreed with a Maſter (unſkilled in Numbers) to ſerve him Eleven Years without any other Reward for his Service but the Produce of one Wheat Corn for the firſt Year; and that Product to be ſowed the ſecond Year, and ſo on from Year to Year until the end of the Time, allowing the Increaſe to be but in a ten-fold Proportion.

It is required to find the Sum of the whole Produce.

Firſt $\begin{cases} 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5. \text{ Indices or Years.} \\ 10 \,.\, 100 \,.\, 1000 \,.\, 10000 \,.\, 100000 \text{ Wheat Corns in } \div\!\div \end{cases}$

Then $\begin{cases} \text{As } 4 + 2 = 6. \\ \text{So } 10000 \times 100 = 1000000. \text{ the 6th Year's Produce.} \end{cases}$

And $\begin{cases} 6 + 5 = 11. \\ 1000000 \times 100000 = 100000000000. \end{cases}$ The eleventh or laſt Year's Produce.

Then (either by *Theorem* 1, or 2) the Sum of all the Series will be 111111111110 Corns. Now it may be computed from Page 31 and 34, that 7680 Wheat Corns, round and dry out of the middle of the Ear, will fill a Statute Pint. If ſo,

Then 7680) 111111111110 (14467592 Pints, but 64 Pints are contained in a Buſhel.

Therefore 64) 14467592 (226056 $\frac{1}{8}$ Buſhels. Suppoſe it to be ſold for 3 Shillings the Buſhel;

Then $\begin{array}{r} 226056\frac{1}{8} \\ 3 \\ \hline \end{array}$

Shillings - 678168 $\frac{3}{8}$ = 33908 *l.* 8 *s.* 4 $\frac{1}{2}$ *d.* A very good Recompence for Eleven Years Service.

There are ſeveral pretty Queſtions reſolved by Numbers in Arithmetical Progreſſion, and by thoſe in $\div\!\div$, which the ingenious Learner will eaſily perceive hereafter; *viz.* When we come to the Solution of Queſtions relating to Intereſt and Annuities, *&c.*

There is alſo a third Kind of Proportion, called Muſical, which being but of little or no common Uſe, I ſhall therefore give but a ſhort Account of it.

Muſical Proportion or Habitude is, when of three Numbers; the firſt hath the ſame Proportion to the third, as the Difference between the firſt and ſecond hath to the Difference between the ſecond and third.

As in theſe, 6 . 8 . 12. *viz.* 6 : 12 :: 8 — 6 : 12 — 8

If there are four Numbers in Muſical Proportion; The firſt will have the ſame Proportion to the fourth, as the Difference between the firſt and ſecond hath to the Difference between the third and fourth.

As in theſe 8 . 14 . 21 . 84.
Here 8 : 84 :: 14 — 8 = 6 : 84 — 21 = 63.
That is, 8 : 84 :: 6 : 63.

The Method of finding out Numbers in Muſical Proportion, is beſt expreſſed by Letters; as ſhall be ſhewed in the *Algebraick Part.*

## Sect. 3. *How to* Change *or* Vary *the Order of Things*, &c.

THIS being a Thing not treated of in any common Books of Arithmetick (that I have had the Opportunity of peruſing), made me think it would be acceptable to the young Learner, to know how oft it is poſſible to vary or change the Order or Poſition of any propoſed Number of Things.

As how many ſeveral Changes may be rung upon any propoſed Number of Bells; or how many ſeveral Variations may be made of any determined Number of Letters, or any other Things propoſed to be varied.

*The Method of finding out the Number of Changes is by a continual Multiplication of all the Terms in a Series of Arithmetical Progreſſions, whoſe firſt Term and common Difference is Unity or* 1. *And the laſt Term the Number of Things propoſed to be varied*, viz. 1 × 2 × 3 × 4 × 5 × 6 × 7, *&c.* As will appear from what follows.

1. If the Things propoſed to be varied are only two, they admit of a double Poſition (as to Order of Place) and no more.

Thus, $\left\{ \begin{matrix} 1 . 2 \\ 2 . 1 \end{matrix} \right\} = 2 = 1 \times 2.$

2. And if three Things are propoſed to be varied, they may be

be changed ſix ſeveral Ways (as to their Order of Place) and no more,

For, beginning with 1, there will be { 1 . 2 . 3 / 1 . 3 . 2

Next, beginning with 2, there will be { 2 . 1 . 3 / 2 . 3 . 1

Again, beginning with 3, it will be { 3 . 1 . 2 / 3 . 2 . 1

Which in all make 6 or 3 Times 2, *viz.* $1 \times 2 \times 3 = 6$

Suppoſe four Things are propoſed to be varied;
Then they will admit of 24 ſeveral Changes, as to their Order of different Places.

For beginning the Order with 1 it will be { 1 . 2 . 3 . 4 / 1 . 2 . 4 . 3 / 1 . 3 . 2 . 4 / 1 . 3 . 4 . 2 / 1 . 4 . 2 . 3 / 1 . 4 . 3 . 2

Here is ſix different Changes.

And for the ſame Reaſon there will be 6 different Changes, when 2 begins the Order, and as many when 3 and 4 begins the Order; which in all is $24 = 1 \times 2 \times 3 \times 4$. And by this Method of proceeding, it may be made evident, that 5 Things admit of 120 ſeveral Variations or Changes; and 6 Things of 720, &c. As in this following Table.

| *The Number of Things propoſed to be varied.* | *The manner how their ſeveral Variations are produced.* | *The different Changes or Variations every one of the propoſed Numbers can admit of.* |
|---|---|---|
| 1 | 1 | = 1 |
| 2 | 1 × 2 | = 2 |
| 3 | 2 × 3 | = 6 |
| 4 | 6 × 4 | = 24 |
| 5 | 24 × 5 | = 120 |
| 6 | 120 × 6 | = 720 |
| 7 | 720 × 7 | = 5040 |
| 8 | 5040 × 8 | = 40320 |
| 9 | 40320 × 9 | = 362880 |
| 10 | 362880 × 10 | = 3628800 |
| 11 | 3628800 × 11 | = 39916800 |
| 12 | 39916800 × 12 | = 479001600 |
| &c. | &c. | &c. |

Theſe may be thus continued on to any aſſigned Number. Suppoſe to 24 the Number of Letters in the Alphabet, which will admit of 620448401733239439360000 ſeveral Variations.

From theſe Computations may be ſtarted ſeveral pretty, and indeed, very ſtrange Queſtions.

## *EXAMPLES.*

Six Gentlemen, that were travelling, met together by Chance at a certain Inn upon the Road, where they were ſo pleaſed with their Hoſt, and each other's Company, that in a Frolick they made a Contract to ſtay at that Place, ſo long as they, together with their Hoſt, could ſit every Day in a different Order or Poſition at Dinner; which by the foregoing Computations will be found near 14 Years. For they being made 7 with their Hoſt, will admit of 5040 different Poſitions; but 5040 being divided by 365 $\frac{1}{4}$ (the Number of the Days in one Year) will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told, that before the Fire of *London* (which happened *Anno* 1666) there were 12 Bells in St *Mary Le Bow*'s Church in *Cheapſide*, *London*. Suppoſe it were required to tell how many ſeveral Changes might have been rung upon thoſe 12 Bells; and at a moderate Computation how long all thoſe Changes would have been ringing but once over.

Firſt, $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 479001600$, the Number of Changes.

Then ſuppoſing there might be rung 10 Changes in one Minute: *viz.* $12 \times 10 = 120$ Strokes in a Minute, which is 2 Strokes in a Second of Time: Now according to that Rate there muſt be allowed 47900160 Minutes to ring them once over in all their different Changes; *viz.* 10) 479001600 (47900160.

In one Year there is 365 Days, 5 Hours, and 49 Minutes; which, being reduced into Minutes, is 525949.

Then 525949) 47900160 (91 Years and 26 Days.

So long would thoſe 12 Bells have been continually ringing without any Intermiſſion, before all their different Changes could have been truly rung but once over. It is ſtrange, and ſeems almoſt incredible, that a few Things ſhould produce ſuch Varieties.

But that which ſeems yet more ſtrange and ſurpriſing (yea, even impoſſible to thoſe who are not verſed in the Power of Numbers) is, is, that if two Bells more had been added to the aforesaid 12, they would have advanced the Number of Changes (and consequently the Time) beyond common Belief. For 14 Bells would require (at the same rate of ringing as before) about 16575 Years to ring all their different Changes but once over.

And if it were possible to ring 24 Bells in Changes (and at the same rate of 10 Changes in a Minute, which is 2 Strokes in one Second) they would require more than 117000000000000000 Years to ring them but once over in all their different Changes; as may easily be computed from the precedent Table.

## CHAP. VII.

## *Of* Proportion Disjunct; *commonly called the* Golden Rule.

PRroportion *Disjunct* or the Golden Rule, is either Direct or Reciprocal, called Inverse. And those are both Simple and Compound.

### SECT. I.

*Direct Proportion* is, when of four Numbers, the first bearing the same Ratio or Proportion to the second; as the third doth to the fourth.

As in these $2 : 8 :: 6 : 24$.

Consequently, the greater the second Term is, in respect to the first; the greater will the fourth Term be, in respect to the third.

That is, as 8 the second Term is 4 Times greater than 2 the first Term: So is 24 the fourth Term, 4 Times greater than 6 the third Term.

Whence it follows, that if four Numbers are in Direct Proportion, the Product of the two Extremes will always be equal to the Product of the two Means, as well in Disjunct as in continued Proportion; according to *Lemma* 2. *page* 77.

For As $2 : 2 \times 4 :: 6 : 6 \times 4$. Or As $3 : 3 \times 5 :: 6 : 6 \times 5$.
But $2 \times 6 \times 4 = 2 \times 4 \times 6$. Or $3 \times 6 \times 5 = 3 \times 5 \times 6$.

That is, the Product of the Extremes is equal to that of the Means.

Again,

Again, the lefs the fecond Term is, in refpect to the firft; the lefs will the fourth Term be in refpect to the third.

As in thefe 18 : 6 :: 12 : 4.

That is, $18 : 18 \div 3 :: 12 : 12 \div 3$.

But $18 \times 12 \div 3 = 18 \div 3 \times 12$. *Viz.* $18 \times 4 = 6 \times 12$.

Confequently 2 . 8 . 6 . 24. And 18 . 6 . 12 . 4. are true Proportionals, *per Corol.* 2. *page* 77.

From thefe Confiderations, comes the Invention of finding a fourth Number in Proportion to any three given Numbers. *Whence it is called the Rule of Three.*

For if the fecond Number multiplied into the third, be equal to the firft multiplied into the fourth, it is eafy to conceive, that if the Product of the fecond and third be divided by the firft, the Quotient muft needs be the fourth Number. For if that Number, which divides another, be multiplied into the Quotient produced by that Divifion; their Product will be equal to the Number divided. *See page* 21.

As in thefe 2 : 8 :: 6 : 24. Here $8 \times 6 = 48 = 24 \times 2$.
But if $24 \times 2 = 48$, then will $48 \div 2 = 24$. Or $48 \div 24 = 2$.

*Note*, Any four Numbers in direct Proportion may be varied feveral Ways. As in thefe.

*Viz.* If 2 : 8 :: 6 : 24. Then 2 : 6 :: 8 : 24.
And 6 : 24 :: 2 : 8. Or 24 : 6 :: 8 : 2. *&c.*

*Thefe Variations being well underftood, will be of no fmall Ufe in the ftating of any Queftion in this Rule of Three.*

When three Numbers are given, and it is required to find a fourth Proportional; the greateft Difficulty (if there be any) will be in the right ftating the Queftion, or abftracting the Numbers out of the Words in the Queftion, and placing them down in their proper Order.

Now this will be very eafy, if it be truly confidered, that always two of the three given Terms, are only fuppofed, and affigned or limit the Ratio or Proportion. The third moves the Queftion; and the fourth gives the Anfwer.

As for inftance; if 3 Yards of Cloth coft 9 Shillings: What will 6 Yards coft at the fame Rate or Proportion?

Here 3 Yards, and 9 Shillings, are two fuppofed Numbers that imply the Rate; as appears by the Word [if] *viz.* If 3 Yards coft 9 Shillings (then comes the Queftion) What will 6 Yards coft?

*N. B.*

*N. B.* The Term, which moves the Queſtion, hath generally ſome of thoſe Words before it; *viz.* **What will? How many? How long? How far? How much?** &c.

Then (carefully obſerve this; *viz.*) The firſt Term in the Suppoſition muſt always be of the ſame kind and Denomination with that Term which moves the Queſtion. And the Term ſought will always be of the ſame kind and Denomination with the ſecond Term in the Suppoſition.

Thus, { *yds* *ſhil.* *yds* *ſhil.* 3 : 9 :: 6 : ——— Then

All Queſtions in direct Proportion may be anſwered by three ſeveral *Theorems.*

Theorem 1. { *Multiply the ſecond and third Terms together, and divide their Product by the firſt Term; the Quotient will be the Anſwer required.*

*yds* *ſhil.* *yds* *ſhil.*

Thus 3 : 9 :: 6 : 18. The Anſwer.
6
——
3) 54 (18 Shillings, { becauſe the ſecond Term was Shillings.

Theorem 2. { *Divide the ſecond Term by the firſt, then multiply the Quotient into the third Term; and their Product will be the Anſwer required.*

*yds* *ſhil.* *yds* *ſhil.*

3 : 9 :: 6 : 18.

Thus 3) 9 (=3. Then 3 × 6 = 18, as before.

Theorem 3. { *Divide the third Term by the firſt, then multiply the Quotient into the ſecond Term, and their Product will be the Anſwer.*

*yds* *ſhil.* *yds* *ſhil.*

3 : 9 :: 6 : 18.

Thus 3) 6 (=2. And 9 × 2 = 18, as before.

Here you ſee that all the three *Theorems* are equally true; but the firſt is moſt general, and uſually practiſed. Yet the two laſt may be readily performed, when either the ſecond or third Term can be divided by the firſt; and will be found of ſingular Uſe in the *Rules of Fellowſhip*, &c. as will appear further on.

*Queſt.*

*Quest.* 2. If 8 Pounds of Tobacco cost 14 *Shillings*; what will half a hundred Weight (*viz.* 56 Pounds) cost at the same Rate?

Thus 8 lb : 14 *s.* : : 56 lb : 4 *l.* 18 *s.* The Answer.

$$\begin{array}{r} 14 \\ \hline 224 \\ 56 \\ \hline 8)\ 784\ (= 98\,s. = 4\,l.\ 18\,s. \end{array}$$

Or thus 8) 56 (= 7. Then 14 × 7 = 98 *s.* as before.

*Quest.* 3. If 14 *Shillings* will buy 8 Pounds of *Tobacco*; how much will 4 *l.* 18 *s.* buy after the same Rate?

Stated thus, 14 : 8 lb : : 4 *l.* 18 *s.* = 98 *s.* : ——
Then 98 × 8 = 784. And 14) 784 (56 lb. The Answer.

*Quest.* 4. If half a hundred Weight of *Tobacco* be worth 4 *l.* 18 *s.* How much may I buy for 14 *Shillings* at that Rate?

Stated thus, 4 *l.* 18 *s.* = 98 *s.* : 56 lb : : 14 *s.* : ——
Then 56 × 14 = 784. And 98) 784 (8 lb. The Answer.

*Quest.* 5. Suppose 4 *l.* 18 *s.* will buy 56 Pounds of *Tobacco*; what will 8 Pounds of the same *Tobacco* cost?

This Question is thus stated, 56 lb : 4 *l.* 18 *s.* = 98 *s.* : : 8 lb : —
Then 98 × 8 = 784. And 56) 784 (= 14 *s.* The Answer.

*Note*, The three last Questions are only the second varied, being proposed purely to give an Instance how any Question in this *Rule of Three* may be varied, according to *page* 86.

*Quest.* 6. What will three quarters of a Yard of *Velvet* cost, when the Price of 21 Yards and a half is worth 22 *l.* 10 *s.* 6 *d.* This Question truly stated will stand

Thus, 21 $\frac{1}{2}$ *yds* : 22 *l.* 10 *s.* 6 *d.* : : $\frac{3}{4}$ to the Answer.

Which may be found three several Ways; *viz.* by *Reduction*; by *Vulgar Fractions*; and by *Decimals*.

1. By *Reduction.* Bring the first and third Terms into one Denomination; *viz.* into Quarters, and reduce the second Term into it's least Denomination, *per Sect.* 4. *page* 42.

Thus 21 $\frac{1}{2}$ = 86 Quarters. And 22 *l.* 10 *s.* 6 *d.* = 5406 *Pence.*
Then 86 : 5406 : : 3 : 15 *s.* 8 $\frac{50}{86}$ *d.* For 5406 × 3 = 16218.

And

And 86) 16218 (= 188 $\frac{50}{86}$ *d.* Then 188 $\frac{50}{86}$ *Pence* = 15 *s.* 8 *d.* 2 $\frac{14}{43}$ *Farthings*; the Anfwer required.

2. The fame Queftion ftated in *Vulgar Fractions* will ftand thus; 21 $\frac{1}{2}$ = $\frac{43}{2}$ : 22 $\frac{21}{40}$ = $\frac{901}{40}$ : : $\frac{3}{4}$ : (See *Sect.* 3. *page* 50.) Then $\frac{901}{40}$ × $\frac{3}{4}$ = $\frac{2703}{160}$. And $\frac{43}{2}$) $\frac{2703}{160}$ (= $\frac{5406}{6880}$. *page* 55, 56.

Thefe $\frac{5406}{6880}$ Parts of a Pound are brought into *Shillings* by multiplying the Numerator with 20, and dividing the Product by it's Denominator, *&c.*

Thus 5406 × 20 = 108120. And 6880) 108120 (15 *s.*
And there remains 4920. Again, 4920 × 12 = 59040.
Then 6880) 59040 (8 *d.* and $\frac{50}{86}$, as before.

3. The fame wrought by *Decimal Fractions* will be thus;

21 $\frac{1}{2}$ = 21,5; 22 *l.* 10 *s.* 6 *d.* = 22,525, and $\frac{3}{4}$ = 0,75
Therefore 21,5 : 22,525 : : 0,75 : to the Anfwer.
Then 22,525 × 0,75 = 16,89375
And 21,5) 16,89375 (0,7857 *l.* = 15 *s.* 8 *d.* 2 *far.* $\frac{272}{1000}$.

*Queft.* 7. If 2 *C.* 3 *qrs.* 21 lb. of *Sugar* coft 6 *l.* 1 *s.* 8 *d.* What will 12 *C.* 2 *qrs.* coft at the fame Rate?

That is, 2 *C.* 3 *qrs.* 21 lb : 6 *l.* 1 *s.* 8 *d.* : : 12 *C.* 2 *qrs.* To what?

| | | |
|---|---|---|
| 4 | 20 | 4 |
| 11 *qrs.* | 121 *s.* | 50 *qrs.* |
| 28 | 12 | 28 |
| 88 | 250 | 1400 lb. |
| 22 | 121 | |

*Viz.* 308 + 21 = 329 lb : 1460 *d.* : : 1400 lb : ——

Then 1460 × 1400 = 2044000. And 329) 2044000 (6212 $\frac{3}{4}$ *d.* = 25 *l.* 17 *s.* 8 $\frac{3}{4}$ *d.* the Anfwer required.

The fame Queftion ftated in Decimals will ftand thus;

2,9375 : 6,0833 : : 12,5 : To the Anfwer.

Then 6,0833 × 12,5 = 76,04125 which being divided by 2,9375 will give 25,8863, *&c.* the Anfwer in Decimals, which brought into Coin, will be 25 *l.* 17 *s* 8 $\frac{3}{4}$ *d.* as before.

Note, *When the firft Term is an Unit or* 1, *the Queftion is anfwered by Multiplication only.*

*Example.* Suppofe I give 5 *Shillings* 4 *Pence* for one Ounce of Silver, What muft I pay for 32 $\frac{1}{2}$ Ounces at the fame Rate?

That is 1 Ounce : 5 *s.* 4 *d.* : : 32 $\frac{1}{2}$ Ounces : To, *&c.*
Which is beft ftated thus 1 : 64 *d.* : : 32,5 :

Then $32,5 \times 64 = 2080d. = 8l.\ 13s.\ 4d.$ the Anfwer required. For 1 neither multiplies nor divides.

When the fecond or third Term is an Unit or 1, then the Queftion is anfwered by Divifion only. As in this *Example.*

If a Silver Tankard weighing 21 Ounces, coft 5 *l.* 19 *s.* What is that an Ounce?

Thus 21 *oz.* : 5 *l.* 19 *s.* = 119 *s.* :: 1 : 5 *s.* 8 *d.* the Anfwer.
That is 21) 119 (= 5 *s.* $\frac{14}{21}$ = 5 *s.* 8 *d.*

The Proof of all Queftions in the *Rule of Three Direct*, may be eafily conceived from what hath been already faid; *viz.* That the Product of the firft and fourth Terms, muft always be equal to the Product of the fecond and third Terms.

Or otherwife, by varying the Queftion, as in the fecond, third, fourth, and fifth Queftions.

I fhall conclude this Section with inferting a few Queftions and their Anfwers; leaving their Work for the Learner's Practice.

*Queft.* 1. What will the Carriage of 17 *C.* 3 *qrs.* 11 lb. come to, at the Rate of 7 *s.* the Hundred?

Anfwer 6 *l.* 4 *s.* 11 $\frac{1}{4}$ *d.*

*Queft.* 2. If 6 *l.* 4 *s.* 11 $\frac{1}{4}$ *d.* be paid for the Carriage of 17 *C.* 3 *qrs.* 11 lb; What was paid for the Carriage of 1 lb?

Anfwer 3 Farthings.

*Queft.* 3. A *Grocer* bought 3 *C.* 1 *qr.* 14 lb. Weight of *Cloves*, at the Rate of 2 *s.* 4 *d.* *per Pound*, and fold them for 52 *l.* 14 *s.* Whether did he gain or lofe by the Bargain, and how much?

Anfwer, he gained 8 *l.* 12 *s.*

*Queft.* 4. A *Draper* bought of a Merchant eight Packs of *Coth*; every Pack had four Parcels in it; and each Parcel contained ten Pieces; every Piece was Twenty-fix Yards; he gave after the Rate of four Pounds fixteen Shillings for 6 Yards. What came the eight Packs to, and what were they worth *per Yard?*

*Anfw.* They came to 6656 *l.* And were worth 16 *s.* *per Yard.*

*Queft.* 5. A Merchant bought 436 Yards of Broad *Cloth* for 8 *s.* 6 *d.* *per Yard*; and fold it again for 10 *s.* 4 *d.* *per Yard.* What did he gain by the 436 *Yards?*

*Anfw.* he gained 39 *l.* 19 *s.* 4 *d.*

*Queft.*

*Quest.* 6. A *Goldsmith* bought a Wedge of *Gold*, which weighed 14 lb. 3 *oz.* 8 *pw.* for 514 *l.* 4 *s.* What did he pay *per Ounce?*

Answ. 3 *l.* *per Ounce.*

*Quest.* 7. What will 48 *oz.* 17 *pw.* 20 *Grains of Silver* Plate come to, at the Rate of 5 *s.* 6 *d.* *per Ounce?*

Answ. 13 *l.* 1 *s.* 10 $\frac{3}{4}$ *d.*

*Quest.* 8. If in four Weeks one spend 13 *s.* 4 *d.* How long will 53 *l.* 6 *s.* last at that Rate?

Answ. 6 Years, 47 Days, 2 Hours, 24'.

*Quest.* 9. What will the one eighth Part of a Ship be worth, when the half is valued at 1015 *l.* 10 *s.*

Answ. 253 *l.* 17 *s.* 6 *d.*

*Quest.* 10. The Sun is said to perform one entire Revolution, (or 360 Degrees) in the Space of 365 Days, 5 Hours, 48 Minutes, and 57 Seconds of Time, called a *Tropical* or *Solar Year*; How much doth it move in one Day?

Answ. 59' . 8'' . 19''' &c.

*Quest.* 11. If $\frac{5}{8}$ of a Yard of *Velvet* cost $\frac{2}{3}$ of a *Pound Sterling*, What will $\frac{1}{16}$ of a Yard cost of the same *Velvet* at that Rate?

Answ. $\frac{16}{240}$ = 1 *s.* 4 *d.*

*Quest.* 12. Suppose 2 *l.* and $\frac{3}{8}$ of $\frac{1}{3}$ of a *Pound Sterling* will buy 3 Yards and $\frac{2}{3}$ of $\frac{4}{5}$ of a Yard of *Cloth*, How much will $\frac{3}{4}$ of a Yard cost at that Rate?

Answ. $\frac{2225}{4896}$ of a Pound = 9 *s.* 4 $\frac{1}{2}$ *d.*

## Sect. 2. *Of* Reciprocal Proportion; *usually called* The Rule *of* Three Inverse.

*Reciprocal Proportion* is, when of four Numbers the third (*viz.* that which moves the Question) beareth the same Ratio to the first: As the second does to the fourth.

Therefore, the less the third Term is, in respect to the first; the greater will the fourth Term be, in respect to the second.

*EXAMPLE* 1.

If sixteen Men can do a Piece of Work in six Days; How many Days must eight Men require to do the same Work, at the same Rate of working?

Here it is plain that eight Men must needs have more Time than 16 Men to do the same Work. Consequently the greater the third Term is, in respect to the first, the lesser will the fourth Term be, in respect to the second.

*Example* 2. If 8 Men can do a Piece of Work in 12 Days, How many Days will 16 Men require to do the same Work? Here it is plain the fourth Term must be less than the second, because 16 Men undoubtedly can do the same Work in less Time than 8 Men can.

*From these Considerations, compared with those in* page 85. *it will be easy to perceive, whether the Terms of any proposed Question are in Direct or Reciprocal Proportion.*

*For when, according to the true Meaning and Design of any Question in Proportion, More requires More, or Less requires Less, the Terms are in Direct Proportion; as in this last Section.*

*But if More require Less, or Less require More* (as above) *then the Terms will be in Reciprocal Proportion.*

The Manner of placing down the proposed Terms is the same in both Rules, *viz.* The first Term in the Supposition must be of the same Kind and Denomination with the third Term which moves the Question; and the Term sought must be of the same Kind and Denomination with the second Term in the Supposition. As in the two last *Examples.*

| | | Men | Days | Men | Days |
|---|---|---|---|---|---|
| Thus, is | *Example* 1. | 16 : | 6 : : | 8 : | — |
| | *Example* 2. | 8 : | 12 : : | 16 : | — |

The Question being truly stated, observe this *Theorem.*

Theorem. *Multiply the first and second Terms together, and divide the Product by the third Term, the Quotient will be the Answer required.*

Thus in the second *Example* $12 \times 8 = 96$.

Then 16) 96 (= 6 Days the Answer required.

That is, 16 Men may do the same Work in 6 Days, as 8 Men can do in 12 Days.

Now the Reason of this Operation (and consequently of the *Theorem*) is grounded upon this Consideration; *viz.* If 8 Men require 12 Days to do the Work, it is plain that one Man would require 8 Times 12 Days = 96 Days to do the same Work; but if one Man can do it in 96 Days, most certain 16 Men can do it in one 16th Part of that Time. Therefore 96 divided by 16 will give the Answer required, *viz.* 16) 96 (6 as before, &c.

*Quest.* 3. Suppose 800 Soldiers were besieged in a Town, and their Victuals were computed to serve them two Months (or 56 Days) How many of those Soldiers must depart the Garrison, that the same Victuals may serve the remaining Soldiers 5 Months.

The

The Queſtion truly ſtated will ſtand

| | *Months* | | *Soldiers* | | *Months* | | *Soldiers* |
|---|---|---|---|---|---|---|---|
| Thus, | 2 | : | 800 | : : | 5 | : | —— |

2

5) 1600 (320 : So many Soldiers may ſtay in the Garriſon.

Conſequently, 800 — 320 = 480 Soldiers that muſt go out of the Garriſon, which is the Anſwer required.

*Queſtion* 4. *A* borrowed of his Friend *B* 250 *l.* for ſix Months, promiſing to do him the like Kindneſs upon Demand: Some Time after *B* deſires *A* to lend him 400 *l.* the Queſtion is, how long *B* muſt keep the 400 *l.* to be fully ſatisfied for his former Kindneſs to *A.*

Thus, 250 *l.* : 6 Months : : 400 *l.* : ——

6

400) 1500 (3 Months.

12

3

28 Days in one Month.

4) 84 (21 Days. Anſw. 3 Months, 21 Days.

*Queſtion* 5. If a Penny White Loaf ought to weigh eight Ounces *Troy Weight*, when Wheat is ſold for ſix Shillings Six-Pence the Buſhel; what muſt it weigh when Wheat is ſold for four Shillings the Buſhel?

Thus 6 *s.* 6 *d.* = 78 *d.* : 8 *oz.* : : 4 *s.* = 48 *d.* : to the Anſwer.

8

48) 624 (13 *oz.* the Anſwer required.

48

144

144

(0)

The Proof of this Inverſe Rule is eaſily deduced from it's Operations; *viz.* The Product of the firſt and ſecond Terms, muſt be equal to the Product of the third and fourth Terms.

*Note*, Any Queſtion that falls under this Inverſe Rule or Reciprocal Proportion, may be ſo ſtated as to have it's Terms in Direct Proportion; by only changing the Places of the firſt and third Terms in the Queſtion. Thus,

*Queſtion*

*Question* 6. If a Field will feed eighteen Horses for seven Weeks: How long will it feed Forty-two Horses at the same Rate of feeding?

First, 18 Horses : 7 Weeks : : 42 Horses : 3 Weeks.
Here the Terms are stated inversely, as before.

Otherwise thus, 42 Horses : 7 Weeks : : 18 Horses : 3 Weeks. Then $18 \times 7 = 126$. And $126 \div 42 = 3$ Weeks. The Answer required.

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## Sect. 3. *Of* Compound Proportion; *commonly called* The Double Rule of Three.

*Compound Proportion* (as it is here meant) is, when there are five Numbers given to find out a sixth Proportional; and this is generally performed by a Double Position; that is, by stating and working the Question at two Operations, either in Direct or Reciprocal Proportion, according as the Question requires.

*And therefore it is called, The Double Golden Rule, or Double Rule of Three.*

The Double Rule Direct is, when the sixth Term or Number sought, is found by two Operations, both of them in Direct Proportion.

*Example* 1. If a Hundred Pounds gain six Pounds Interest in twelve Months; how much will three Hundred Pounds gain in nine Months, at the same Rate?

First 100*l.* : 6*l.* : : 300*l.* : 18*l.*
6
100) 1800 (18*l.* { The Interest of 300*l.* for twelve Months.

*Months* *Months*
Then, 12 : 18*l.* : : 9 : 13*l.* 10*s.*
9
12) 162 (13*l.* 10*s.* The Answer required.

I suppose the Learner will easily conceive the Reason of these two Operations. For, first it is plain by Direct Proportion, that if 100*l.* gain 6*l.* in twelve Months, 300*l.* will gain 18*l.* in the same Time, and at the same Rate.

And

And by the ſame Rule it is plain, that if 12 Months will produce or give 18 *l.* Intereſt for 300 *l.* then 9 Months muſt needs give 13 ½ for the ſame Sum, *viz.* 300 *l.*

The Double Rule of Three Inverſe is, when the ſixth Term, or Number ſought, is found at two Operations (as before). But one of them requires an Anſwer in Reciprocal Proportion.

*Queſtion* 2. If 6 Buſhels of Oats will ſerve 4 Horſe 8 Days, How many Days will 21 Buſhels ſerve 16 Horſes, at the ſame Rate of feeding?

This *Queſtion* being parted into two Poſitions, the firſt will be thus:

If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, How many Days will 21 Buſhels ſerve them?

Here it is plain, that 21 Buſhels will ſerve them longer than 6 Buſhels; therefore the firſt Poſition falls in Direct Proportion.

| | Buſh. | | Days | | Buſh. | | Days |
|---|---|---|---|---|---|---|---|
| Thus, | 6 | : | 8 | : : | 21 | : | 28 |
| | | | | | 8 | | |
| | | | | | 6) 168 (28 Days | | |

*That is, if* 6 *Buſhels will ſerve* 4 *Horſes* 8 *Days,* 21 *Buſhels will ſerve them* 28 *Days.*

The next Poſition muſt be to find how long the ſaid 21 Buſhels will ſerve 16 Horſes at the ſame Rate of feeding: it is plain, that 21 Buſhels cannot ſerve 16 Horſes ſo many Days as they will ſerve 4 Horſes; therefore this ſecond Poſition falls in Reciprocal Proportion.

| | Horſes | | Days | | Horſes | | Days | |
|---|---|---|---|---|---|---|---|---|
| Thus, | 4 | : | 28 | : : | 16 | : | 7 | the Anſwer required. |

After the like manner any Queſtion in the Double Rule of Three may be anſwered by two ſingle Poſitions, if Care be taken in ſtating them right, *viz.* Whether their Operation muſt be performed by the ſingle Rule Direct, or Inverſe.

But all Queſtions in this Double Rule, where five Numbers are propoſed to find a ſixth, may more eaſily and readily be anſwered by one general Theorem; which compriſeth both the Direct and Inverſe Rules; without conſidering either of them being deduced from the ſingle Operations before-going.

But firſt you muſt carefully note, that in all Queſtions of this Nature, three of the five propoſed Terms are always conditional and

and ſuppoſed; and that the other two move the Queſtion. As for Inſtance in *Example* 1.

*Viz.* If 100 *l.* will gain 6 *l.* in 12 Months; theſe three Terms are only ſuppoſed or conditional. Then come the Queſtion; What will 300 *l.* gain in 9 Months? Now, in Order to raiſe the general Theorem, let us ſuppoſe, inſtead of Numbers, theſe Letters.

*Viz.* Let $\begin{cases} P = 100. \\ T = 12. \\ G = 6. \end{cases}$ The *Principal.* The *Time.* The *Gain.* } In the Suppoſition of any propoſed Queſtion.

And, $\begin{cases} p = 300. \\ t = 9. \\ g = 13{,}5. \end{cases}$ The *Principal.* The *Time.* The *Gain.* } The three Terms wherein the Queſtion lies.

The $P : G :: p : \frac{Gp}{P} =$ { The Product of the two Means divided by the firſt Extreme.

That is, $100 : 6 :: 300 : \frac{300 \times 6}{100} = 18.$ { Which is the firſt Part of the Queſtion.

Then $T : \frac{Gp}{P} :: t : g$

*Viz.* $12 : 18 :: 9 : 13{,}5$ { Which is the ſecond Part of the Queſtion.

*Ergo* $Tg = \frac{Gpt}{P}$ { That is, the Product of the Extremes is equal to that of the Means.

Conſequently, $TgP = Gpt$ is the *Theorem.*

This *Theorem* affords two Rules, by which all Queſtions in this Double Rule of Three, or rather of five Numbers, may be reſolved; due Regard being had to the true placing down of the propoſed Terms, which muſt be thus:

Always place the three conditional Terms in this Order; let that Number which is the principal Cauſe of Gain, Loſs, or Action, &c. (*viz.* *P.*) be put in the firſt Place; that Number which denotes the Space of Time, or Diſtance of Place, &c. (*viz.* *T.*) be put in the ſecond Place. And that Number which is the Gain, Loſs, or Action, &c. (*viz.* *G.*) be put in the third Place. Now according to theſe Directions, the conditional Terms of the laſt Queſtion will ſtand thus; *P. T. G.*

That done, place the other two Terms which move the Queſtion, underneath thoſe of the ſame Name,

Thus, $\begin{cases} P.\ T.\ G. \\ p.\ t. \end{cases}$

Then

Then if the Blank or Term ſought, fall under the third Place, as in this Queſtion,

It will be $\left\{ \frac{Gpt}{TP} = g. \right.$ Which gives this Rule.

Rule 1. { *Multiply the three laſt Terms together for a Dividend, and the two firſt together for a Diviſor; the Quotient ariſing from them will be the ſixth Term.*

That is, in our propoſed *Example* 1.

Thus 6 × 300 × 9 = 16200 the Dividend.
And 100 × 12 = 1200 the Diviſor.
Then 1200) 16200 (13½ the Anſwer, as before.

But if the Blank or Term ſought fall under the firſt Place, then

It will be $\left\{ \frac{TgP}{tG} = p. \right.$

Or if the Blank fall under the ſecond Place,

It will be $\left\{ \frac{TgP}{Gp} = t. \right.$ Either of theſe give this Rule.

Rule 2. { *Multiply the firſt, ſecond, and laſt Terms together for a Dividend, and the other two together for a Diviſer; the Quotient ariſing from them will be the ſixth Term.*

And becauſe our *Example* 2. falls under the Conſideration both of Direct and Reciprocal Proportion, let it be here propoſed again.

*Viz.* If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days; how many Days will 21 Buſhels ſerve 16 Horſes, *&c.*

If the Terms of this Queſtion be placed down as before directed, they will ſtand

| | *Horſes.* | *Days.* | *Buſhels.* | |
|---|---|---|---|---|
| Thus | 4 | . 8 . | 6 | Terms in the Suppoſition. |
| | 16 | | 21 | |

Here the Blank falls under the ſecond Place, therefore it muſt be found by the ſecond Rule.

Thus 4 × 8 × 21 = 672 the Dividend.
And 16 × 6 = 96 the Diviſor.
Then 96) 672 (7 the Anſwer, as before.

*Queſt.* 3. What Principal or Stock will gain 20 *l.* in 8 Months at 6 *per Cent. per Annum?*

| *Prin.* | | *Time.* | | *Gain.* | |
|---|---|---|---|---|---|
| 100 | . | 12 | . | 6 | Terms in the Suppoſition. |
| | . | 8 | | 20 | |

In this Queſtion the Blank falls under the firſt Place, therefore it muſt be found by the ſecond Rule.

Thus $100 \times 12 \times 20 = 24000$ the Dividend.
And $8 \times 6 = 48$ the Diviſor.
Then 48) 24000 (500 *l.* the Anſw. required.

The Proof of all Queſtions in this Double Rule of five Numbers, is beſt performed by varying the Queſtion; *viz.* by ſtating it in another Order, as in the laſt *Example:* Thus,

If 100 *l.* gain 6 *l.* in 12 Months, what will 500 *l.* gain in 8 Months?

The Anſwer to this Queſtion muſt be 20 *l.* if the Work of the laſt *Example* be true.

| | *Prin.* | | *Time.* | | *Gain.* | |
|---|---|---|---|---|---|---|
| Stated thus { | 100 | . | 12 | . | 6 | } then, *per* Rule 1, |
| | 500 | . | 8 | . | | |

$500 \times 8 \times 6 = 24000$. And $100 \times 12 = 1200$.
Then 1200) 24000 (20 *l.* the Anſwer, &c.

*Queſt.* 4. If two Men can do 12 Rods of Ditching in 6 Days, How many Rods may be done by 8 Men in 24 Days, at the ſame Rate of working?

Anſw. 192 Rods.

*Queſt.* 5. If the Carriage of 5 *C.* 3 *qrs. Weight*, 150 Miles, coſt 3 *l.* 7 *s.* 4 *d.* What muſt be paid for the Carriage of 7 *C.* 2 *qrs.* 25 lb. *Weight*, 64 Miles, at the ſame Rate?

Anſw. 1 *l.* 18 *s.* $7\frac{1}{2}$ *d.*

*Queſt.* 6. If 8 Men deſerve 2 *l.* Wages for 5 Days Work, How much will 32 Men deſerve for 24 Days, at the ſame Rate?

Anſw. 38 *l.* 8 *s.*

*Queſt.* 7. Suppoſe a Hundred Pounds would defray the Expences of five Men for Twenty-two Weeks and ſix Days, How long would twelve Men be in ſpending of one Hundred and Fifty Pounds, at the ſame Rate?

Anſw. 14 Weeks and 2 Days.

CHAP.

# CHAP. VIII.

## *Of Trading in Company, usually called the* Rule of Fellowship; *also* Bartering, *and* Exchanging *of Coins, &c.*

THE *Rule of Fellowship* is that by which the Accompts of several Partners trading in a Company, are so adjusted or made up, that every Partner may have his just Part of the Gain, or sustain his just Part of the Loss; according to the Proportion or Share of Money he hath in the Joint-Stock: Now this falls under two Considerations, called the *Single* and *Double Rules of Fellowship.*

### Sect. 1. *The* Single Rule of Fellowship, viz. *That without* Time.

BY the *Single Rule of Fellowship* is adjusted the Accompts of those Partners that put all their several and perhaps different Sums of Money, into a common Stock at one and the same Time; and therefore it is usually called the *Rule of Fellowship* without Time: Now all Questions of this Nature are answered by so many several Operations in the *Rule of Three Direct*, as there are Partners in the Stock.

*For, as the Total Sum of Money in the Stock is in Proportion to the whole Gain, or Loss: so is every Man's particular Part of that Stock; to his particular Share of that Gain, or Loss.*

*Quest.* 1. Three Partners, suppose *A*, *B*, and *C*, make a Joint-Stock of 96 *l.* in this manner.

*A*, puts in 24 *l.* *B*, puts in 32 *l.* and *C*, puts in 40 *l.* with this 96 *l.* they trade and gain 12 *l.* It is required to find each Man's true Part of that Gain.

The Operation will stand, thus

$$96\,l. : 12\,l. :: \left\{\begin{array}{l} 24\,l. : 3\,l. = A\text{'s} \\ 32\,l. : 4\,l. = B\text{'s} \\ 40\,l. : 5\,l. = C\text{'s} \end{array}\right\} \text{Part of the Gain.}$$

Proof $3\,l. + 4\,l. + 5\,l. = 12\,l.$ the whole Gain.

*That is, if the Sum of each Man's particular Gain, amount to the whole Gain, the Work is true; if not, some Error is committed which must be found out.*

*Note*, These Operations will be very much abbreviated, if you work them by *Theorem* 2. *page* 87. For here 96 is a common Antecedent, and 12 is the common Consequent in all the three Proportions.

Therefore 96 : 12 :: 1 : 0,125 a common Multiplicator.

$$\text{Then} \left\{\begin{matrix}24\\32\\40\end{matrix}\right\} \times 0{,}125 = \left\{\begin{matrix}3\,l.\\4\,l.\\5\,l.\end{matrix}\right\} \text{ for } \left\{\begin{matrix}A,\\B,\\C,\end{matrix}\right\} \text{ as before.}$$

Now this Method is more readily performed than the other, eſpecially when the Partners are many; becauſe one Single Diviſion ſerves for all the Work.

*Queſt.* 2. Three Merchants, *A*, *B*, and *C*, freight a Ship with 248 Tons of Wine: Thus, *A*, loaded 98 Ton, *B*, 86 Ton, and *C*, 64 Ton. By Extremity of Weather the Seamen were forced to caſt or throw 93 Ton of it over-board. How much of this Loſs muſt each Merchant ſuſtain?

Firſt, 248 : 93 :: 1 : 0,375 the common Multiplier.

$$\text{Then} \left\{\begin{matrix}98\\86\\64\end{matrix}\right\} \times 0{,}375 = \left\{\begin{matrix}36{,}75 \text{ for } A\text{'s}\\32{,}25 \text{ for } B\text{'s}\\24{,}00 \text{ for } C\text{'s}\end{matrix}\right\} \text{ Loſs.}$$

Proof 93,00 = the whole Loſs.

Now if the Queſtion were to find how much of the remaining Wine that was ſaved, belongs to *A*, to *B*, and to *C*.

$$\text{Then} \left\{\begin{matrix}98 - 36{,}75 = 61{,}25\\86 - 32{,}25 = 53{,}75\\64 - 24{,}00 = 40{,}00\end{matrix}\right\} \text{ belongs to } \left\{\begin{matrix}A.\\B.\\C.\end{matrix}\right.$$

That is, *A* ought to have 61 Tons and 63 Gallons. *B*, ought to have 53 Tons and 189 Gallons. And *C*, ought to have 40 Tons of what was left.

*Queſt.* 3, Suppoſe ſix Men, *viz.* *A*, *B*, *C*, *D*, *E*, and *F*, make a Joint-Stock of 2558 *l.*

| | | *l.* | *s.* | *Decimals.* |
|---|---|---|---|---|
| Thus *A* | puts in | 654 | 10 | = 654,50 |
| *B* | | 543 | 15 | = 543,75 |
| *C* | | 480 | 00 | = 480,00 |
| *D* | | 254 | 10 | = 254,50 |
| *E* | | 365 | 05 | = 365,25 |
| *F* | | 260 | 00 | = 260,00 |
| The whole Stock | - | 2558 | 00 | = 2558,00 |

according to the *Queſtion.*

With

With this Stock of 2558 *l.* they Trade eighteen Months, and Gain 831 *l.* 7 *s.* It is required to find every Man's Part or Share of that Gain.

Note, *Although the Time of Trading*, viz. *eighteen Months, be mentioned in the Question, yet it is no Way concerned in answering of it; as you may observe in the following Work.*

First, 2558 *l.* : 831,35 *l.* : : 1 *l.* : 0,325 Decimal Parts.
Consequently, 1 *l.* : 0,325 : : 654,5 : 212,7125. That is,

| | | | | |
|---|---|---|---|---|
| 654,50 | × 0,325 = | 212,71250 | for | *A.* |
| 543,75 | | 176,71875 | | *B.* |
| 480,00 | | 156,00000 | | *C.* |
| 254,50 | | 82,71250 | | *D.* |
| 365,25 | | 118,70625 | | *E.* |
| 260,00 | | 84,50000 | | *F.* |

| | | | *l.* Parts. | | *l.* | *s.* | *d.* |
|---|---|---|---|---|---|---|---|
| That is, | *A.* | gains | 212,71250 | = | 212 | 14 | 03 |
| | *B.* | | 176,71875 | = | 176 | 14 | 04½ |
| | *C.* | | 156,00000 | = | 156 | 00 | 00 |
| | *D.* | | 82,71250 | = | 82 | 14 | 03 |
| | *E.* | | 118,70625 | = | 118 | 14 | 01½ |
| | *F.* | | 84,50000 | = | 84 | 10 | 00 |
| Proof. | | Sum | 831,35 | = | 831 | 07 | 00 |

I have omitted resolving this Question according to the usual Method (as before directed) of finding every Man's particular Part of the Gain by the Golden Rule, as in the first Work of *Example* 1. leaving that for the Learner's Practice.

---

## Sect. 2. *The* Double Rule of Fellowship; *or that with* Time.

THIS is usually called the *Double Rule of Fellowship*, because every particular Man's Money is to be considered with Relation to the Time of it's Continuance in the Joint-Stock.

*Question* 1. *A*, and *B*, join in Partnership upon these Terms, *viz.* *A*, agrees to lay down 100 *l.* and to employ it in Trade 3 Months: Then *B*, is to lay down his 100 *l.* and with the whole Stock of 200 *l.* they are to trade 3 Months more. Now at the End of that Time, they find their whole Gain to be 21 *l.* It is required to know what each Man's Part of the Gain ought to be, according to his Stock, and the Time of employing it.

Here

Here it is but reasonable to conclude, that *A*, ought to gain more than B, notwithstanding their Stocks of Money are equal; because *A* employed his Money a longer Time than *B*.

Now for solving of this Question, let us suppose *A*'s 100 *l.* employed the first 3 Months to gain Z = a Sum as yet unknown; then it must gain 2 Z in 6 Months; and to find what *B*, must gain, it will be,

| *l.* | *Months.* | | |
|---|---|---|---|
| 100 . | 6 . . | 2 Z = *A*'s Gain | } *per Rule* 1. *Page* 97. |
| 100 . | 3 . | to *B*'s Gain | |

$$Ergo\ \frac{100 \times 3 \times 2Z}{100 \times 6} = B\text{'s Gain.}$$

But *A*'s Gain added to *B*'s Gain must = 21 *l.* the whole Gain by the Question.

$$\text{Therefore } 2Z + \frac{100 \times 3 \times 2Z}{100 \times 6} = 21\ l.$$

That is, $100 \times 6 \times 2Z + 100 \times 3 \times 2Z = 21 \times 100 \times 6$.
Which contracted is, $900 \times 2Z = 21 \times 600$.

Consequently, $2Z = \frac{21 \times 600}{900}$, which gives the following Analogy.

*Viz.* 900 : 21 :: 600 : 2 Z = 14 *l.* for *A*'s Gain.
And 900 : 21 :: 100 × 3 = 300 : 7 *l.* for *B*'s Gain.

Now this way of arguing hath not only resolved the present Question, but it also affords (and demonstrates) a general Rule for resolving all Questions of this Nature, be the Partners never so many.

Rule. { *Multiply every particular Man's Stock, with the Time it is employed, then it will be, as the Sum of all those Products; is to the whole Gain (or Loss). So is every one of those Products: to it's proportional Part of that whole Gain (or Loss).*

*Question* 2. Three Merchants *A*, *B*, and *C*, enter into Partnership, thus; *A* puts into the Stock 65 *l.* for 8 Months; *B* puts in 78 *l.* for 12 Months; and *C* puts in 84 *l.* for 6 Months. With these they traffick, and gain 166 *l.* 12 *s.* It is required to find each Man's Share of the Gain, proportionable to the Stock and Time of employing it.

1. *A*'s

| | | | | |
|---|---|---|---|---|
| 1. *A*'s | Stock | 65 *l.* × 8 | *Months, the Time it was employed* = | 520 |
| 2. *B*'s | | 78 *l.* × 12 | | 936 |
| 3. *C*'s | | 84 *l.* × 6 | | 504 |
| | | | The Sum of thoſe Products is, | 1960 |

Then, according to the Rule, the ſeveral Proportions will ſtand thus,

| | | | |
|---|---|---|---|
| 1960 : 166,6 :: | 520 : 44,20 = 44 *l.* 4 *s.* 0 *d.* | for | *A.* |
| | 936 : 79,56 = 79 *l.* 11 *s.* $2\frac{1}{2}$ *d.* | | *B.* |
| | 504 : 42,84 = 42 *l.* 16 *s.* $9\frac{1}{2}$ *d.* | | *C.* |
| | The whole Gain = 166 *l.* 12 *s.* 0 *d.* | | |

Or you may work as in ſome of the former *Examples*, viz. by finding the proportional Part of the Gain due to one Pound, &c.

Thus 1960 : 166,6 :: 1 : 0,085 the common Multiplier.

| | | | | | | |
|---|---|---|---|---|---|---|
| Then | 520 | × 0,085 = | 44,2 | for | *A.* | &c. As before. |
| | 936 | | 79,56 | | *B.* | |
| | 504 | | 42,84 | | *C.* | |

*Queſtion* 3. Six Merchants, *viz.* *A*, *B*, *C*, *D*, *E*, and *F*, enter into Partnerſhip, and compoſe a Joint-Stock in this manner;

| | | | *l.* | *s.* | | | |
|---|---|---|---|---|---|---|---|
| *Viz.* | *A* | puts in | 64 | 10 | for | $4\frac{1}{2}$ | Months. |
| | *B* | | 78 | 15 | | 6 | |
| | *C* | | 100 | 00 | | $8\frac{1}{4}$ | |
| | *D* | | 80 | 10 | | 12 | |
| | *E* | | 74 | 12 | | $9\frac{1}{2}$ | |
| | *F* | | 125 | 15 | | 7 | |

They traffick, and gain 258 *l.* 18 *s.* $4\frac{1}{2}$ *d.* It is required to find every Man's Share of the Gain, according to the Stock and Time it was employed.

The ſeveral Stocks of Money, and their reſpective Times being firſt brought into Decimals, and then multiplied together, will produce theſe following Products.

| | | *l.* × *Months.* | | |
|---|---|---|---|---|
| *A*'s | Stock | 64,50 × 4,50 | *The Time it was employed* = | 290,25 |
| *B*'s | | 78,75 × 6,00 | | 472,50 |
| *C*'s | | 100,00 × 8,25 | | 825,00 |
| *D*'s | | 80,50 × 12,00 | | 966,00 |
| *E*'s | | 74,6 × 9,50 | | 708,70 |
| *F*'s | | 125,15 × 7,00 | | 880,25 |
| | | | The Sum of thoſe Products = | 4142,70 |

Then

Then if you work by the common Way; it will be 4142,7 : 258,91875 :: 290,25 : 18,140625 = 18*l.* 2*s.* 9¾*d.* for *A*'s part of the Gain; and so on for the rest.

But if you work by the easiest Way, *viz.* by finding the proportional Part of the Gain due to one Pound.

Thus 4142,7 : 258,91875 :: 1 : 0,0625.

Then

| | | | | | |
|---|---|---|---|---|---|
| | | | *l.* *s.* *d.* | | |
| 290,25 | × 0,0625 = | 18,140625 | = 18 . 02 . 09¾ | for | *A* |
| 472,50 | | 29,531250 | = 29 . 10 . 07½ | | *B* |
| 825,00 | | 51,562500 | = 51 . 11 . 03 | | *C* |
| 966,00 | | 60,375000 | = 60 . 07 . 06 | | *D* |
| 708,70 | | 44,293750 | = 44 . 05 . 10½ | | *E* |
| 880,25 | | 55,015625 | = 55 . 00 . 03¾ | | *F* |

The whole Gain = 258 . 18 . 04½

These few *Examples* being well understood, are sufficient to shew the whole Business of Fellowship, &c.

## Sect. 3. *Of* Bartering.

WHEN Merchants, or Tradesmen, exchange one Commodity for another, it is called *Bartering*; and the only Difficulty in this way of dealing, lies in duly proportioning the Commodities to be exchanged, so as that neither Party may sustain Loss.

*Question* 1. Two Merchants, *A*, and *B*, *Barter*; *A* would exchange 5 *C.* 3 *qrs.* 14 *pound* of *Pepper*, which is worth 3*l.* 10*s.* *per C.* with *B* for *Cotton*, worth 10*d.* *per pound* weight; how much *Cotton* must *B* give to *A* for his *Pepper*?

Note, *In order to the resolving of this Question (and all other Questions of this Nature) you must first find, by the Rule of Three (or otherwise) the true Value of that Commodity whose Quantity is given (which in this Question is Pepper). And then find how much of the other Commodity will amount to that Sum, at the Rate proposed.*

First 5 *C.* 3 *qrs.* 14 lb. = 5,875 } in Decimals.
And 3 *l.* 10 *s.* 0 *d.* = 3,500 }

Then 1 : 3,5 :: 5,875 : 20,5625 = 20*l.* 11*s.* 3*d.* the true Value of the *Pepper*.

Next, it is easy to conceive, that *A* ought to have as much *Cotton* at 10*d.* *per Pound*, as will amount to 20*l.* 11*s.* 3*d.* which may be thus found;

10*d.* : 1 lb. :: 20*l.* 11*s.* 3*d.* = 4235*d.* : 493,5 lb.

That

That is, 4 *C.* 1 *qr*, 17½ *pound* of *Cotton.* And ſo much *B* muſt give to *A* in exchange for his 5 *C.* 3 *qrs.* 14 *pound* of *Pepper.*

*Queſtion* 2. Two Merchants *A* and *B* barter thus; *A* hath 86 Yards of Broad Cloth worth 9 *s.* 2 *d.* *per Yard* ready Money: but in Barter he will have 11 *s.* *per Yard.* *B* hath Shalloon worth 2 *s.* 1 *d.* *per Yard* ready Money; it is required to find how many Yards of the Shalloon *B* muſt give to *A* for his Cloth, to make his Gain in the Barter equal to that of *A*'s.

The Method of reſolving this, and the like Queſtions, differs a little from the laſt Caſe; for in this you muſt firſt find what Advance *B* ought to make *per Yard* upon his Shalloon, in proportion to what *A* hath done upon a Yard of his Cloth.

Thus $\left\{\begin{matrix} s. & d. & & d. & & s. & & d. & & s. & d. & & d. & & s. & d. & & d. \\ 9 & .\ 2 & = & 110 & : & 11 & = & 132 & :: & 2 & .\ 1 & = & 25 & : & 2 & .\ 6 & = & 30 \end{matrix}\right.$

the advanced Price for a Yard of *B*'s Shalloon. Then proceed as before in the laſt *Example.*

Thus 1 Yard : 11 *s.* :: 86 Yards : 946 *s.* = 47 *l.* 6 *s.* the advanced Value of all the Cloth.

Next, If 2 *s.* 6 *d.* will buy one Yard of Shalloon, at it's advanced Price, how many Yards will 47 *l.* 6 *s.* buy.

Thus 2,5 : 1 :: 946 : 378,4 Yards.

That is, *B* muſt give 378⅖ Yards of his Shalloon to *A*, for his 86 Yards of Broad Cloth.

Theſe two Examples are ſufficient to ſhew the Learner, that the Method of bartering, or exchanging Commodities for Commodities, wholly depends upon a clear underſtanding of the Golden Rule; which indeed is ſo called, becauſe of it's Univerſal Uſe.

## Sect. 4. *Of* Exchanging Coins.

EXchanging the Coins of one Country for thoſe of another, is like the Buſineſs of bartering Commodities. That is, it conſiſts in finding what Sum of one Country Coin will be equal in Value to any propoſed Sum of another Country Coin. And, in order to perform that, it will be very neceſſary to have a true Account at all times of the juſt Value of thoſe Foreign Coins which are to be exchanged, as they are compared in Value with our *Engliſh* Coin.

I ſay, at all times, becauſe the Par of Exchange (as the Merchants call it) differs almoſt every Day from *London* to other Countries. That is, it riſes and falls, according as Money is plenty or ſcarce; or according to the Time allowed for Payment of the Money in Exchange, &c.

Thofe that defire to be fully fatisfied in the common Values of Foreign Coins, Weights, Meafures, *&c.* may find them in a Book called the *Merchants Map of Commerce*, which for Brevity fake I have omitted tranfcribing, and only collected thefe few of Coins.

| | *Foreign Coins.* | *English Coin.* *l.* | *s.* | *d.* |
|---|---|---|---|---|
| French Coin. | A *Denier* = | 0 | 0 | 0 $\frac{3}{40}$ |
| | 12 *Deniers* = 1 *Soulz* = | 0 | 0 | 0 $\frac{9}{10}$ |
| | 12 *Soulz* = 1 *Livre* = | 0 | 1 | 6 |
| | 3 *Livres* = 1 *Crown* = | 0 | 4 | 6 |
| Low-Country Coin. | A *Stiver* = | 0 | 0 | 1 $\frac{1}{5}$ |
| | 6 *Stivers* = 1 *Flemifh Shilling* = | 0 | 0 | 7 $\frac{1}{5}$ |
| | 20 *Stivers* = 1 *Gilder* = | 0 | 2 | 0 |
| | 10 *Gilders* = 33 $\frac{1}{3}$ *Shillings* or a *Flemifh Pound* = | 1 | 0 | 0 |
| | A *Embden Dollar* = | 0 | 2 | 3 $\frac{2}{5}$ |
| | A *Campen Dollar* = | 0 | 2 | 7 $\frac{1}{5}$ |
| | A *Zealand Dollar* = | 0 | 3 | 0 |
| | A *Lyons Dollar* = | 0 | 4 | 0 |
| | A *Specie Dollar* = | 0 | 5 | 0 |
| | A *Duccatoon* = | 0 | 6 | 3 $\frac{3}{5}$ |
| Germany. | A *Rixdollar* of the *Empire* = | 0 | 4 | 5 $\frac{3}{4}$ |
| Germany. | A *Gilder* of *Nuremberg* = | 0 | 7 | 1 |
| *In* Italy and Spain. | The *Livre* at *Leghorn* = | 0 | 0 | 9 |
| | *Florence Crown Current* = | 0 | 5 | 3 |
| | *Venice Ducat de Banco* = | 0 | 4 | 4 |
| | The *Current Ducat* = | 0 | 3 | 4 |
| | The *Naples Ducat* = | 0 | 5 | 0 |
| | The *Cadiz Ducat* = | 0 | 5 | 6 $\frac{1}{4}$ |
| | The *Barcelona Ducat* = | 0 | 6 | 0 |
| | The *Valencia Ducat* = | 0 | 5 | 3 |
| | The *Bergonia Ducat* = | 0 | 4 | 4 |
| | The *Portugal Teftoon* = | 0 | 1 | 3 |
| | The Piece of Eight = | 0 | 4 | 6 |

*Note*, The *Englifh* generally reckon their Exchange with other Countries by Pence, *viz.* other Countries value their Crowns, Dollars, or Ducats, *&c.* by *Englifh* Pence. Except with fome Parts of the *Low-Countries*, with whom the Exchange is in Pounds Sterling.

*Queft.* 1. How many Dollars at 4*s.* 6*d.* *per Dollar*, may one have for 162*l.* 18*s.* Anfwer 724 Dollars.

Thus

Thus 162 *l.* 18 *s.* = 3258 *s.* and 4 *s.* 6. = 54 *d.*

Then 54 : 1 : : 3258 : 724 the Anſwer.

*Queſt.* 2. How many *Saragoſſa* Ducats, of 5 *s.* 6 *d.* the Ducat, may be had for 275 *Bergonia* Ducats, at 4 *s.* 4 *d.* the Piece?

Anſwer 216 and 3 *s.* 8 *d.* over.

Thus 5 *s.* 6 *d.* = 66 *d.* and 4 *s.* 4 *d.* = 52 *d.*

Then 275 × 52 = 14300 *d.* = 275 Ducats.

Conſequently 66) 14300 (216 $\frac{2}{3}$ the Anſwer required.

*Queſt.* 3. A Traveller would change 233 *l.* 16 *s.* 8 *d.* Sterling Money; for *Venice* Ducats at 4 *s.* 9 $\frac{1}{2}$ *d. per Ducat*; How many Ducats muſt he have? Anſwer 976 Ducats.

Thus 4 *s.* 9 $\frac{1}{2}$ *d.* = 57,5 *d.* and 233 *l.* 16 *s.* 8 *d.* = 56120 *d.*

Then 57,5 *d.*) 56120 *d.* (976 the Anſwer required.

*Queſt.* 4. A Caſhier hath received 759 Ducats, at 7 *s.* 6 *d. per Ducat*; And 579 Dollars at 4 *s.* 8 *d. per Dollar:* Which he would exchange for *Flemiſh* Marks at 14 *s.* 3 *d. per* Piece: How many ought he to have?

Anſwer 589 Marks, and 15 *d.* over.

For 7 *s.* 6 *d.* = 90 *d.* and 4 *s.* 8 *d.* = 56 *d.*

Then { 759 × 90 = 68310 *d.* the Value of the Ducats.
579 × 56 = 32424 *d.* the Value of the Dollars,

their Sum = 100734 *d.*

And 14 *s.* 3 *d.* = 171 *d.* the *Flemiſh* Mark in Pence.

Conſequently 171) 100734 (589 *&c.* the Anſwer required.

*Queſt.* 5. A Bill of Exchange was accepted at *London* for the Payment of 400 *l.* Sterling, for the like Value delivered in *Amſterdam*, at 1 *l.* 13 *s.* 6 *d.* for 1 *l.* Sterling; How much money was delivered at *Amſterdam*?

Anſwer. 670 *l. Flemiſh.*

For 1 *l.* = 240 *d.* and 1 *l.* 13 *s.* 6 *d.* = 402 *d.*

Then 240 : 402 : : 400 : 670 the Anſwer required.

*Queſt.* 6. When the Exchange from *Antwerp* to *London* is at 1 *l.* 4 *s.* 7 *d. Flemiſh*, for 1 *l.* Sterling; How many Pounds Sterling muſt be paid at *London*; to ballance 236 *l. Flemiſh* at *Antwerp*.

Anſwer 192 *l.* Sterling.

Thus 1 *l.* 4 *s.* 7 *d.* = 295 *d.* and 1 *l.* = 240 *d.*

Then 295 : 240 : : 236 : 192 the Anſwer.

*Queſt.* 7. A Merchant delivered at *London* 120 *l.* Sterling to receive 147 *l.* *Flemiſh* at *Amſterdam*; How much was 1 *l.* Sterling valued at, in *Flemiſh* Money?

Anſwer. 1 *l.* 4 *s.* 6 *d.*

Thus 120 : 147 : : 240 *d.* : 294 *d.* = 1 *l.* 4 *s.* 6 *d.* *&c.*

*Queſt.* 8. A Factor hath ſold Goods at *Cadiz* for 1468 Pieces of Eight, valued at 4 *s.* $6\frac{1}{2}$ *d.* Sterling per Piece; How much Sterling Money do thoſe Pieces of Eight amount to?

Anſwer 333 *l.* 7 *s.* 2 *d.*

Thus, if 1 = 54,5 *d.* then 1468 × 54,5 = 83006 *d.* *&c.*

*Queſt.* 9. A Traveller would have an equal Number of Crowns at 5 *s.* 6 *d.* *per* Crown; and Dollars at 4 *s.* 5 *d.* *per* Piece; How many of each ſort may he have for 309 *l.* 8 *s.*?

Anſwer 624 of each.

Thus 309 *l.* 8 *s.* = 74256 *d.*
And 5 *s.* 6 *d.* + 4 *s.* 5 *d.* = 119 *d.*
Then 119) 74256 (624 the Anſwer required.

*Queſt.* 10. Suppoſe I would exchange 527 *l.* 17 *s.* 6 *d.* for Dollars at 4 *s.* 6 *d.* a Piece, Ducats at 5 *s.* 8 *d.* a Piece, and Crowns at 6 *s.* 1 *d.* a Piece; and would have 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns. How many of each ſort muſt I have?

Anſwer 927 Dollars, $463\frac{1}{2}$ Ducats, and 618 Crowns.

For { 54 *d.* = 1 Dollar. 68 *d.* = 1 Ducat. 73 *d.* = 1 Crown. } *per Queſtion.*

And 126690 *d.* = 527 *l.* 17 *s.* 6 *d.*

Now if the Crowns, Dollars, and Ducats, were to be equal in Number; then 73 + 54 + 68 muſt have been the Diviſor, by which 126690 muſt have been divided, and the Quotient would have been the Anſwer to the Queſtion. As in the laſt *Example.*

But here inſtead of their Sum, ſuch Parts of them muſt be taken as are aſſigned or limited by the Queſtion; that ſo the Number of ſome one of them may be found.

And becauſe there muſt be { 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns,

Therefore it will be $\frac{1}{2}$ of a Ducat for one Dollar, and $\frac{2}{3}$ of a Crown for one Dollar.

Conſequently,

Confequently, $54 + \frac{68}{2} : + \frac{2}{3}$ of $73 = 136\frac{2}{3}$, or $\frac{410}{3}$ will be the Divifor to find the Number of Dollars.

Thus $\frac{410}{3}$) 126690 (927 the Number of Dollars.

Then $\frac{1}{2}$ of $927 = 463\frac{1}{2}$ is the Number of Ducats.

And $\frac{2}{3}$ of $927 = 618$ is the Number of Crowns.

Or if you pleafe you may form Divifors to find either the Ducats or Crowns firft: For if it be 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns, as before;

Then will 6 Dollars be for 3 Ducats, and 6 Dollars for 4 Crowns.

Therefore, $\left\{ \begin{array}{l} \frac{2}{3} \text{ of a Dollar} \\ \frac{3}{4} \text{ of a Ducat} \end{array} \right\}$ will be for 1 Crown.

Confequently, $\frac{2}{3}$ of $54 : + \frac{3}{4}$ of $68 : + 73 = 205$ will be the Divifor to find the Crowns firft, *&c.*

*Queft.* 11. A Cafhier is to receive 500 *l.* He is offered Crowns at 6 *s.* $1\frac{1}{2}$ *d. per* Crown, which are worth but 6 *s.* Or he may have Dollars at 4 *s.* 5 *d.* the Piece, which are worth but 4 *s.* 4 *d.* Which of thefe fhall he receive to have the leaft Lofs? And how much will he lofe in the Payment?

1 $\left\{ \begin{array}{l} 1 \text{ Crown} = 72\,d. \\ 1 \text{ Dollar} = 52\,d. \end{array} \right\}$ according to their true Values.

2 $\left\{ \begin{array}{l} 1 \text{ Crown} = 73{,}5\,d. \\ 1 \text{ Dollar} = 53{,}0\,d. \end{array} \right\}$ the advanced Values.

Now to find which will be the leaft Lofs; find what the advanced Value of a Dollar ought to be in Proportion to that of 1 Crown.

Thus 72 : 73,5 : : 52 : 53,083 *&c.* But he may have Dollars at 53 *d. per* Piece, therefore the Payment in Dollars will be the leaft Lofs; *viz.* 53 is lefs than 53,083 *&c.*

Next, to find what the whole Lofs will be by receiving Dollars. Becaufe the 500 *l.* = 120000 *d.* is advanced as much above the true Value, as 53 *d.* is above 52 *d.* Therefore fay, If 53 *d.* advance 1 *d.* = 53 *d.* — 52 *d.*; what will 120000 *d.* advance? *i. e.*

53 *d.* : 1 *d.* : : 120000 *d.* : $2264\frac{8}{53}$ *d.* = 9 *l.* 8 *s.* $4\frac{8}{53}$ *d.* = the Lofs.

*Queft.* 12. Suppofe I exchange 4 *l.* 10 *s.* 10 *d.* for 11 Crowns and 7 Dollars; and at another Time I have 4 Crowns and 3 Dollars for 1 *l.* 15 *s.* each being of the fame Value with the firft. What is the Value of a Crown, and of a Dollar?

First

First 11 Crowns + 7 Dollars = 1090 *d.* } by the Question.
Second 2 Crowns + 3 Dollars = 420 *d.* }

Then in order to find the Value of 1 Crown, you must cast off the Dollars by making them of the same Number; Thus,

33 Crowns + 21 Dollars = 3270 *d.* the first multipl. with 3.
28 Crowns + 21 Dollars = 2940 *d.* the second multipl. with 7.
Then 5 Crowns = 330 *d.* being the Difference.

Consequently 5) 330 (66 *d.* = 5 *s.* 6 *d.* is the Value of 1 Crown. And 4 Crowns = 264 *d.*

Then will 3 Dollars = 420 *d.* — 264 *d.* = 156 *d.*
Consequently 3) 156 (52 *d.* = 4 *s.* 4 *d.* the Value of 1 Dollar.

## CHAP. IX.

## *Of* Alligation.

WHEN it is required to mix several Sorts of Ingredients together; as different sorts of Corn, Wines, Wool, Spices, or Metals; or to compose Medicines, *&c.* the Method of proportioning such Mixtures, is called the *Rule of Alligation*; and is divided into two Parts or Branches; called *Medial* and *Alternate.*

### Sect. 1. *Of* Alligation Medial.

*ALligation Medial*, is that by which the Mean Rate or Price of any Mixture is found, when the particular Quantities of the Mixtures and Rates are given; and is thus performed.

First find the Sum of all the Quantities proposed to be mixed? And also the Sum of all their particular Rates.

Then the Proportion will be,

Rule { *As the Sum of all the Quantities : Is to the Sum of all their Rates : : So is any Part of the Mixture : To the Mean Rate or Price of that Part.*

*Quest.* 1. Suppose 15 Bushels of Wheat at 5 *s.* the Bushel, and 12 Bushels of Rye at 3 *s.* 6 *d.* the Bushel, were mixed together; What

What is the Mean Rate or Price, it may be ſold for a Buſhel, without Loſs or Gain?

This Queſtion prepared as directed above, will ſtand

| | | |
|---|---|---|
| Thus | 15 Buſhels of Wheat at 5 *s.* *per* Buſhel, comes to | 900 *d.* |
| | 12 Buſhels of Rye at 3 *s.* 6 *d.* each, comes to | 504 *d.* |
| | 27 = their Sum. And their total Value = | 1404 *d.* |

Then 27 Buſhels : 1404 *d.* : : 1 Buſhel : 52 *d.* = 4 *s.* 4 *d.* the Anſwer required.

*Queſt.* 2. A Grocer mixeth 36 Pounds of Tobacco, worth 1 *s.* 6 *d.* a Pound, with 12 Pounds of another ſort at 2 *s.* a Pound, and 12 Pounds of a third ſort at 1 *s.* 10 *d.* the Pound. How may he ſell the Mixture *per* Pound?

| | lb. | | *s.* | *d.* | | *d.* |
|---|---|---|---|---|---|---|
| Firſt | 36 | at | 1 . | 6 | *per* Pound amounts to | 648 |
| | 12 | at | 2 . | 0 | | 288 |
| | 12 | at | 1 . | 10 | | 264 |
| | 60 = the Number of Pounds. | | | | Their Value = | 1200 |

Then 60 lb : 1200 *d.* : : 1 lb : 20 *d.* = 1 *s.* 8 *d.* the Anſwer required.

*Queſt.* 3. A Vintner mixeth 31 Gallons and a half of *Malaga* Sack worth 7 *s.* 6 *d.* the Gallon; with 18 Gallons of Canary at 6 *s.* 9 *d.* the Gallon; 13 Gallons and a half of Sherry at 5 *s.* the Gallon; and 27 Gallons of White Wine at 4 *s.* 3 *d.* the Gallon. It is required to find what one Gallon of this Mixture is worth.

| | *Gal.* | | *s.* | *d.* | | *Pence.* |
|---|---|---|---|---|---|---|
| Firſt | $31\frac{1}{2}$ | at | 7 . | 6 | *per* Gallon comes to | 2835 |
| | 18 | at | 6 . | 9 | | 1458 |
| | $13\frac{1}{2}$ | at | 5 . | 0 | | 810 |
| | 27 | at | 4 . | 3 | | 1377 |
| | 90 = the Number of Gall. | | | | Their Value = | 6480 |

Then 90 : 6480 : : 1 : 72 *d.* = 6 *s.* the Rate or Price of one Gallon, as was required.

The Proof of all Operations in theſe ſort of Mixtures, is done by comparing the Value of all the Mixture (being ſold at the Mean Rate) with the total Value of all the particular Quantities, ſuppoſing they had been ſold at their reſpective Rates unmixed; if thoſe Sums are equal, the Work is true.

Sect.

## Sect. 2. *Of* Allegation Alternate.

ALligation Alternate, is that by which the particular Quantities of every Ingredient concerned in any Mixture are found; when the particular Rates of every one of thoſe Ingredients, and the mean Rate are given; being (as it were) the Converſe to Alligation Medial; as will appear by the following Operations, which admit of three Caſes.

*Caſe* I. The Particular Rates of any Ingredients propoſed to be mixed, and the Mean Rate of the whole Mixture being given. To find how much of each Ingredient is requiſite to compoſe the Mixture; when the whole Quantity, or any Part thereof, is not limited.

*Queſt.* 1. How much Wheat at 5*s.* the Buſhel, and Rye at 3*s.* 6*d.* the Buſhel, will compoſe a Mixture that may be ſold for 4*s.* 4*d.* the Buſhel?

Note, *In all Queſtions of this Nature, it will be convenient to place the Mean Rate ſo, as that it may be eaſily compared with the Particular Rates, in order to find every one of their Differences from the Mean Rate, by Inſpection only.*

Thus, the Mean Rate = 52 *d.* $\begin{cases} \text{Wheat } 60\,d. \\ \text{Rye } 42\,d. \end{cases}$

*Then take the ſeveral Differences between the Mean Rate, and the Particular Rates; ſetting down thoſe Differences alternately, and they will be the Quantities required.*

$$\text{Thus } 52 \begin{cases} 60 \\ 42 \end{cases} \begin{cases} 10 = 52 - 42 \\ 8 = 60 - 52 \end{cases}$$

That is 52 — 42 = 10 for the Quantity of Wheat.
And 60 — 52 = 8 for the Quantity of Rye, that will compoſe the Mixture required.

The Proof by *Alligation Medial.*

Add $\begin{cases} \text{10 Buſhels of Wheat at 60 } d. \textit{ per} \text{ Buſhel} = 600\,d. \\ \text{8 Buſhels of Rye at 42 } d. \textit{ per} \text{ Buſhel} = 336\,d. \end{cases}$

18 = the Number of Buſhels. = 936 *d.*

Then 18 : 936 : : 1 : 52 *d.* = 4*s.* 4*d.* the Mean Rate.

*Note*, Although 10 and 8 do anſwer the Queſtion, as plainly appears by the Proof, yet they are not the only two Numbers; for this Queſtion, and all others of this kind, will admit of various Anſwers, and all whole Numbers; for any two Numbers that are in the ſame Proportion to one another, as 10 is to 8, will as truly anſwer the Queſtion.

*Viz.*

$$Viz.\ 10 : 8 :: \left\{ \begin{array}{c} 5 : 4 \\ 15 : 12 \\ 20 : 16 \\ 25 : 20 \end{array} \right\}$$ *&c. ad infinitum.*

*Quest.* 2. A Grocer would mix three ſorts of Tobacco together, *viz.* One Sort of 18 *d. per* lb. another Sort of 22 *d. per* lb. and a third Sort of 2 *s.* the lb. How much of each Sort muſt he take, that the whole Mixture may be ſold for 20 *d.* the Pound?

Having ſet down the given Rates, as before, then find each of their Differences from the propoſed Mean Rate, and place thoſe Differences alternately. Thus,

$$\text{Mean Rate } 20 \left\{ \begin{array}{c} 18 \\ 22 \\ 24 \end{array} \right\} \left\{ \begin{array}{l} 4 + 2 = 24 - 20 \text{ and } 22 - 20 \\ 2 = 20 - 18 \\ 2 = 20 - 18 \end{array} \right.$$

Theſe Differences, *viz.* 6 . 2 . 2 are the Quantities required.

*Proof.* { 6 lb. of Tobacco at 18 *d.* / 2 lb. ——— at 22 *d.* / 2 lb. ——— at 24 *d.* } the Pound come to { 108 *d.* / 44 *d.* / 48 *d.*

10 = the Number of Pounds. Their Value = 200 *d.*

Then 10) 200 (20 the Mean Rate.

Or indeed any three Numbers that have the ſame Ratio to one another as 6 and 2 have, will anſwer the Queſtion.

$$\text{That is, } 6 : 2 :: \left\{ \begin{array}{c} 9 : 3 \\ 12 : 4 \\ 15 : 5 \end{array} \right\}$$ *&c.*

But if only one of the three given Rates had been greater than the Mean Rate; as ſuppoſe 14 *d. per* Pound, 18 *d. per* Pound, and 24 *d. per* Pound, and the Mean Rate 20 *d.* as before. Then their Differences muſt have been placed,

$$\text{Thus, } 20 \left\{ \begin{array}{c} 14 \\ 18 \\ 24 \end{array} \right\} \left\{ \begin{array}{c} 4 \\ 4 \\ 6 + 2 \end{array} \right\}$$ *&c.* as before.

*Quest.* 3. A Vintner would make a Mixture of Malaga, worth 7 *s.* 6 *d. per* Gallon, with Canary at 6 *s.* 9 *d. per* Gallon, Sherry at 5 *s. per* Gallon, and White Wine at 4 *s.* 3 *d. per* Gallon; What Quantity of each Sort muſt he take, that the Mixture may be ſold for 6 *s. per* Gallon?

In all Queſtions of this Kind, wherein it is required to mix four Things together, two of them having their Prices greater, and two leſſer than the mean Rate: you muſt always alligate or compare a greater and leſſer Price with the mean Price, ſetting down their Differences alternately, as in the firſt *Example* of this *Section.*

| | | | |
|---|---|---|---|
| Thus, Mean Rate = 72 *d.* | Malaga 90 *d.* | 21 = 72 — 51 | |
| | White 51 *d.* | 18 = 90 — 72 | |
| | Sherry 60 *d.* | 9 = 81 — 72 | |
| | Canary 81 *d.* | 12 = 72 — 60 | |

Hence 21 Gallons of Malaga, 12 of Canary, 9 of Sherry, and 18 of White will compoſe the Mixture required.

| | | | |
|---|---|---|---|
| Or thus, 72 | Malaga 90 *d.* | 12 Malaga | will, &c. |
| | Sherry 60 *d.* | 18 Sherry | |
| | Canary 81 *d.* | 21 Canary | |
| | White 51 *d.* | 9 White | |

Either of theſe Mixtures equally anſwer the Queſtion, which may be eaſily tried as before in the laſt, &c.

*Caſe* II. The particular Rates of all the Ingredients propoſed to be mixed, the Mean Rate of the whole Mixture, and any one of the Quantities to be mixed being given: Thence to find how much of every one of the other Ingredients is requiſite to compoſe the Mixture.

*Note*, This is uſually called *Alligation Partial.*

*Queſt.* 4. How much Wheat at 5 *s.* the Buſhel, muſt be mixed with 12 Buſhels of Rye at 3 *s.* 6 *d.* a Buſhel; that the whole Mixture may be ſold for 4 *s.* 4 *d.* the Buſhel?

In this Caſe you muſt ſet down all the particular Rates, with the Mean Rate, and find their Differences juſt as before; without any regard had to the Quantity given.

| | | |
|---|---|---|
| Thus, Mean Rate 52 *d.* | Wheat 60 *d.* | 10 |
| | Rye 42 *d.* | 8 |

Then { *As the Quantity found by the Differences of the ſame Name with the Quantity given : Is to the Quantity given :: So is any of the other Quantities found by the Differences : To the Quantity of it's Name.*

Thus 8 : 12 :: 10 : 15, the Quantity or Number of Buſhels of Wheat required.

*Queſt.* 5. How much Malaga at 7 *s.* 6 *d.* the Gallon, Sherry at 5 *s.* the Gallon, and White Wine at 4 *s.* 3 *d.* the Gallon, muſt be mixed with 18 Gallons of Canary at 6 *s.* 9 *d.* the Gallon; that the whole Mixture may be ſold for 6 *s.* the Gallon?

The

The Terms being set down, &c. as before, will stand

Thus, Mean Rate 72 *d.* { Malaga 90 *d.* | White 51 *d.* } { 21 | 18 }; { Sherry 60 *d.* | Canary 81 *d.* } { 9 | 12 }

Then, as 12 : 18 :: { 21 : $31\frac{1}{2}$ Gallons of Malaga. | 18 : 27 Gallons of White. | 9 : $13\frac{1}{2}$ Gallons of Sherry. }

That is, $31\frac{1}{2}$ Gallons of Malaga, 27 of White Wine, and $13\frac{1}{2}$ of Sherry, being mixed with 18 Gallons of Canary, will make the Mixture required.

Or thus, 72 { Malaga 90 | Sherry 60 } { 12 | 18 }; { Canary 81 | White 51 } { 21 | 9 }

Then, as 21 : 18 :: { 12 : $10\frac{6}{21}$ the Malaga. | 18 : $15\frac{9}{21}$ the Sherry. | 9 : $7\frac{15}{21}$ the White. } &c.

| | *Gallons.* | | *Pence.* |
|---|---|---|---|
| *Proof.* | $10\frac{6}{21}$ at 90 *d.* | each | $925\frac{15}{21}$ |
| | $15\frac{9}{21}$ at 90 *d.* | | $925\frac{1}{21}$ |
| | $7\frac{15}{21}$ at 51 *d.* | | $393\frac{9}{21}$ |
| | 18 at 81 *d.* | | 1458 |
| Sum | $51\frac{9}{21}$ | Value = | $3702\frac{18}{21}$ |

Then $51\frac{9}{21}$) $3702\frac{18}{21}$ (72 *d.* = 6 *s.* the Mean Rate.

Therefore the Quantities are as truly assigned here, as in the last Work.

*Case* III. The particular Rates of all the Ingredients proposed to be mixed; and the Sum of all their Quantities with the Mean Rate of that Sum being given; to find the particular Quantities of the Mixture.

This is called *Alligation Total*, and is thus performed.

Set down all the particular Rates, with the Mean Rate, and find their Differences, as before: add together all the Differences into one Sum;

Then { *As the Sum of all the Differences : Is to the Sum of all the Quantities given :: So is every particular Difference : To it's particular Quantity.* }

*Quest.* 6. Let it be required to mix Wheat at 5 *s.* the Bushel, with Rye at 3 *s.* 6 *d.* the Bushel; so that the whole Quantity may be 27 Bushels, to be sold for 4 *s.* 4 *d.* a Bushel; what Quantity of each must be taken to make up the Mixture?

| | | | |
|---|---|---|---|
| Mean Rate 52 | Wheat 60 *d.* | | 10 |
| | Rye 42 *d.* | | 8 |
| | | | 18 = their Sum. |

Then 18 : 27 :: { 10 : 15; 8 : 12 } the Quantities required.

*Question* 7. Suppose it were required to mix Malaga at 7 *s.* 6 *d.* the Gallon, with Canary at 6 *s.* 9 *d.* the Gallon; Sherry at 5 *s.* the Gallon, and White Wine at 4 *s.* 3 *d.* the Gallon; so that the whole Mixture may be 90 Gallons; to be sold for 6 *s.* the Gallon: How much of each sort will compose that Mixture?

| | | |
|---|---|---|
| Mean Rate = 72 *d.* | Malaga 90 | 21 |
| | White 51 | 18 |
| | Canary 81 | 9 |
| | Sherry 60 | 12 |
| | | 60 = their Sum. |

| | | | |
|---|---|---|---|
| Then 60 : 90 :: | 21 : 31 ½ | the Gallons of | Malaga. |
| | 18 : 27 | | White Wine. |
| | 9 : 13 ½ | | Sherry. |
| | 12 : 18 | | Canary. |

| | | |
|---|---|---|
| Or thus, 72 | Malaga 90 | 12 |
| | Sherry 60 | 18 |
| | Canary 81 | 21 |
| | White 51 | 9 |
| | | 60 = their Sum. |

| | | | |
|---|---|---|---|
| Then 60 : 90 :: | 12 : 18 | Gallons of | Malaga. |
| | 28 : 27 | | Sherry. |
| | 21 : 31 ½ | | Canary. |
| | 9 : 13 ½ | | White Wine. |

Either of these Ways do equally answer the Question, as may be easily tried by *Alligation Medial.* As before, &c.

Note, *The Work of these Proportions may be much shortened (especially when there are many Ingredients to be mixed) if you observe the same Method as was proposed in the* Rule of Fellowship, *page* 99, &c.

I have made Use of the very same *Examples* both in *Alligation Medial*, and *Alternate*, throughout the three *Cases*; being, as I presume, much better than if they had been different ones; because the Learner may (if he considers them a little) easily perceive, not only the Difference between the two Rules, but also wherein the

the chief Difference of each Case in the *Alternate Rule* depends, *&c.* Not but that I could have inserted many various *Examples*, as also the Manner of composing Medicines, *&c.* which, for Brevity sake, I have omitted, and refer those that desire to see into that Business to Sir *Jonas More*'s *Arithmetick*, wherein he will find it largely handled. And so I shall conclude with *Alligation Alternate*, which altho' it gives true Answers to Questions of that Kind, with some little Variety, according as the Ingredients are more or less in Number; as appears by the foregoing *Examples*; yet it will not give all the Answers such Questions are capable of, nor perhaps those which suit best with the present Occasion: Nor can this Imperfection be remedied by common *Arithmetick*; but by an *Algebraick* way of arguing it may; whereby all the possible Answers to any Question may be clearly and easily discovered; as shall be shewed further on in the Second Part.

## CHAP. X.

## *Of* Metals *and their* Specifick Gravities, *&c.*

### Sect. 1. *Of* Gold *and* Silver.

PURE Gold, free from Mixture with other Metals, usually called Fine Gold, is of such a Nature and Purity that it will endure the Fire without wasting, although it be kept continually melted: And therefore some of the ancient Philosophers have supposed the Sun to be a Globe of liquid or melted Gold.

Silver having not the Purity of Gold, will not endure the Fire like it: Yet Fine Silver will waste but a very little by being in the Fire any reasonable time; whereas Copper, Tin, Lead, *&c.* will not only waste, but may be calcined or burnt to a Powder.

Both Gold and Silver in their Purity, are so very flexible or soft (like new Lead, *&c.*) that they are not so useful either in Coin, or otherwise (except to beat in Leaf-Gold or Silver) as when they are allay'd, or mixed and hardened with Copper or Brass. And altho' most Places differ more or less in the Quantity of such Allay, yet in *England* it is generally agreed on, that,

Standard

## Standard for Gold.

22 Carats of Fine Gold, and 2 Carats of Copper, being melted together, ſhall be eſteemed the true Standard for Gold Coin, &c. (*The French and Spaniſh Gold being very near of the ſame Standard.*) That is, if any Quantity or Weight of Fine Gold, be divided into Twenty-four equal Parts, and 22 of thoſe Parts be mixed with 2 of the like Parts of Copper; that Mixture is called Standard Gold.

Whence you may obſerve, that a Carat is not any certain Quantity or Weight, but $\frac{1}{24}$ part of any Quantity or Weight; and the *Minters* and *Goldſmiths* divide it into 4 equal Parts, which they call Grains of a Carat; alſo they ſubdivide one of thoſe Grains, into Halves, Quarters, &c.

## Standard for Silver.

Eleven Ounces and Two Penny-weight of Fine Silver, and Eighteen Penny-weight of Copper being melted together, is eſteemed the true Standard for Silver Coin, called Sterling Silver. And ſo in Proportion for a greater or leſſer Quantity; which is a leſs Proportion of Allay for Silver, than the other is for Gold.

*Note*, When either Silver or Gold is finer than Standard, it is called Better; if coarſer, it is called Worſe; and that Betterneſs or Worſeneſs, is reckond by Carats and Grains of a Carat in Gold, and by Penny-weights in Silver; and is thus diſcovered: The *Goldſmiths* or *Refiners*, &c. take a ſmall Quantity of ſuch Gold as they intend to try (which they call making an *Aſſay*) and weigh it very exactly, then they put it into a Crucible, and melt it in a ſtroug Fire, ſo long, that if there be any Copper, or other Allay mixt with it, that Allay may be conſumed or burnt away: When it is cold they weigh it very exactly again, and if it have loſt nothing of it's firſt Weight, they conclude it is Fine Gold, but if the Loſs be $\frac{1}{24}$ Part, they call it 23 Carats Fine, or one Carat better than Standard: If it have loſt $\frac{2}{24}$ Parts it is 22 Carats fine, or Standard: If $\frac{3}{24}$ Parts, it is ſaid to be 21 Carats fine, or rather one Carat worſe than Standard, and ſo in Proportion as it happens to be better or worſe.

In the ſame Manner they make their Aſſay on Silver, only they compute it's Loſs by Penny-weights, &c.

The Author of the *Preſent State of England*, mentioned before (*page* 32.) ſays,

' That

'That the *Engliſh* Coin may want neither the Purity nor 'Weight required, it is moſt wiſely and carefully provided, that 'once every Year the chief Officers of the *Mint* appear before the 'Lords of the Council in the *Star-Chamber* at *Weſtminſter*, with 'ſome Pieces of all ſorts of Monies coined the foregoing Year, 'taken at adventure out of the *Mint*, and kept under ſeveral 'Locks, by ſeveral Perſons, 'till that Appearance, and then by 'a Jury of 24 able *Goldſmiths*, in the Preſence of the ſaid Lords, 'every Piece is moſt exactly weighed and aſſay'd.'

This if it were conſtantly practiſed would keep our Coin to it's true Standard, *&c.*

Many pretty Queſtions may be ſtarted concerning the Fineneſs of Gold and Silver, *&c.*

### *EXAMPLE* 1.

If an Ingot of Silver weighing 787 *Oz.* 14 *Pwt.* 6 *Grains*, be 11 *Oz.* 6 *Pwt.* fine; How much fine Silver is there in it, and what amounts it to, at 5 *s.* $1\frac{1}{2}$ *d.* the Ounce?

This Ingot is better than Standard by 4 *Pwt.* For 11 *Oz.* 2 *Pwt.* = 222 *Pwt.* the fine Silver in 12 *Oz.* of Standard. But 11 *Oz.* 6 *Pwt.* = 226 *Pwt.* the fine Silver in 12 *Oz.* according to the Queſtion.

Firſt 787 *Oz.* 14 *Pwt.* 6 *Grains* = 378102 *Grains.*
And 12 *Ounces* = 240 *Pwt.*
Then, As 240 : 226 : : 378102 : $356046\frac{1}{20}$ = 741 *Oz.* 15 *Pwt.* $6\frac{1}{20}$ *Grains* the fine Silver in that Ingot.

Which at 5 *s.* $1\frac{1}{2}$ *d.* the Ounce, amounts to 190 *l.* 1 *s.* 6 *d.* and near a Half-penny.

### *EXAMPLE* 2.

If an Ingot of Gold weighing 115 *Oz.* 13 *Pwt.* 18 *Grains*; be $\frac{1}{4}$ of a Grain worſe than Standard: How much Standard Gold is there in it, and what comes it to at 3 *l.* 11 *s.* an Ounce?

Firſt 115 *Oz.* 13 *Pwt.* 18 *Grains* = 55530 Grains *Troy.*
Then 24) 55530 (2313,75 = a Carat of that Quantity.
And 4) 2313,75 (578,4375 = a Grain of that Carat.
Conſequently 4) 578,4375 (144,609375 = $\frac{1}{4}$ of a Grain.

Again, 2313,75 × 22 = 50902,5 ought to be the fine Gold in that Ingot, if it had been Standard:

But

But 50902,5 − 144,609375 = 50757,890625 is the Quantity of find Gold according to the Queſtion. Therefore 509025 : 50757,890625 : : 55530 : 55372,244 *&c.* Grains = 115 *Oz.* 7 *Pwt.* 4,244 *&c.* Grains *Troy*, being the Quantity of Standard Gold in that Ingot, as was required.

Next for the Value of it at 3 *l.* 11 *s. per Ounce*; 1 *Oz.* = 480 Grains; and 3 *l.* 11 *s.* = 71 *s.* Conſequently 480 : 71 : : 55372,244 *&c.* : 8190,4777 *&c.* = 409 *l.* 10 *s.* 5 $\frac{1}{4}$ *d.* very near; being the Value of that Ingot, as was required.

Or the laſt Queſtion may be otherwiſe wrought thus; 115 *Oz.* 13 *Pwt.* 18 *Grains* = 115,6875. And $\frac{1}{4}$ of a Grain of a Carat is $\frac{1}{16}$ (*viz.* the $\frac{1}{4}$ of $\frac{1}{4}$) Then 22 — $\frac{1}{16}$ = 21 $\frac{15}{16}$ = 21,9375. Conſequently 22 : 21,9375 : : 115,6875 : 115,358842 *&c.* = 115 *Oz.* 7 *Pwt.* 4,244 *Grains*, &c. as before.

Next for the Value; as 1 : 3,55 : : 115,358842 : 409,523,889 = 409 *l.* 10 *s.* 5 $\frac{1}{4}$ *d.* very near: as before.

## Sect. 2. *The* Specifick Gravity *of* Metals, &c.

I Take an Enquiry made about the different Gravities, or Weights of Metals, and other Bodies, to be (not only a Work of Curioſity, but alſo) of very good Uſe upon many Occaſions. Therefore ſeveral Authors have given us ſuch Proportions, or Difference of their Weights, as they are ſaid to have one to another; ſuppoſing every one of them to be of the ſame Magnitude or Bigneſs. Some of which I ſhall here inſert.

1. *Henry Van Etten*, in his *Mathematical Recreations*, printed *Anno* 1633, ſets down the Proportion of their Weights thus; Gold 1875. Lead 1165. Silver 1040. Copper 910. Iron 810. Tin 750. Water 100.

2. One *Aiſted*, in his *Encyclopædia*, printed *Anno* 1649, hath them thus: Gold 1875. Quickſilver 1500. Lead 1165. Silver 1040. Copper 910. Iron 806. Tin 750. Honey 150. Water 100. Oil 90. Theſe ſeem to be taken from thoſe of *Van Etten*'s, with ſome Additions only.

3. The ingenious Mr *Oughtred*, in his *Circles of Proportions*, printed *Anno* 1660, hath their Proportions (according to the Experiments of one *Marinus Ghetaldi*, in his Tract called *Archimedes Promotus*) thus: Gold 3990. Quickſilver 2850. Lead 2415. Silver 2170. Braſs 1890. Iron 1680. Tin 1554.

4. In

4. In the Philosophical Transactions, (*Number* 169 and 199) there is an Account of a great many Experiments of this Kind; from whence I collected these following, *viz.* Gold 18888. Mercury 14019. Lead 11343. Silver 11087. Copper 8843. Hammered Brass 8349. Cast Brass 8100. Steel 7852. Iron 7643. Tin 7321. Pump-water 1000.

These last Proportions being approved of and published by Order of the *Royal Society* seem to be unquestionably true: Nevertheless, because they differ so much from the beforementioned (*and those from one another*) I have for my own Satisfaction made several Experiments of that Kind: And have (*I presume*) obtained the Proportions of Weight that one Body bears to another of the same Bulk or Magnitude, as nicely as the Nature of such Matter, which may be contracted or brought into a lesser Body (*viz.* either by Drying, or Hammering, or otherwise) will admit of; which are as follow:

| | | Ounces Troy. | | Ounces Avoird. |
|---|---|---|---|---|
| A Cubick Inch of | *Fine Gold*, is - - | 10,359273 | = | 11,365602 |
| | *Standard Gold* - - | 9,962625 | = | 10,930422 |
| | *Quickfilver* - - - | 7,384411 | = | 8,101753 |
| | *Lead* - - - - - | 5,984010 | = | 6,553885 |
| | *Fine Silver* - - - | 5,850035 | = | 6,418324 |
| | *Standard Silver* - - | 5,556769 | = | 6,096569 |
| | *Rose Copper* - - - | 4,747121 | = | 5,208369 |
| | *Plate Brass* - - - | 4,404273 | = | 4,832116 |
| | *Cast Brass* - - - | 4,272409 | = | 4,630300 |
| | *Steel* - - - - - | 4,142127 | = | 4,544505 |
| | *Common Iron* - - | 4,031361 | = | 4,422979 |
| | *Block Tin* - - - | 3,861519 | = | 4,236638 |
| | *Fine Marble* - - | 1,429411 | = | 1,568859 |
| | *Common Glass* - - | 1,360841 | = | 1,493037 |
| | *Alabaster* - - - | 0,988456 | = | 1,084477 |
| | *Dry Ivory* - - - | 0,962083 | = | 1,055542 |
| | *Dry Box-wood* - - | 0,543282 | = | 0,596057 |
| | *Sea Water* - - - | 0,542742 | = | 0,594894 |
| | *Common clear Water* | 0,527458 | = | 0,578697 |
| | *Red Wine* - - - - | 0,523766 | = | 0,574646 |
| | *Proof Spirits of Brandy* | 0,489268 | = | 0,536796 |
| | *Sound Dry Oak* - - | 0,489008 | = | 0,536569 |
| | *Linseed Oil* - - | 0,491591 | = | 0,539345 |
| | *Oil Olive* - - - | 0,481569 | = | 0,528350 |

R In

In this Table you have the Specifick Gravity or Weight of a Cubic Inch, of various sorts of Bodies, both in *Troy* Ounces and *Avoirdupois* Ounces, and Decimal Parts of an Ounce, which I can assure you required more Charge, Care, and Trouble, to find out nicely, than I was at first aware of.

Now from hence it will be easy to determine the Weight of any proposed Quantity, of the same Matter and Kind with those in the Table; it's Solid Content being given in Cubic Inches. For it is plain, that if the Number of Cubic Inches contained in any given Quantity, be multiplied with the tabular Weight of one Inch, (*of the same Kind of Matter*) the Product will be the Weight of that Quantity in Ounces, *&c.*

## *EXAMPLE.*

Suppose it were required to find the Weight of a Piece of Marble, containing three Solid Feet, and 40 Cubic Inches.

First $1728 \times 3 = 5184$ the Cubic Inches in 3 Solid Feet.

And $5184 + 40 = 5224$ the Number of Cubic Inches in the Piece of Marble.

Then $5224 \times 1,429411 = 7410,066624$ Ounces *Troy*.
Or $5224 \times 1,568859 = 8195,719416$ Ounces *Avoirdupois*.

The Weight of that Piece of Marble, in Ounces, *&c.* which is easily brought into Pounds, *&c.* The like for any of the rest.

The Converse of this Work is as easy; *viz.* if the Weight of any proposed Quantity be given, thence to find the Solid Content of that Quantity in Cubic Inches, *&c.*

Thus, divide the given Weight of the proposed Quantity (*it being first reduced into Ounces*, &c.), by the tabular Weight of one Inch (*of the same Kind of Matter*), and the Quotient will be the Number of Cubic Inches contained in that Quantity.

*Note*, If you would find what Weight any Quantity of those Bodies mentioned in the Table will have, when it is immersed or put into Water, you must substract the Weight of an equal Quantity of Water (with that of the Body), from the Weight of the proposed Body (if it be heavier than Water), and there will remain the Weight required. As for Instance,

A Cubic Inch of Lead $= 5,984010$ } Ounces *Troy*, &c.
A Cubic Inch of Sea Water $= 0,542742$ }

. their Difference is $= 5,441268$ the Weight of a Cubic Inch of Lead in the Water, *&c.*

CHAP.

# CHAP. XI.

## Evolution, *or Extracting the* Roots *out of all* Single Powers; *by one Geometrical Method.*

### SECT. I.

*EVolution* is the Unravelling, or as it were the Unfolding and Refolving any propofed Power or Number, into the fame Parts of which it was compofed, or fuppofed to be made up. Now in order to perform that, it will be convenient to confider how thofe Powers are compofed, *&c.*

A Square Number is that which is equally equal; or which is contained under two equal Numbers. *Euclid.* 7. *Def.* 18. Thus the Square Number 4 is compofed of the two equal Numbers 2 and 2. *viz.* $2 \times 2 = 4$. Or the Square Number 9 is compofed of the two equal Numbers 3 and 3. *viz.* $3 \times 3 = 9$: according to *Euclid.* That is, if any Number be multiplied into itfelf; that Product is called a Square Number.

A Cube is that Number which is equally equally equal, or which is contained under three equal Numbers. *Eucl.* 7. *Def.* 19. Thus the Cube Number 8 is compofed of the three equal Numbers 2 and 2 and 2. *viz.* $2 \times 2 \times 2 = 8$, *&c.* That is, if any Number be multiplied into itfelf, and that Product be multiplied with the fame Number; the fecond Product is called a Cube Number.

Thefe two, *viz.* the Square and Cube Numbers, borrow their Names from *Geometrical* Extenfions or Figures; as from the three Signal Quantities mentioned in *page* 2. That is, a *Root* is reprefented by a Line or Side, having but one Dimenfion, *viz.* that of Length only. The Square is a Plane or Figure of two Dimenfions, having equal Length and Breadth. The Cube is a Solid Body of three Dimenfions; having equal Length, Breadth, and Thicknefs: But beyond thefe three, Nature proceeds not, as to *Local* Extenfion. That is, the Nature of Place or Space, admits no Room for other ways of Extenfion, than Length, Breadth, and Thicknefs. Neither is it poffible to form, or compofe any Figure or Body beyond that of a Solid.

And therefore all the fuperior Powers above the Cube or third Power; as the *Biquadrat* or fourth Power, the *Surfolid* or fifth Power, *&c.* are beft explained and underftood by a Rank or Series of Numbers in *Geometrical Proportion.* For Inftance: Suppofe any Rank of *Geometrical Proportionals,* whofe firft Term and Ratio are the fame; and to them let there be affigned a Series of Numbers in *Arithmetical Progression*, beginning with an Unit or 1, whose common Difference is also 1, as in *page* 79.

Thus, $\begin{cases} 1 . 2 . 3 . 4 . 5 . 6 . 7 \text{ Indices.} \\ 2 . 4 . 8 . 16 . 32 . 64 . 128 \text{ \&c. in} \div\div \end{cases}$

Then are those Numbers in ∺ produced by a continued Multiplication of the first Term or Root into itself; and those in *Arithmetical Progression* or **Indices**, do shew what Degree or Power each Term in the *Geometrical Proportion* is of. For Example; In this Series of ∺ 2 is both the first Term or Root, and common Ratio of the Series. Then $2 \times 2 = 4$ the second Term or Square; and $2 \times 2 \times 2 = 8$, or $4 \times 2 = 8$, the Cube or third Term; $2 \times 2 \times 2 \times 2 = 16$, or $8 \times 2 = 16$ the fourth Term or Biquadrat. And so on for the rest.

Note, *This is called* **Involution**, viz. *When any Number is drawn into itself, and afterwards into that Product*, &c. *it is said to be so often involved into itself; and the Indices are the Exponents of their respective Powers so involved.*

And according to these Involutions, is formed the following Table of Powers; wherein the Root is only one single Figure.

| *Root, or single Side.* | *Square, or second Power.* Index (2) | *Cube, or the third Power.* Index (3) | *Biquadrat, or Square squared; being the fourth Power.* Index (4) | *Sursolid, or the fifth Power.* Index (5) | *Square cubed, or Cube squared; the sixth Power.* Index (6) | *The second Sursolid, or seventh Power.* Index (7) | *The Biquadrat squared, or the eighth Power.* Index (8) | *The Cube cubed, or the ninth Power, &c.* Index (9) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| 3 | 9 | 27 | 81 | 243 | 725 | 2187 | 6561 | 19683 |
| 4 | 16 | 64 | 256 | 1024 | 4096 | 16384 | 65536 | 262144 |
| 5 | 25 | 125 | 625 | 3125 | 15625 | 78125 | 390625 | 1953125 |
| 6 | 36 | 216 | 1296 | 7776 | 46656 | 279936 | 1679616 | 10077696 |
| 7 | 49 | 343 | 2401 | 16807 | 117649 | 823543 | 5764801 | 40353607 |
| 8 | 64 | 512 | 4096 | 32768 | 262144 | 2097152 | 16777216 | 134217728 |
| 9 | 81 | 729 | 6561 | 9049 | 531441 | 4782969 | 4304[illegible] | 387420489 |

This Table plainly shews (by Inspection) any Power (under the Tenth) of all the nine Figures; and from thence may be taken the nearest Root of any Square, Cube, Biquadrat, &c. of any Number whose Root or Side is a single Figure.

But

But if the Root conſiſts of two, three, or more places of Figures, then it muſt be found by piece-meal, or Figure after Figure, at ſeveral Operations.

The Extraction of all Roots, above the Square (*viz.* of the Cube, Biquadrat, Surſolid, *&c.*) hath heretofore been a very tedious and troubleſome Piece of Work: All which is now very much ſhortened, and rendered eaſy, as will appear further on.

When any Number is propoſed to have it's Root extracted, the firſt Work is to prepare it, by Points ſet over (or under) their proper Figures; according as the given Power, whoſe Root is ſought doth require; and that is done by conſidering the Index of the given Power, which for the Square is 2, for the Cube 3, for the Biquadrat is 4, *&c.* (as in the precedent Table) Then allow ſo many Places of Figures in the given Power, for each ſingle Figure of the Root, as it's Index denotes; always beginning thoſe Points over the Place of Unity, and aſcend towards the Left-Hand if the given Number be Integers, and deſcend towards the Right-Hand in Decimal Parts. As in theſe following.

Suppoſe any given Number; as 75640387246 which I ſhall all along hereafter call the Reſolvend.

Then if it be required to extract any of the following Roots, it muſt be pointed (according to the forementioned Conſideration) in this manner:

| | | |
|---|---|---|
| *Viz.* For the | *Square Root* Thus | 75640387246 (pointed every 2nd figure) |
| | *Cube Root* | 75640387246 (pointed every 3rd figure) |
| | *Biquadrat Root* | 75640387246 (pointed every 4th figure) |
| | *Surſolid Root* | 75640387246 (pointed every 5th figure) |

Or ſuppoſe the Number to be 0,674035982

| | | |
|---|---|---|
| Then for the | *Square Root* Thus | 0,6740359820 |
| | *Cube Root* | 0,674035982 |
| | *Biquadrat Root* | 0,674035982000 |

Now the Reaſon of pointing the given Reſolvend in this manner; *viz.* the allowing two Figures in the Square; three Figures in the Cube, and four Figures in the Biquadrat, *&c.* for one Figure in the Root, may be made evident ſeveral ways; but I think it is eaſily conceived from the Table of ſingle Powers, wherein you may obſerve that all the Powers of the Figure 9

(which

(which is but a ſingle Figure) have the ſame Number of Places of Figures, as the Index of thoſe Powers denotes: Therefore ſo many Places of Figures muſt be taken or aſſigned for every ſingle Figure in the Root. Conſequently by theſe Points is known how many Places of Figures there will be in the Root, *viz.* So many Points as there are, ſo many Figures there muſt be in the Root, and whether they muſt be Integers or Decimal Parts, is eaſily determined by the reſpective Places of the Points.

## Sect. 2. *To* Extract *the* Square Root.

AND firſt how to extract the Square Root, according to the common Method.

Having pointed the given Reſolvend into Periods of two Figures as before directed; then by the Table of Powers (or otherwiſe) find the greateſt Square that is contained in the firſt Period towards the Left-Hand (ſetting down it's Root, like a Quotient Figure in *Diviſion*) and ſubſtract that Square out of the ſaid Period of the Reſolvend: To the Remainder bring down the next Period of Figures, for a Dividend, and double the Root of the firſt Square for a Diviſor; enquiring how oft it may be had in that Dividend, ſo as when the Quotient Figure is annexed to the Diviſor, and that increaſed Diviſor multiplied with the ſame Quotient Figure, the Product may be the greateſt Number that can be taken out of that Dividend; which ſubſtract from the ſaid Dividend, and to the Remainder bring down the next Period of Figures, for another new Dividend: Then ſee how often the laſt increaſed Diviſor, can be had in the new Dividend (*with the ſame Caution as before*, viz.), ſo as that the Quotient Figure being annexed to the Diviſor, and that increaſed Diviſor multiplied with the ſame Quotient Figure, their Product may be the greateſt Number that can be ſubſtracted from the new Dividend. (As before) And ſo proceed on from Period to Period (*viz.* from Point to Point) in the very ſame Manner, until all be finiſhed.

An Example or two being well obſerved will render the Work of forming the new Diviſors, &c. more plain and eaſy than can be expreſſed in a Multitude of Words.

*Example* 1. Let it be required to extract the Square Root out of 572199960721. This Reſolvend being prepared or pointed as before directed, will ſtand

Thus,

```
                  . . . . . .
          Thus, 572199960721 (756439 the Root.
                49 = the greatest Square in 57.
1. Divisor  145)     821
              5      725 = 145 × 5
2. Divisor  1506)     9699
               6      9036 = 1506 × 6
3. Divisor  15124)     66396
                4      60496 = 15124 × 4
4. Divisor  151283)     590007
                 3      453849 = 151283 × 3
5. Divisor  1512869)     13615821
                  9      13615821 = 1512869 × 9
```

Proof 756439 × 756439 = 572199960721 the Resolvend.

*Example* 2. What is the Square Root of 1850701,764025?

```
                . . . .  . . .
Operation 1850701,764025 (1360,405
          1
23)        85
 3         69
266)       1607                  { Hence 1360,405 is the
  6        1596                  {   Root required.
17204)      1101,76
    4       1088 16
1720805)      13 604025
      5       13 604025
                 (0)
```

*Ex.* 3. What is the Square Root of 0,06076225 Decimal Parts?

```
                  . . . . .
Operation 0,06076225 (0,2465 the Root required.
           ,04 = ,2 × ,2
,44)       207
  4        176                    { 0,2465 × 0,2465 =
,486)       3162           Proof  {   0,06076225 the
   6        2916                  {   Resolvend.
,4925)       24625
    5        24625
              (0)
```

What

What is here done in whole Numbers, mixed Numbers, and Decimals, may alfo be done in Vulgar Fractions; if you firft change the given Fraction into Decimals. (As in *Sect.* 5. *p.* 68.)

*Example* 4. Let it be required to extract the Square Root of $\frac{16}{25}$. Firft $\frac{16}{25} = 0,64$

Then $\dot{0},\dot{6}4$ (,8 *the Root required.*
,64
(0)

In thefe four Examples the Refolvend hath been a perfect Square; and therefore the Root hath been extracted without leaving any Remainder: But it very often happens that the Refolvend is not a true Figurate Number, according to the propofed Power. That is, it is not a perfect Square, Cube, Biquadrat, &c. and then fomething will remain after the Extraction hath been made throughout all the Points. Such Numbers are called **Surd** Numbers, and their Roots can never be truly found, but will become a continued Series, *ad infinitum:* If to the Remainder there be ftill annexed Cyphers according as the propofed Power requires, *viz.* by two's in the Square; three's in the Cube, four's in the Biquadrat, &c. And the Operations continued on as before.

*Example* 5. Suppofe it were required to extract the Square Root of 6968.

| | |
|---|---|
| Operation | 6968 (83,4745, &c. |
| | 64 |
| 163) | 568 |
| 3 | 489 |
| 1664) | 79,00 |
| 4 | 66 56 |
| 16687) | 12 4400 |
| 7 | 11 6809 |
| 166944) | 759100 |
| 4 | 667776 |
| 1669485) | 9132400 |
| 5 | 8347425 |
| 1669490) | 784975 &c. |

Then the Root of any Surd Number may be continued on to what Exactnefs you pleafe, but cannot be truly found.

In my *Compendium of Algebra*, Chap. 9. I have propofed another Way of extracting the Square Root, and there given Examples of the Work; which to avoid Prolixity is thus;

Having

Having pointed the given Refolvend, and taken the greateft Square to the firft Point from it, as before. Then divide the Remainder of the whole Refolvend by 2 (that is, half it) and point it a-new. (This I call a new Dividend) Then make the Root of the firft Square a Divifor, inquiring how oft it may be found in the new Dividend to the next figure forward, referving that Figure under the next Point for the half Square of the Quotient Figure. Which being found, multiply the Divifor with it, adding to that Product the Tens of the half Square if there be any, as in plain Divifion. Then annex the Quotient Figure to the laft Divifor for a new Divifor, with which proceed in all Refpects as with the laft Divifor; and fo on until all be finifhed.

*Example* 6. What is the Square Root of 2990667969

```
                  .  .  .  .  .
Operation    2990667969
           — 25           (5 The firft fingle Root
             2) 490667969 The Remainder to be divided by 2.
                   .  .  .  .
Firft Root 5)    2453339⁸4,5 (54687
           + 4   208 = 5 × 4 : + ½ the Square of 4, viz. 16/2 = 8.
Divifor   54)    3733
          +  6   3258 = 54 × 6 : + ½ the Square of 6.
Divifor  546)    47539
         +   8   43712 = 546 × 8 : + ½ the Square of 8.
Divifor 5468)    382784,5
        +    7   382784,5 = 5468 × 7 : + ½ the Square of 7.
                  (0)
```

Hence the Root is found to be 54687, as was required.

All the Difficulty in this Method is only the true placing of the half Square of the Quotient Figure, when it happens to be an odd Number: In that Cafe you muft bring down one Figure more of the Dividend; *viz.* of the next Period; under which, place the odd 5 that will always arife from the half Square of an odd Number: As 7 whofe Square is 49; the Half of which is 24,5 to be placed as in the laft Operation of this Example.

N.B. *When the Number of Figures in the Root of any Surd Number is limited; you need not proceed in extracting the whole Root as before; but only to one Figure more than half the defigned Number of Figures; for the reft may be obtained by plain Divifion only.*

*Example* 7. Suppose it were required to extract the Square Root of 7 (a Surd Number) to have 12 Places of Figures in it.

| | | |
|---|---|---|
| | 7 (2,645751 | First part of the Root. |
| | 4 | |
| Remainder | 3 | |
| 2) | 1,50 = Half the Remainder. | |
| + ,6 | 1,38 = 2 × ,6 : + ½ the Square of 0,6 = 0,18 | |
| 2,6) | 1200 | |
| + ,04 | 1048 | |
| 2,64) | 152000 | |
| + ,005 | 132125 | |
| 2,645) | 1987500 | |
| + ,0007 | 1851745 | |
| 2,6457) | 13575500 | |
| + ,00005 | 13228625 | |
| 2,64575) | 34687500 | |
| + ,000001 | 26457505 | |
| 2,645751 | 8229995 | |

Having thus got 7 of the 12 Figures required in the Root; the rest may be easily found by the contract Way of Division proposed in *page* 68.

| | | |
|---|---|---|
| Thus 2,645751) | 8229995 | (2,64575131106 |
| .... | 7937253 | |
| | 292742 | |
| | 264575 | |
| | 28167 | |
| | 26457 | |
| | 1710 | |
| | 1097 | |
| | (13) | |

Hence I find the Root of 7 to be 2,64575131106, *as was required.*

Thus you have two ways of extracting the Square Root, either of which may be practised as every one likes best.

Sect.

## Sect. 3. *To extract the* Cube Root.

THE Method I ſhall here propoſe for extracting the Cube Root admits of two Caſes; both which are to be very well obſerved.

Having pointed the given Reſolvend, (as before directed) *viz.* into Periods of three Figures; then ſeek a Cube Number by the Table of Powers (or otherwiſe) that comes neareſt to the firſt Period of the Reſolvend, whether it be greater or leſs than that Period.

*Caſe* 1. If the Cube Number ſo taken, be leſs than the firſt Period of the Reſolvend, call it's Root Leſs than Juſt: And ſubſtract that Cube from the firſt Period of the Reſolvend.

*Caſe* 2. But if that Cube be greater than the firſt Period of the Reſolvend, call it's Root More than Juſt: And ſubſtract the Reſolvend from that Cube, annexing Cyphers to it, that ſo Subſtraction may be made.

To the firſt Root, whether it be leſs or more than juſt, annex ſo many Cyphers as there are remaining Points over the whole Numbers of the Reſolvend, and multiply it with 3: Then making that Product a Diviſor, by which you muſt divide the Difference between the Reſolvend and the foreſaid Cube; that Quotient will be the Reſolvend depreſſed to a Square, and therefore muſt be pointed as ſuch, *viz.* into Periods of two Figures each. That being done, make the firſt Root (without thoſe Cyphers that were annexed to it) a Diviſor, enquiring how oft it may be found in the firſt Period of the new Reſolvend (as before in extracting the Square Root) with this Conſideration, that if the Root (now a Diviſor) be leſs than juſt, as in *Caſe* 1. you muſt annex the Quotient Figure to it, and then multiply the Root ſo increaſed, into the ſaid Quotient Figure; ſetting down the Unit's Place of their Product under the pointed Figure of that Period, ſubſtracting it, as in Diviſion. And ſo on from one Period to another, as before.

But if the ſaid Root (now a Diviſor) be more than juſt, as in *Caſe* 2. Then you muſt ſubſtract the Quotient Figure from a Cypher annexed, or ſuppoſed to be annexed, to the ſaid Diviſor; multiplying the Root ſo decreaſed into the Quotient Figure; ſetting down their Product as before, *&c.* An *Example* or two in each *Caſe* will render the Work plain and eaſy.

Note, *Each Quotient Figure ought always to be twice added to the Diviſor, if the Tabular Cube was taken leſs than juſt, or twice ſubſtracted from it, if greater*; viz. *once before you multiply by it, and once with the next Quotient Figure: as will be ſhewn in the following Examples; which are therefore more exact and conciſe than as done by the Author in all the former Editions of his Work.*

S 2 *Example*

*Ex.* 1. What is the Cube Root of 146363183 the given Refelvend, to be pointed thus 146363183 (the firſt Root, leſs than juſt.

125 = the neareſt Cube to 146

500 × 3 = 1500) 21363183 (14242,12 new Reſolvend

First Root 5)

+ 2

1 Diviſor 52) 14242,12 (527 the Root required.

+ 27 104

2 Diviſor 547) 3842

3829

13 the Remainder to be rejected.

Here the Root 527 is the true Root at the firſt Operation, as may be eaſily tried by involving it.

That is 527 × 527 × 527 = 146363183 the given Reſolvend. But if it had not been the true Root, then every thing that hath been here done muſt have been repeated; only inſtead of the firſt ſingle Root (*viz.* 5) you muſt have taken the increaſed Root (*viz.* 527) and this call a ſecond Operation; which would increaſe the laſt Root to nine Places of Figures; *viz.* every Operation triples the Number of Places in the laſt Root; as will appear further on.

N. B. *It often happens that four, five, and ſometimes more Places of Figures may be taken into the Root: eſpecially when the ſecond Place proves to be a Cypher. That is, when the firſt Cube comes very near to the firſt Period of the Reſolvend.*

*EXAMPLE* 2.

What is the Cube Root of 67507824239 (4000 Root leſs than juſt.

Firſt neareſt Cube = 64

Root 4000 × 3 = 12000) 3507824239 (292318,68

Firſt Root 4)

+ 07

1 Diviſor 407) 292318,68 (4071,79

+ 071 2849

2 Diviſor 4141) 418

+ 1,7 4141

3 Diviſor 4142,7) 3277,68

+ ,79 2899,81

4 Diviſor 4143,49) 377,79 &c.

In this *Example* I have taken ſix Figures into the Roots, becauſe the ſecond Place proved to be a Cypher. And in theſe ſix the

the Excess is not an Unit in the last Place; for if there were made a second Operation, the Root would be 4071,78 &c. as may be easily tried.

### EXAMPLE 3.

Let it be required to extract the Cube Root out of this Number; *Viz.* 976379602989073960279630298890

The nearest Cube to 976 is 1000 whose Root is 10 more than just, it's Cube 1000000000000000000000000000000
— 976379602989073960279630298890 the Resolvend,

Remains 23620397010926039720369701110

The first Root 10000000000 × 3 = 30000000000 the Divisor.

Then 30000000000) 23620397010926039720369701110 (787346567030867990 for a new Resolvend.

1st Root 10
— 007
1 Div. 993) 787346567030867990 . . . . . . . . . . ( 10000000000 = 1st Root. 0079364 &c. substract.
— 79   6951
2 Div. 9851) 92246
— 93   88659
3 Div. 98417) 358756
— 36   295251
4 Div. 984134) 6350570
— 64   5904804
5 Div. 9841276 &c.   44576630 &c.

Remains 9920636000 the Root true to the sixth Figure, and only too little by an Unit at the seventh, at the first Operation.

For a second Operation (if you require no more than ten Places of Figures true in the Root) you need only assume 9920000000; which being less than just, proceed with as follows:

From the given Resolvend, = 976379602989073960279630298890
Sub. the Cube of 9920000000 = 976191488000000000000000000000

Remainder 188114989073 96 &c.

Then 3×992 &c. = 2976 &c.) 18811498907396 &c. (6321068181 &c. for a new Resolvend.

| | |
|---|---|
| 99200) | |
| + 06 | . . . . . |
| 99206) | 6321068181 |
| + 63 | 595236 |
| 992123) | 3687081 |
| + 37 | 2976369 |
| 9921267) | 71071281 |
| + 7 &c. | 69448869 |
| 992127*) | 1622412 |
| .... | 992127 |
| | 630285 |
| | 595276 |
| | 35009 |
| | 29763 |
| | 5246 |
| | 4960 |
| | &c. |

(9920000000 the Root aſſumed,
637163,5 add

9920637163,5 the Root true to the tenth Figure, and only too much by an Unit in the eleventh.

* Here the Additions of the Quotient Figure being of no Conſequence, therefore the Diviſion is carried on from hence, as in *page* 68.

In the ſame manner the Cube Roots of Decimal Parts; or of Vulgar Fractions, being firſt changed into Decimals; may be extracted.

## Sect. 4. *To extract the* Biquadrat Root.

IN extracting the Biquadrat Root, or that of the Fourth Power; (and indeed the Roots of all even Powers) there are ſome ſmall Difficulties, not ſo eaſily expreſſed and explained in a few Words, as they are by an *Algebraick Theorem*; (ſuch as ſhall be ſhewed further on) I have therefore in this Place made choice of extracting ſuch Roots by two ſeveral Extractions, and the rather, becauſe I preſume the Reader by this Time thoroughly acquainted with the Buſineſs of extracting the Sqoare Root, by which this may eaſily be performed. Thus:

Firſt, Extract the Square Root of the propoſed Reſolvend, then the Square Root of that firſt Root will be the Biquadrat Root required.

*Example* 1. What is the Biquadrat Root of 4857532416? Firſt extract it's Square Root,

Thus

```
Thus  4857532416
    — 36 = the greateſt Square, whoſe Root is 6.
      1257532416 Remainder to be divided by 2.

First Root 6)        628766208 (69696
         + 9         5805
          69)         4826
         +  6         4158
          696)         668620
         +   9         626805
          6969           418158
                         418158
                          (0)

Then  69696 { being the firſt Root, whoſe Square Root
  —       4 {      muſt now be extracted.
      29696  Remainder to be divided by 2.

Firſt Root 2)   14848 (264 the Biquadrat Root as was required.
       + 6      138
        26)     1048
       +  4     1048
        264     (0)
```

This is ſo eaſy I need not inſert any more Examples.

## Sect. 5. *To extract the* Surſolid Root.

HAVING pointed the given Reſolvend according as it's Index denotes; *viz.* into Periods of five Figures; ſeeking ſuch a Surſolid Number in the Table of Powers (or otherwiſe) as comes the neareſt to the firſt Period of the Reſolvend, whether greater or leſs, and call it's reſpective Root accordingly, *viz.* more than juſt, or leſs than juſt; annexing ſo many Cyphers to it, as there are remaining Periods of whole Numbers in the Reſolvend; as before in extracting the Cube Root: Then find the Difference between the Reſolvend, and the Surſolid Number ſo taken, by ſubſtracting the leſſer from the greater (as before in the Cube). Next find the Cube of the aforeſaid Surſolid Root with it's annexed Cyphers, (which you may alſo do by the Table of Powers) and multiply that Cube with 5 the Index of the Surſolid, the Product muſt be a Diviſor, by which the Difference between the Reſolvend and the Surſolid Number muſt be divided; that ſo it may be depreſſed to to a Square (as before in the Cube) which muſt be pointed into Periods of two Figures each, calling it the new Reſolvend (as before). Then make the firſt Root, without it's Cyphers, a Diviſor, enquiring how oft it may be found in the firſt Period of the new Reſolvend, with this Conſideration, if the Root (now a Diviſor) be leſs than juſt, you muſt annex twice the Quotient Figure to it; but if it be more than juſt, you muſt ſubſtract twice the Quotient Figure from a Cypher either annexed, or ſuppoſed to be annexed to that Diviſor or Root, multiplying it ſo increaſed or diminiſhed, with the ſaid Quotient Figure, ſetting down their Product, &c. as before. An *Example* in each *Caſe* will render it plain and eaſy.

*Example* 1. Suppoſe it be required to extract the Surſolid Root out of this Number 12309502009375.

12309502009375 the Reſolvend pointed.

The neareſt Surſolid Number to 1230, the firſt Period of the Reſolvend, is 1024, whoſe Root is 4 being leſs than juſt.

Therefore 12309502009375<br>
— 1024<br>
2069502009375 their Difference.

Next the Cube of 400 is 64000000 per Table, &c. And 64000000 × 5 = 320000000 the Diviſor.

Then 320000000) 2069502009375 (6497 &c.

Firſt Root 400<br>
+ 2 × 10 = + 20<br>
1 Diviſor 420) 6467 (400 + 15 = 415 Root true<br>
+ 20 + 2 × 5 = + 30 42<br>
450 2267<br>
2250<br>
17 the Remainder to be rejected.

That is 415 is the Surſolid Root of the given Reſolvend. As may be eaſily tried by involving it to the fifth Power. *Viz.* 415 × 415 × 415 × 415 = 12309502009375 the given Reſolvend.

Note, *Here again the double Quotient Figure ought to be twice added or ſubſtracted, in the ſame Manner as the ſingle one was directed for the Cube Root*, page 131, *and the Operation for the Surſolid Root in theſe two Examples is performed accordingly: contrary to what was heretofore done by the Author.*

*Example*

*Example* 2. What is the Surfolid Root of 2327834559873

The neareft Surfolid Number to 232 is 243 whofe Root is 3 being more than juft.

| | |
|---|---|
| Therefore | 2430000000000 |
| — | 2327834559873 |
| Remains | 102165440127 For a Dividend. |

The Cube of 300 is 27000000 and 27000000 × 5 = 135000000
Then 135000000) 102165440127 (756,7810 new Refolvend.

| | | | |
|---|---|---|---|
| Firft Root | 300 | | |
| — 2 × 2 = | — 4 | . . . . | (300 |
| 1 Divifor | 296) | 756,7810 | (—2,566 |
| —4—2 × 0,5 = | — 5,0 | 592 | 297,434 The Root only too little by 2 in the loweft Figure. |
| 2 Divifor | 291,0) | 164,78 | |
| —1—2 × 0,06 = | — 1,12 | 145,50 | |
| 3 Divifor | 289,88) | 19,2810 | |
| | &c. | &c. | |

Now the Reafon why this Root comes out to fo many Places of Figures at the firft Operation; is becaufe the firft Surfolid Number was fo near the Refolvend, *&c.* As before.

## Sect. 6. *To extract the* Root *of the* Square cubed.

THIS may be eafily performed by two Extractions, according as it's Name denotes. Thus, firft extract the Square Root of the given Refolvend; then extract the Cube Root of that Square Root, and it will be the Root required: That is, it will be the Root of the fixth Power. Or thus, firft extract the Cube Root of the Refolvend; then extract the Cube Root of that Cube Root, and it will be the Root required.

### *EXAMPLE* 1.

Let it be required to extract the Square cubed Root out of this Number 1452205373535155625 the Refolvend.

Firft I extract the Square Root of this Refolvend, which I take to be the beft and eafieft Way.

```
Thus       14522053735351 5625
         — 9
Remains    5522053735351 5625   to be halfed.

Then  3)      27610268676757812,5 (381078125
     + 8      272
      38)      4102
     + 10      3805
      3810)     2976867
     +    7     2667245
      38107)     3096226
     +     8     3048592
      381078)     47634757
     +      1     38107805
      3810781)     95269528
     +       2     76215622
      38107812)     1905390612,5
     +        5     1905390612,5
      381078125        (0)
```

Having found the Square Root of the given Resolvend, I proceed to extract the Cube Root of that Square Root.

```
        That is, of  381078125
                   — 343 = the nearest Cube, it's Root is 700
Then 700 x 3 = 2100) 38078125 (18161

        First Root   7 . .
                   + 2
        1 Divisor    72 .)        18161  ( 700
                   + 25           144    (+25
        2 Divisor    745)          3761    725
                                   3725
                                   (36)
```

Hence I find 725 to be the Square cubed Root required; as may easily be tried by involving it to the sixth Power. That is, 725 x 725 x 725 x 725 x 725 x 725 will be found = 14522053-73535 15725 the given Resolvend.

## Sect. 7. *To* extract *the* Root *of the* ſeventh Power.

HAving pointed the given Reſolvend, as it's Index denotes, *viz.* into Periods of ſeven Figures, ſeek out ſuch a Number of the ſeventh Power, by the Table of Powers, as comes neareſt to the firſt Period of the Reſolvend; whether it be greater or leſſer, calling it's reſpective Root more than juſt, or leſs than juſt, annexing it's proper Number of Cyphers, *&c.* as in the Cube and Surſolid.

Then find the Difference between the given Reſolvend, and that Number of the ſeventh Power (found by the Table of Powers) by ſubſtracting the leſſer from the greater.

Next find the Surſolid or fifth Power of that Root with it's annexed Cyphers (which you may alſo do by the Table of Powers) and multiply that Surſolid Number with 7, the Index of the given Reſolvend; that Product muſt be a Diviſor, by which the foreſaid Difference muſt be divided, that ſo it may be depreſſed to a Square, to be pointed, *&c.* as before in the Cube, *&c.* then make the firſt Root, without it's Cyphers, a Diviſor; [illegible] new Reſolvend (as before) only here you muſt increaſe, or diminiſh the Diviſor with thrice the Quotient Figure *.

*Example.* What is the ſecond Surſolid Root, or that of the ſeventh Power,

of 3829865539550781 25 the Reſolvend pointed.
— 2187 the neareſt of the ſeventh Power.
1642865539550781 25 their Difference.

The firſt Root is 300 being leſs than juſt, and the fifth Power of 300 is 2430000000000, which being multiplied with 7 is 17010000000000 for a Diviſor, by which the aforeſaid Difference muſt be divided; which contracted may ſtand thus,

1701) 16428655 (9658,23 *&c.*

| | | | |
|---|---|---|---|
| Firſt Root | 300 | | |
| + 3 × 20 = + | 60 | | 300 |
| 1 Diviſor | 360) | 9658 | + 25 |
| 60 + 3 × 05 = + | 75 | 72 | 325 = the true Root required. |
| 2 Diviſor | 435) | 2458 | |
| | | 2175 | |
| | | 283 the Remainder to be rejected, as before. | |

* That is, by twice adding or ſubſtracting the triple Quotient Figure, as was done with the double Quotient Figure for the Root of the fifth Power, *page* 136; and the ſingle Quotient Figure for the Cube Root *page* 131.

Hence I have found 325 to be the true Root required, that is, the true Root of the ſeventh Power.

I think it needleſs to proceed farther; *viz.* to inſert *Examples* of higher Powers. For if what is already done be well underſtood, it will be eaſy to conceive how to proceed in extracting the Root of any ſingle Power how high ſoever it be (for the Method is general and alike in all Powers), due Regard being had to their Indices; and to the firſt ſingle Side or Root. That is, whether it be more, or leſs than juſt, &c.

Yet methinks I hear the young Learner ſay, it is poſſible to follow the Directions and Examples, as they are here laid down; but ſtill here is not the Reaſon why they are ſo, and ſo, performed; and why there ſhould be a Remainder left after the Root is found; *viz.* when the given Reſolvend hath a true Root of it's Kind.

It is true, the Reaſons of theſe are not here laid down; neither indeed can they be rendered ſo plain and intelligible by Words, as by an Algebraick Proceſs, from whence the *Theorems* or *Rules* here given, had their firſt Invention; as ſhall be ſhewed in the next Part, when I come to treat of reſolving compounded or adfected Equations; however, take this ſhort and general Account of this Method.

This, and all other of the new Methods of Converging Series (as they are called), are very different from the former (and ſtill common) Methods of extracting Roots, which require the firſt ſingle Side or Root of the firſt Period (in any Reſolvend) to be taken exactly true, and then by involving, and other tedious Ways of ordering it, there is formed a Diviſor; which helps to grope out by Trials a ſecond Figure in the Root. And ſo proceed on from Point to Point; ſtill repeating the whole Work for every ſingle Figure that comes into the Root. And if by Chance there be a Miſtake or Error committed in any one Figure (as it is poſſible there may) it ſpoils the whole Proceſs, which muſt then be wholly begun anew, or at leaſt from that Part of it where the Error firſt entered.

But the Nature and Deſign of the Method which I have here laid down is quite otherwiſe; it being ſo contrived, as to gradually leſſen the Difference betwixt any propoſed Power, and the like Power of another Number aſſumed; *viz.* it leſſens that Difference until it is either quite vanquiſhed, or becomes ſo infinitely ſmall as to be inſignificant.

Therefore when any Number is propoſed to have it's Root extracted; it is here required to take the next neareſt Root of the firſt Period in the Reſolvend; that ſo the Difference betwixt the

given

given Resolvend, and the Homogeneal Power (*viz.* the like Power) of the Root thus taken, may be less either in Excess, or Defect. Which Difference being reduced, or depressed lower, becomes so prepared, that by plain Division (comparatively) there will arise such Quotient Figures, as will both correct and increase the first Root to three Places of Figures at least, sometimes to four or five Places of Figures; according as the said first Difference happens to be more or less (of which you may have observed Instances): But yet there will be a Remainder left, and perhaps an Excess or Defect in the Root so increased, *viz.* in the last Figure of it.

Now to rectify the said Excess or Defect in the Root, and to discover whether the given Resolvend be a true Figurate Number, or not: That is, whether it have a true Root of it's kind; it will be necessary to make a second Operation; by taking the Root so increased, and proceeding with it and the given Resolvend, in all respects as in the first Work (like to the third *Example* of extracting the Cube Root): I say, if the given Resolvend have a true Root, it will appear at this second Operation, and all the aforesaid Differences, &c. will be vanquished; provided the Root required is not to have more than three (or four) Places of Figures in it.

But if the Root be to have more than three Figures in it; or, that the given Resolvend prove to be a Surd Number. Then there will be a Difference as before; which will afford Quotient Figures to rectify and increase the Root last taken, to three Times as many Places of Figures, as it had at the Beginning of that second Operation. As you may see in the aforesaid *Example* 3. of the Cube Root; wherein that Root is increased to twelve Places of Figures at two Operations; which if it were to be extracted the old (and still common) way, it would require at least forty times the Number of Figures I have here used.

Again, if there chance to be a Mistake committed in any Operation performed by the Method here laid down, that Mistake will not destroy the precedent Work, but will be rectified in the next Operation, although it were not discovered before. And thus you may proceed on to a third Operation, which will afford 27 Places of Figures in the Root, &c. with very little Trouble, if compared with former Methods.

The brief Account, which I have here given (*by Way of explaining the Nature of this Method of extracting Roots*) being well considered and compared with the several Operations of the foregoing *Examples*, must needs help the Learner to form such an Idea of it, that he cannot (I presume) but understand how to proceed

proceed in extracting the Root out of any single Power, how high soever it be; without the Help of an *Algebraick Theorem.* Not but when that comes to be once understood; the Work will be much readier and easier performed: As will appear in the next Part.

I did intend to have here inserted the whole Business of Interest and Annuities; but finding that it would require too large a Discourse, to shew the Grounds and Reasons of the several *Theorems* useful therein, I have therefore reserved that Work for the Close of the next Part. Neither indeed can the raising of those *Theorems* be so well delivered in Words, as by an *Algebraick* Way of arguing; which renders them not only much shorter, but also plainer and easier to be understood.

I have also omitted that Rule in *Arithmetick*, usually called the *Rule of Position*, or *Rule of False:* Because all such Questions, as can be answered by that guessing Rule, are much better done by any one who hath but a very small smattering of *Algebra*. I shall therefore conclude this Part of *Numerical Arithmetick*; and proceed to that of *Algebraick Arithmetick*, wherein I would advise the young Learner not to be too hasty in passing from one Rule to another, and then he will find it very easy to be attained.

# AN INTRODUCTION TO THE Mathematicks.

## PART II.

## PROËM.

HAVING *formerly wrote a ſmall Tract of* Algebra, *perhaps it may ſeem (to ſome) very improper to write again upon the ſame Subject; but only (as the uſual Cuſtom is) to have referred my Reader to that Tract. However, becauſe the following Parts of this Treatiſe are managed by an Algebraick Method of arguing; which may fall into the Hands of thoſe who have not ſeen that Tract, or any other of that Kind; I thought it convenient to accommodate the young Geometer with the firſt Elements, or principal Rules, by which all Operations in this Art are performed; that ſo he may not be at a Loſs as he proceeds farther on: Beſides, what I formerly wrote was only a Compendium of that which is here fully handled at large.*

The principal Rules are Addition, Subſtraction, Multiplication, Diviſion, Involution, and Evolution, as in common Arithmetick, but differently performed; and therefore ſome call it Algebraick Arithmetick. Others call it Arithmetick in Specie, becauſe all the Quantities concerned in any Queſtion, remain in their ſubſtituted Letters (howſoever managed by *Addition*, *Subſtraction*, or *Multiplication*, &c.) without being deſtroyed or changed into others, as Figures in common *Arithmetick* are.

Mr. *Harriot* called it Logiſtica Specioſa, or *Specious Computation.*

CHAP.

## CHAP. I.

### *Concerning the* Method *of* noting *down* Quantities; *and* tracing *their* Steps, *&c.*

### Sect. 1. *Of* Notation.

THE Method of noting down Letters for Quantities is various, according to every one's Fancy; but I ſhall here follow the ſame as in my former Tract, and repreſent the Quantity ſought (be it Line or Number, *&c.*) by the ſmall (*a*), and if more Quantities than one are ſought, by the other ſmall Vowels, *e. u.* or *y.*

The given Quantities are repreſented by the ſmall Conſonants, *b. c. d. f. g. &c.*

And for Diſtinction ſake, mark the Points or Ends of Lines in all Schemes, with the capital or great Letters, *viz. A. B. C. D. &c.*

When any Quantity (either given or ſought) is taken more than once, you muſt prefix it's Number to it; as 3*a* ſtands for *a* taken three times, or three times *a*, and 7*b* ſtands for ſeven times *b*, *&c.*

All Numbers thus prefixt to any Quantity, are called Coëfficients or Fellow-Factors; becauſe they multiply the Quantity; and if any Quantity be without a Coëfficient, it is always ſuppoſed or underſtood to have an Unit prefixed to it; as *a* is 1*a*, or *b* is 1*b*, *&c.*

The Signs by which Quantities are chiefly managed are the ſame, and have the ſame Signification, with thoſe in the firſt Part, *page* 5. which I here preſume the Reader to be very well acquainted with. To them muſt be here added theſe three more;

| | | | |
|---|---|---|---|
| *Viz.* | ⓔ | the Sign of | Involution. |
| | *ω* | | Evolution, or extracting Roots. |
| | $\surd$ | | Irrationality, or Sign of a Surd Root. |

All Quantities that are expreſſed by Numbers only (as in *Vulgar Arithmetick*) are called *Abſolute Numbers.*

Thoſe Quantities that are repreſented by ſingle Letters, as, *a. b. c. d. &c.* or by ſeveral Letters that are immediately joined together; as *ab. cd.* or 7*bd. &c.* are called Simple or Single whole Quantities.

But when different Quantities repreſented by different or unlike Letters, are connected together by the Signs (+ or —); as $a+b$, $a-b$, or $ab-dc$, *&c.* they are called Compound whole Quantities.

And

And when Quantities are expreſſed or ſet down like Vulgar Fractions, Thus $\frac{a}{b}$, or $\frac{a+b}{d}$, or $\frac{ab+dc}{b-c}$, &c. they are called Fractional or Broken Quantities.

The Sign wherewith Quantities are connected, always belongs to that Quantity which immediately follows it; and therefore all the Quantities concerned in any Queſtion, may ſtand in any order at Pleaſure, *viz.* the moſt convenient for the next Operation. As $a+b-d$ may ſtand thus $b-d+d$, or thus $a-d+b$, or $-d+a+b$ &c. theſe being ſtill the ſame, tho' differently placed.

That Quantity which hath no Sign before it (as generally the leading Quantity hath not) is always underſtood to have the Sign $+$ before it. As $a$ is $+a$, or $b-d$ is $+b-d$, &c. for the Sign $+$ is the Affirmative Sign, and therefore all leading or poſitive Quantities are underſtood to have it, as well as thoſe that are to be added.

But the Sign $-$ being the Negative Sign, or Sign of Defect, there is a Neceſſity of prefixing it before that Quantity to which it belongs, wherever the Quantity ſtands.

## Sect. 2. *Of* tracing *the* Steps *uſed in bringing* Quantities *to an* Equation.

THE Method of tracing the Steps, uſed in bringing the Quantities concerned in any Queſtion to an Equation, is beſt performed by regiſtering the ſeveral Operations with Figures and Signs placed in the Margin of the Work, according as the ſeveral Operations require; being very uſeful in long and tedious Operations.

For Inſtance: If it be required to ſet down and regiſter the Sum of the two Quantities, $a$ and $b$, the Work will ſtand,

| | | |
|---|---|---|
| Thus | 1 | $a$ |
| | 2 | $b$ |
| $1+2$ | 3 | $a+b$ |

Firſt ſet down the propoſed Quantities, $a$ and $b$, over-againſt the Figures 1, 2, in the ſmall Column (which are here called Steps), and againſt 3 (the third Step), ſet down their Sum, *viz.* $a+b$. Then againſt that third Step, ſet down $1+2$ in the Margin; which denotes that the Quantities againſt the firſt and ſecond Steps are added together, and that thoſe in the third Step are their Sum.

To illuſtrate this in Numbers, ſuppoſe $a=9$, and $b=6$. Then it will be,

| | | |
|---|---|---|
| Thus | 1 | $a \quad =9$ |
| | 2 | $b \quad =6$ |
| $1+2$ | 3 | $a+b=9+6=15$ being the Sum of 9 and 6. |

 Again,

Again, If it were required to ſet down the Difference of the ſame two Quantities; then it will be,

| | | |
|---|---|---|
| Thus | 1 | $a = 9$ |
| | 2 | $b = 6$ |
| 1 — 2 | 3 | $a - b = 9 - 6 = 3$ the Diff. between 9 and 6. |

Or if it were required to ſet down their Product. Then it will be,

| | | |
|---|---|---|
| Thus | 1 | $a = 9$ |
| | 2 | $b = 6$ |
| 1 x 2 | 3 | $a \times b$ or $ab = 9 \times 6 = 54$ the Prod. of 9 into 6. |

&c.

Note, *Letters ſet or joined immediately together (like a Word) ſignify the Rectangle or Product of thoſe Quantities they repreſent; as in the laſt* Example, *wherein* ab = 54 *is the Product of* a = 9 *and* b = 6. &c.

## Axioms.

1. If equal Quantities be added to equal Quantities, the Sum of theſe Quantities will be equal.
2. If equal Quantities be taken from equal Quantities, the Quantities remaining will be equal.
3. If equal Quantities be multiplied with equal Quantities, their Products will be equal.
4. If equal Quantities be divided by equal Quantities, their Quotients will be equal.
5. Thoſe Quantities, that are equal to one and the ſame Thing, are equal to one another.

Note, *I adviſe the Learner to get theſe five Axioms perfectly by Heart.*

Theſe Things being premiſed, and a perfect Knowledge of the Signs and their Significations being gained, the young *Algebraiſt* may proceed to the following *Rules*. But firſt I muſt make bold to adviſe him here (as I have formerly done) that he be very ready in one Rule before he undertakes the next.

That is, He ſhould be expert in *Addition*, before he meddles with *Subſtraction*; and in *Subſtraction*, before he undertakes *Multiplication*, &c. becauſe they have a Dependency one upon another.

CHAP.

# CHAP. II.

## *Concerning the* Six *Principal* Rules, *of* Algebraick Arithmetick, *of whole* Quantities.

### Sect. 1. Addition *of whole* Quantities.

*ADDITION* of whole Quantities admits of three Cases.

*Case* 1. If the Quantities are like, and have like Signs; add the Co-ëfficients or prefixt Numbers together, and to their Sum adjoin the Quantities with the same Sign.

| | | *Exam.* 1. | *Exam.* 2. | *Exam.* 3. | *Exam.* 4. |
|---|---|---|---|---|---|
| | 1 | $a$ | $-a$ | $5b$ | $-7bc$ |
| | 2 | $a$ | $-a$ | $3b$ | $-8bc$ |
| 1 + 2 | 3 | $2a$ | $-2a$ | $8b$ | $-15bc$ |

| Thus | | *Exam.* 5. | *Exam.* 6. | *Exam.* 7. |
|---|---|---|---|---|
| | 1 | $3a+5b$ | $3a-5b$ | $6ab+12$ |
| | 2 | $2a+7b$ | $2a-7b$ | $3ab+24$ |
| 1 + 2 | 3 | $5a+12b$ | $5a-12b$ | $9ab+36$ |

*The Reason of these Additions is evident from the Work of Common* Arithmetick. *For suppose* a, *to represent one Crown, to which if I add one Crown, the Sum will be two Crowns, or* 2 a. *as in Example* 1.

*Or if we suppose* — a, *to represent the Want or Debt of one Crown, to which if another Want or Debt of one Crown be added, the Sum must needs be the Want or Debt of two Crowns, or* — 2 a; *as in Example* 2. *And so for all the rest.*

*Case* 2. If the Quantities are alike, and have unlike Signs; substract the Co-ëfficients, from each other, and to their Difference join the Quantities with the Sign of the greater.

| | | *Exam.* 8. | *Exam.* 9. | *Exam.* 10. | *Exam.* 11. |
|---|---|---|---|---|---|
| | 1 | $+5a$ | $-5a$ | $7bc$ | $-9abd$ |
| | 2 | $-3a$ | $+3a$ | $-6bc$ | $+7abd$ |
| 1 + 2 | 3 | $+2a$ | $-2a$ | $-1bc$ | $-2abd$ |

| | | *Example* 12. | *Example* 13. |
|---|---|---|---|
| | 1 | $7a-5b$ | $-8ab-7bc+15$ |
| | 2 | $-5a+7b$ | $+12ab+7bc-24$ |
| 1 + 2 | 3 | $2a+2b$ | $4ab-9$ |

*The Reason of the Operations in this Case may be easily understood, by any one that duly considers the comparing of Stock and Debts together, or the ballancing of Accounts betwixt Debtor and Creditor. That is, the Affirmative Quantities represent the Stock or Creditor: The Negative Quantities represent the Debts; and their Sum represents the Ballance,* &c.

*Case* 3. When the Quantities are unlike, set them all down, without altering their Signs; and thence will arise compound Quantities, which can be no otherwise added but by their Signs.

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $a$ | $a$ | $5b + 7dc$ |
| | 2 | $b$ | $-b$ | $4a - 20$ |
| 1 + 2 | 3 | $a + b$ | $a - b$ | $5b + 7dc + 4a - 20$ |

Here follow a few Examples wherein all the 3 *Cases* are promiscuously concerned.

| | | | |
|---|---|---|---|
| | 1 | $aa + 2ab + bb$ | $8ab + bc - 37$ |
| | 2 | $-4ab$ | $-7ab - bc + 42 - 6d$ |
| 1 + 2 | 3 | $aa - 2ab + bb$ | $ab + 5 - 6d$ |

| | | | |
|---|---|---|---|
| | 1 | $aa - 2ab + bb$ | $9bc + 7ab - 45$ |
| | 2 | $+ 4ab + bb$ | $4d - 6bc - 7ab + da$ |
| 1 + 2 | 3 | $aa + 2ab + bb$ | $3bc + 4d - 45 + da$ |

| | | | |
|---|---|---|---|
| | 1 | $5a$ | $a + b - ab$ |
| | 2 | $-7a$ | $7c - d$ |
| | 3 | $+3a$ | $4e + f$ |
| 1 + 2 + 3 | 4 | $a$ | $a + b - ab + 7c - d + 4e + f$ |

| | | |
|---|---|---|
| | 1 | $3aa + 4abc - bb + 30$ |
| | 2 | $2bb - 3aa - 2abc - 25$ |
| | 3 | $dd + 2aa - 3abc - 3$ |
| 1 + 2 + 3 | 4 | $+dd + 2aa + bb - abc + 2$ |

## Sect. 2. Substraction *of whole* Quantities.

*SUBSTRACTION* of whole Quantities is performed by one general Rule.

RULE.

*Change all the Signs of the Substrahend* (viz. *of those Quantities which are to be substracted*) *or suppose them in your Mind to be changed; then add all the Quantities together, as before in* Addition, *and their Sum will be the true Remainder or Difference required.*

This

This general Rule is deduced from theſe evident Truths.

To ſubſtract an Affirmative Quantity, from an Affirmative, is the ſame as to add a Negative Quantity to an Affirmative: that is $+2a$ taken from $+3a$, is the ſame with $-2a$ added to $+3a$. Conſequently, to ſubſtract a Negative Quantity from an Affirmative, will be the ſame as to add an Affirmative Quantity to an Affirmative: that is $-2a$ taken from $+3a$ will be the ſame with $+2a$ added to $+3a$.

| | | *Exam.* 1. | *Exam.* 2. | *Exam.* 3. | *Exam.* 4. |
|---|---|---|---|---|---|
| | 1 | $2a$ | $-2a$ | $8b$ | $-15bc$ |
| | 2 | $a$ | $-a$ | $3b$ | $-8bc$ |
| 1 — 2 | 3 | $a$ | $-a$ | $5b$ | $-7bc$ |

| | | *Exam.* 5. | *Exam.* 6. | *Exam.* 7. |
|---|---|---|---|---|
| | 1 | $5a+12b$ | $5a-12b$ | $9ab+36$ |
| | 2 | $2a+7b$ | $2a-7b$ | $3ab+24$ |
| 1 — 2 | 3 | $3a+5b$ | $3a-5b$ | $6ab+12$ |

| | | *Exam.* 8. | *Exam.* 9. | *Exam.* 10. | *Exam.* 11. |
|---|---|---|---|---|---|
| | 1 | $+2a$ | $-2a$ | $bc$ | $-2abd$ |
| | 2 | $-3a$ | $+3a$ | $-6bc$ | $+7abd$ |
| 1 — 2 | 3 | $+5a$ | $-5a$ | $+7bc$ | $-9abd$ |

| | | *Example* 12. | *Example* 13. |
|---|---|---|---|
| | 1 | $2a+2b$ | $4ab-9$ |
| | 2 | $-5a+7b$ | $-8ab-7bc+15$ |
| 1 — 2 | 3 | $7a-5b$ | $12ab+7bc-24$ |

If theſe 13 *Examples* be compared with thoſe in *Addition*, the Work will appear very evident, theſe being only the Converſe or Proof of thoſe; according to the Nature of *Addition* and *Subſtraction* in common *Arithmetick*.

## More *Examples* in *Subſtraction*.

| | | | | |
|---|---|---|---|---|
| | 1 | $a+b$ | $5bc+3da$ | $8a+5bd+25$ |
| | 2 | $a-b$ | $5bc-4da$ | $7a-3bd-12$ |
| 1 — 2 | 3 | $+2b$ | $+7da$ | $a+8bd+37$ |

1 — 2

| | | | | |
|---|---|---|---|---|
| | 1 | $c + 13$ | $a$ | $0$ |
| | 2 | $3a - b - 2c$ | $b$ | $2a - 4b$ |
| 1 — 2 | 3 | $3c + 13 - 3a + b$ | $a - b$ | $-2a + 4b$ |

| | | | |
|---|---|---|---|
| | 2 | $a + b - 54$ | $76$ |
| | 1 | $d - 3b - bc - 75$ | $a - b - 5d + 7c$ |
| 1 — 1 | 3 | $a + 4b + bc + 21 - d$ | $76 - a + b + 5d - 7c$ |

That $a - b$ taken from $a + b$ leaves $+ 2b$ for the Remainder, as in the firſt of theſe *Examples*, may be thus proved;

| | | | |
|---|---|---|---|
| Let | 1 | $a + b = z$ | |
| And | 2 | $a - b = x$ | |
| $2 + b$ | 3 | $a = x + b$ | *per Axiom* 1. |
| $1 - 3$ | 4 | $b = z - x - b.$ | *per Axiom* 2. |
| $4 + b$ | 5 | $2b = z - x$ which was to be proved. | |

The Truth of all Operations in *Subſtraction*, where any Doubt ariſes, may be proved, by adding the Subſtrahend to the Remainder, as in Common *Arithmetick*.

*EXAMPLE.*

| | | | | | |
|---|---|---|---|---|---|
| From | 1 | $+ 5a$ | $0$ | $+ 9bc$ | |
| Take | 2 | $- 2a$ | $+ 3b$ | $- 6da$ | Subſtrahend. |
| 1 — 2 | 3 | $+ 7a$ | $- 3b$ | $+ 6da - 9bc$ | Remainder. |
| 2 + 3 | 4 | $+ 5a$ | $0$ | $9bc$ | Proof. |

## Sect. 3. Multiplication *of whole* Quantities.

*MULTIPLICATION* of whole Quantities admits of three Caſes.

*Caſe* 1. When the Quantities have like Signs, and no Coëfficients, ſet or join them together, and prefix the Sign + before them; and that will be their Product.

| | | Exam. 1. | Exam. 2. | Exam. 3. | Exam. 4. |
|---|---|---|---|---|---|
| Thus { | 1 | $a$ | $- a$ | $a + b$ | $- a - b$ |
| | 2 | $b$ | $- b$ | $d$ | $- d$ |
| 1 × 2 | 3 | $ab$ | $+ ab$ | $ad + bd$ | $+ ad + bd$ |

*Caſe* 2. If there be Coëfficients; multiply them, and to their Product adjoin the Quantities ſet together as before.

Thus

| | | Exam. 5. | Exam. 6. | Exam. 7. | Exam. 8. |
|---|---|---|---|---|---|
| Thus { | 1 | $5a$ | $-6d$ | $3a + 2b$ | $a + b$ |
| | 2 | $3b$ | $-7b$ | $6$ | $5b$ |
| 1 × 2 | 3 | $15ab$ | $+42db$ | $18a + 12b$ | $5ab + 5bb$ |

*Case* 3. When the Quantities have unlike Signs; join them and the Product of the Coëfficients together (as before), but prefix the Sign — before them;

| | | Exam. 9. | Exam. 10. | Exam. 11. | Exam. 12. |
|---|---|---|---|---|---|
| Thus { | 1 | $+a$ | $-6d$ | $4a - 7b$ | $4a - 7b$ |
| | 2 | $-b$ | $+7b.$ | $3f$ | $-3f$ |
| 1 × 2 | 3 | $-ab$ | $-42db$ | $12af - 21bf$ | $-12af + bf$ |

That is, + into +, or — into —, gives + } in the Product.
But + into —, or — into +, gives — }

That + into + will produce + in the Product, is evident from *Multiplication* in Common *Arithmetick*: viz. + 5 into + 7 will give + 35 *&c.* But that + into —, or — into + ſhould produce the Sign —, as in the four laſt Examples; and that — into — ſhould produce the Sign +, as in the ſecond, fourth, and ſixth Examples, may perhaps ſeem ſomewhat hard to be conceived; and requires a Demonſtration.

Firſt to prove that $-7b$ into $+3f = -21bf$. As in *Example* 11.

| | | | |
|---|---|---|---|
| Suppoſe | 1 | $4a - 7b = 0$ | |
| Then will | 2 | $4a = 7b$ | *per Axiom* 1. |
| But | 3 | $+3f = +3f$ | |
| 2 × 3 | 4 | $12af = 21bf$ | *per Axiom* 3. |
| 4 — 21 $bf$ | 5 | $12af - 21bf = 0$ | *per Axiom* 2. |

Conſequently + into —, or — into + produces —, which was the Thing to be proved.

Secondly to prove that $-7b$ into $-3f$ gives $+21bf$ as in *Example* 12.

| | | | |
|---|---|---|---|
| Let | 1 | $4a - 7b = 0$ | } as before. |
| Then | 2 | $4a = 7b$ | } |
| But | 3 | $-3f = -3f$ | |
| the 2 × 3 is | 4 | $-12af = -21bf$ | by what is proved above. |
| 4 21 $bf$ | 5 | $-12af + 21bf = 0.$ | *per Axiom* 1. |

Conſequently — into — gives +, which was to be proved.

Or

Or these may be otherwise proved by Numbers.

Thus, suppose $\left\{\begin{array}{l} a = 20 \\ b = 14 \end{array}\right\}$ and $\left\{\begin{array}{l} c = 12 \\ d = 8 \end{array}\right\}$ or any other Numbers.

Then $a - b = 6$ $\quad c - d = 4$ *per Axiom* 2.

Consequently, $\overline{a - b} \times \overline{c - d} = 6 \times 4 = 24$, *per Axiom* 3. but $\overline{a - b} \times \overline{c - d}$, according to the precedent Rules, will be, $ac - cb + bd - da$, which, if true, must be equal to 24.

Proof $\left\{\begin{array}{l} ac = 20 \times 12 = 240 \\ bd = 14 \times 8 = 112 \end{array}\right.$ $\quad \begin{array}{l} cb = 12 \times 14 = 168 \\ da = 8 \times 20 = 160 \end{array}$

Hence $ac + bd = 352$ *per Axiom* 1.

And $cb + da = 328$ which being subtracted,

Leaves $ac + bd - cb - da = 352 - 328 = 24$, which plainly shews,

That + into — produces — } in the Product.
And — into — produces + }

*Q. E. D.*

*Note*, If the Multiplier consists of several Terms, then every one of those Terms must be multiplied into all the Terms of the Multiplicand; and the Sum of those particular Products, will be the Product required, as in Common *Arithmetick*.

### *EXAMPLES.*

| | | | |
|---|---|---|---|
| | 1 | $a + b - d$ | $7b + 5d$ |
| | 2 | $a - b$ | $3a - 5f$ |
| $1 \times a$ | 3 | $aa + ba - da$ | $21ba + 15da$ |
| $1 \times b$ | 4 | $- ba - bb + db$ | $- 35bf - 25df$ |
| $3 + 4$ | 5 | $aa - da - bb + db$ | $21ba + 15da - 35bf - 25df$ |

| | | | |
|---|---|---|---|
| | 1 | $aa - ba$ | $2c - 2d$ |
| | 2 | $a + b$ | $3a - 4b$ |
| $1 \times 2$ | 3 | $aaa - abb$ | $6ca - 9da - 8bc + 12db$ |

| | | | |
|---|---|---|---|
| | 1 | $aa + 2a + 4$ | $aa - ba + bb$ |
| | 2 | $a - 2$ | $a + b$ |
| | | $aaa + 2aa + 4a$ | $aaa - baa + bba$ |
| | | $- 2aa - 4a - 8$ | $+ baa - bba + bbb$ |
| $1 \times 2$ | 3 | $aaa - 8$ | $aaa + bbb$ |

Sect.

## Sect. 4. Division *of whole* Quantities.

*Division* of Species, is the converse or direct contrary to that of *Multiplication*, and consequently is performed by converse Operations (as in common *Arithmetick*), and admits of four *Cases*.

*Case* 1. When the Quantities in the Dividend, have like Signs to those in the Divisor, and no Co-efficients in either; cast off or expunge all the Quantities in the Dividend, that are like those in the Divisor; and set down the other Quantities with the Sign + for the Quotient required.

| | | | | | |
|---|---|---|---|---|---|
| Thus | 1 | $ab$ | $-ab$ | $ad+bd$ | $-ad-bd$ |
| | 2 | $b$ | $-b$ | $d$ | $-d$ |
| 1 ÷ 2 | 3 | $a$ | $+a$ | $a+b$ | $a+b$. |

*Case* 2. When the Quantities in the Dividend have unlike Signs to those in the Divisor; then set down the Quotient Quantities found as before, with the Sign — before them.

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $+ab$ | $-ab-bd$ | $abc+bcd+bcf$ |
| | 2 | $-b$ | $+b$ | $-bc$ |
| 1 ÷ 2 | 3 | $-a$ | $-a-d$ | $-a-d-f$ |

*Case* 3. If the Quantities in the Dividend and Divisor, have Co-efficients; divide the Numbers (as in common *Arithmetick*) and to their Quotients adjoin the Quotient Quantities.

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $15ab$ | $42db$ | $12af-21bf$ |
| | 2 | $3b$ | $-7b$ | $3f$ |
| 1 ÷ 2 | 3 | $5a$ | $-6d$ | $4a-7b$ |

*Note*, When the Quantities and Co-efficients in the Divisor and Dividend are all the same, the Quotient will be an Unit, or 1.

| | | | | | |
|---|---|---|---|---|---|
| Thus | 1 | $ab$ | $9bc$ | $7ab+5bc$ | $8ab+4d$ |
| | 2 | $ab$ | $-9bc$ | $7ab+5bc$ | $-8ab-4d$ |
| 1 ÷ 2 | 3 | 1 | $-1$ | 1 | $-1$ |

*Case* 4. When the Quantities in the Divisor cannot be exactly found in the Dividend; then set them both down like a *Vulgar Fraction*, as in common *Arithmetick*.

| Thus | | | | | |
|---|---|---|---|---|---|
| | 1 | $a$ | $6bc$ | $5b + aa$ | $8adc$ |
| | 2 | $b$ | $3d$ | $5d + 7b$ | $4abc$ |
| $1 \div 2$ | 3 | $\frac{a}{b}$ | $\frac{2bc}{d}$ | $\frac{5b+aa}{5d+7b}$ | $\frac{2d}{b}$ |

*N. B.* In Division one thing must be very carefully observed; *viz.* that like Signs give + and unlike Signs give — in the Quotient; which needs no other Proof than that already laid down in the last Section, if duly compared with what hath been said concerning *Multiplication* and *Division*, in Vulgar *Arithmetick*.

*Examples* of *Division* at large.

| | | |
|---|---|---|
| | 1 | $21ba + 15da - 35bf - 25df \; (+ 3a$ |
| | 2 | $7b + 5d$ |
| $2 \times 3a$ | 3 | $21ba + 15da$ |
| $1 - 3$ | 4 | $0 \quad 0 \quad - 35bf - 25df \; (- 5f$ |
| $2 \times - 5f$ | 5 | $- 35bf - 25df$ |
| $4 - 5$ | 6 | $0 \quad 0$ |
| $1 \div 2$ | 7 | $3a - 5f$ the Quot. collected from the 3 and 5 Steps. |

Or *Division* of Quantities may stand as Numbers in common *Arithmetick* do; thus

$$
\begin{array}{l}
3a - 6)\; 6aaaa - 96 \quad (2aaa + 4aa + 8a + 16 \\
\qquad\quad 6aaaa - 12aaa \\
\hline
\qquad\quad 0 + \quad 12aaa - 96 \\
\qquad\qquad + \quad 12aaa - 24aa \\
\hline
\qquad\qquad\qquad 0 \quad + 24aa - 96 \\
\qquad\qquad\qquad\quad + 24aa - 48a \\
\hline
\qquad\qquad\qquad\qquad 0 \quad + 48a - 96 \\
\qquad\qquad\qquad\qquad\quad + 48a - 96 \\
\hline
\qquad\qquad\qquad\qquad\qquad 0 \quad 0
\end{array}
$$

That is, $\overline{6aaa - 96} \div \overline{3a - 6}$ gives $2aaa + 4aa + 8a + 16$ for the Quotient, as may easily be proved by *Multiplication*, viz. $\overline{2aaa + 4aa + 8a + 16} \times \overline{3a - 6}$ will produce $6a^4 - 96$; and so for the rest.

## Sect. 5. Involution *of whole* Quantities.

INvolution is the raising or producing of Powers, from any proposed Root, and is performed in all respects like *Multiplication*, save only in this; *Multiplication* admits of any different Factors, but Involution still retains the same.

*EXAMPLES.*

## *EXAMPLES.*

| | | | | |
|---|---|---|---|---|
| | 1 | $a$ | $-a$ | the Root, or ſingle Power. |
| 1 ⊛$^2$ | 2 | $aa$ | $+aa$ | Square, or ſecond Power. |
| 1 ⊛$^3$ | 3 | $aaa$ | $-aaa$ | Cube, or third Power. |
| 1 ⊛$^4$ | 4 | $aaaa$ | $+aaaa$ | Biquadrat, or fourth Power. |
| 1 ⊛$^5$ | 5 | $aaaaa$ | $-aaaaa$ | Surſolid, or fifth Power, &c. |

*Note*, The Figures placed in the Margin, after the Sign (⊛) of Involution, ſhew to what Height the Root is involved; and are called Indices of the Power; and are uſually placed over the involved Quantities, in order to contract the Work, eſpecially when the Powers are any thing high.

$$\text{Thus}\begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases} \qquad \text{And}\begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5\ b^5 = aaaaabbbbb \\ a^3\ b^3\ d^3 = aaabbbddd \end{cases}$$

If the Quantities have Co-ëfficients, the Co-ëfficients muſt be involved along with the Quantities, as in theſe,

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $2a$ | $-3a$ | $5bc$ |
| 1 ⊛$^2$ | 2 | $4aa$ | $+9aa$ | $25bbcc$ |
| 1 ⊛$^3$ | 3 | $8aaa$ | $-27aaa$ | $125bbbccc$ |
| 1 ⊛$^4$ | 4 | $16aaaa$ | $+81aaaa$ | $625b^4c^4$ |
| 1 ⊛$^5$ | 5 | $32aaaaa$ | $-243a^5$ | $3125b^5c^5$ &c. |

Involution of Compound Quantities is performed in the ſame manner, due regard being had to their Signs and Co-ëfficients, if there be any. As for inſtance, ſuppoſe $a+b$ were given to be involved to the fifth Power.

| | | |
|---|---|---|
| Thus | 1 | $a+b$ called a Binomial Root |
| | | $a+b$ |
| 1 × $a$ | 2 | $aa+ab$ |
| 1 × $b$ | 3 | $\quad +ab+bb$ |
| 1 ⊛$^2$ | 4 | $aa+2ab+bb$, the Square of $a+b$ |
| | | $a+b$ |
| 4 × $a$ | 5 | $aaa+2aab+abb$ |
| 4 × $b$ | 6 | $\quad +aab+2abb+bbb$ |
| 1 ⊛$^3$ | 7 | $aaa+3aab+3abb+bbb$, the Cube of $a+b$ |

| | | |
|---|---|---|
| | 7 | $aaa + 3aab + 3abb + bbb$ |
| | | $a + b$ |
| $7 \times a$ | 8 | $a^4 + 3a^3b + 3aabb + abbb$ |
| $7 \times b$ | 9 | $+ a^3b + 3aabb + 3abbb + b^4$ |
| $1\ ⊗^{5}$ | 10 | $a^4 + 4a^3b + 6aabb + 4abbb + b^4$ |
| | | $a + b$ |
| $10 \times a$ | 11 | $a^5 + 4a^4b + 6a^3bb + 4aab^3 + ab^4$ |
| $1 \times b$ | 12 | $a^4b + 4a^3bb + 6aab^3 + ab^4 + b^5$ |
| $1\ ⊗^{5}$ | 13 | $a^5 + 5a^4b + 10a^3bb + 10aab^3 + 5ab^4 + b^5$ |
| | | &c. |

Again, Let $a - b$, called a Refidual Root, be given.

| | | |
|---|---|---|
| Then | 1 | $a - b$ |
| | | $a - b$ |
| $1 \times a$ | 2 | $aa - ab$ |
| $1 \times - b$ | 3 | $- ab + bb$ |
| $1\ ⊗^{2}$ | 4 | $aa - 2ab + bb$ the Square of $a - b$ |
| | | $a - b$ |
| $4 \times a$ | 5 | $aaa - 2aab + abb$ |
| $4 \times - b$ | 6 | $- aab + 2abb - bbb$ |
| $1\ ⊗^{3}$ | 7 | $aaa - 3aab + 3abb - bbb$, the Cube of $a - b$ |
| | | $a - b$ |
| $7 \times a$ | 8 | $aaaa - 3aaab + 3aabb - abbb$ |
| $7 \times - b$ | 9 | $- aaab + 3aabb - 3abbb + bbbb$ |
| $1\ ⊗^{4}$ | 10 | $aaaa - 4aaab + 6aabb - 4abbb + bbbb$ |
| | | $a - b$ |
| $10 \times a$ | 11 | $a^5 - 4a^4b + 6a^3bb - 4aab^3 + ab^4$ |
| $10 \times - b$ | 12 | $- a^4b + 4a^3bb - 6aab^3 + 4ab^4 \quad b^5$ |
| $1\ ⊗^{5}$ | 13 | $a^5 - 5a^4b + 10a^3bb - 10aab^3 + 5ab^4 - b^5$ |
| | | &c. |

By comparing thefe two Examples together, you may make the following Obfervations.

1. That the Powers raifed from a Refidual Root (*viz.* the Difference of two Quantities) are the fame with their like Powers raifed from a Binomial Root (or the Sum of two Quantities) fave only in their Signs; *viz.* the Binomial Powers have the Sign + to every Term, but the Refidual Powers have the Signs + and — interchangeably to every other Term.

2. The Indices of the Powers of the leading Quantity ($a$) continually decreafe in Arithmetical Progreffion; *viz.* in the Square

it

it is *a a*, *a:* In the Cube *a a a*, *a a*, *a:* In the Biquadrat *a a a a*, *a a a*, *a a*, *a*, &c.

3. The Indices of the other Quantity *b* do continually increaſe in Arithmetical Progreſſion; *viz.* In the Square it is *b*, *b b:* In the Cube *b*, *b b*, *b b b:* In the Biquadrat *b*, *b b*, *b b b*, *b b b b*, &c.

4. The firſt and laſt Terms, are always pure Powers of the ſingle Quantities, and are both of the ſame Height.

5. The Sum of the Indices of any two Letters joined together in the intermediate Terms, are always equal to the Index of the higheſt Power, *viz.* of the firſt or laſt Term.

Theſe Obſervations being duly conſidered, it will be eaſy to conceive how the Terms of any propoſed Power raiſed from a Binomial or Reſidual Root muſt ſtand, without their Unciæ or Numerical Figures.

For Inſtance, ſuppoſe it were required to raiſe the Binomial Root $a + b$ to the ſeventh Power; then the Terms of that Power will ſtand without their Unciæ in this Order.

Viz. $a^7 + a^6 b + a^5 b^2 + a^4 b^3 + a^3 b^4 + a^2 b^5 + a\ b^6 + b^7$.

And becauſe the Uncia (not only of any ſingle Letter, but alſo) of every ſingle Power, how high ſoever it be, is an Unit or 1 (which neither multiplies nor divides) and all the Powers of any Binomial or Reſidual Root are naturally raiſed by multiplying of the precedent Power into it's original Root, which is done by only joining each Letter in the Root to the precedent Power, with it's Unciæ, and then removing the ſaid Power, when it is ſo joined to the ſecond Letter, one place forward (either to the left or right Hand) it muſt needs follow,

That the Unciæ of the ſecond Terms (in any ſuch Power) will always be the Sum of ſo many Units added together more one, as there have been Multiplications of the firſt Root; which will always be determined by the Index of the firſt Term in the Power.

And becauſe the Unciæ of all the intermediate Terms, are only removed along with their Letters, it alſo follows; that if they are added together, their reſpective Sums will produce the true Unciæ of the intermediate Terms in the new raiſed Power. As doth plainly appear from the following Numbers ſo removed without their Letters; which both ſhews and demonſtrates an eaſy Way of producing the Unciæ of any ordinary Power (*viz.* of one not very high) raiſed from either a Binomial or Reſidual Root.

Thus

Thus

Add { 1 . 1 .  The two Unciæ of the Root.
      1 . 1

Add { 1 . 2 . 1  The Unciæ of the Square.
      1 . 2 . 1

Add { 1 . 3 . 3 . 1  The Unciæ of the Cube.
      1 . 3 . 3 . 1

Add { 1 . 4 . 6 . 4 . 1  The Unciæ of the fourth Power.
      1 . 4 . 6 . 4 . 1

Add { 1 . 5 . 10 . 10 . 5 . 1  Unciæ of the fifth Power.
      1 . 5 . 10 . 10 . 5 . 1

Add { 1 . 6 . 15 . 20 . 15 . 6 . 1  Unciæ of the 6th Power.
      1 . 6 . 15 . 20 . 15 . 6 . 1

1 . 7 . 21 . 35 . 35 . 21 . 7 . 1  Unciæ of 7th Pow.

And ſo on in this manner *ad infinitum.*

Now if theſe Numbers are prefixed to the aforeſaid Letters, all the Terms will be compleated with their reſpective Unciæ, and will ſtand thus;

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b.$$

But that the Buſineſs of finding theſe Unciæ, may be rendered yet more eaſy for Practice, it will be convenient to conſider what Series or Progreſſion, the Unciæ of each Term do make from the aforeſaid additions.

| Unciæ of the firſt Term. | Unciæ of the ſecond Term. | Unciæ of the third Term. | Unciæ of the fourth Term. | Unciæ of the fifth Term. | Unciæ of the ſixth Term. | Unciæ of the ſeventh Term. | Unciæ of the eighth Term, &c. | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | Unciæ of the ſingle Quantities. |
| 1 | 2 | 1 | | | | | | Unciæ of the Square. |
| 1 | 3 | 3 | 1 | | | | | Unciæ of the Cube. |
| 1 | 4 | 6 | 4 | 1 | | | | Unciæ of the 4th Power. |
| 1 | 5 | 10 | 10 | 5 | 1 | | | Unciæ of the 5th Power. |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | Unciæ of the 6th Power. |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | Unciæ of the 7th Power, &c. |

The Unciæ of the firſt Term are only a Series of Units, whoſe Sum is every where the Unciæ of the ſecond Term. The Unciæ of the ſecond Term, are a Series of Numbers in Arithmetick Progreſſion, whoſe Sum is every where the Unciæ of the next Superior Power in the third Term, and may be found by Propoſition

tion 1. Chap. 6. Part 1. For Inſtance, in the ſeventh Power it will be $\frac{\overline{0+1}\times 6}{2} = 21$ = the Uncia of the third Term.

The reſt of the Unciæ are a compounded Series, whoſe reſpective Sums may be obtained from the Unciæ of their precedent Terms.

Thus $\frac{21\times 5}{3} = 35$. Then $\frac{35\times 4}{4} = 35$. Again $\frac{35\times 3}{5} = 21$. And $\frac{21\times 2}{6} = 7$ &c.

From hence may be deduced this general Rule.

RULE.

*If the Index of the firſt Letter of any Term be multiplied into it's own Uncia, and that Product be divided by the Number of Terms to that Place; the Quotient will be the Uncia of the next ſucceeding Term forward.*

That is, by the help of thoſe Indices that belong to the ſeveral Powers of the firſt or leading Letter only (as $a$) the true Unciæ of every Term may be eaſily underſtood.

*EXAMPLE* 2.

Let it be required to compleat all the Terms of the aforeſaid ſeveral Powers, *viz.* $a^7 + a^6\ b + a^5\ b^2 + a^4\ b^3 + a^3\ b^4 + a^2\ b^5 + a\ b^6 + b^7$, with their proper Unciæ.

1. The Index of $a^7$ the firſt Term will be the Uncia of the ſecond Term. Thus $a^7 + 7a^6 b$.

2. Then half the ſecond Term's Index into it's Uncia, *viz.* $\frac{7\times 6}{2} = 21$ will be the third Term's Uncia. Thus $a^7 + 7a^6\ b + 21\ a^5\ b^2$ will be the three firſt Terms.

3. Again $\frac{21\times 5}{3} = 35$ is the Uncia of the fourth Term, whence $a^7 + 7a^6\ b + 21a^5\ b^2 + 35a^4 b^3$ will be the four firſt Terms.

4. And $\frac{35\times 4}{4} = 35$ will be the Uncia of the fifth Term, whence $a^7 + 7a^6\ b + 21a^5\ b^2 + 35a^4\ b^3 + 35a^3 b^4$ will be the five firſt Terms.

And ſo proceed 'till all the Terms are compleated with their reſpective Unciæ; which will ſtand, thus $a^7 + 7a^6\ b + 21a^5\ b^2 + 35a^4\ b^3 + 35a^3\ b^4 + 21a^2\ b^5 + 7a\ b^6 + b^7$.

Now

Now here it may be further obferved, that the Unciæ do only increafe until the Indices of the two Letters become equal, or change Places; and then the reft of the Unciæ will return or decreafe in the fame order. That is, wherever the Indices of the Letters are alike, there the Unciæ will be alike.

And therefore one needs to find the Unciæ (as before) but to half the Number of Terms in any Power.

If what hath been faid, and the Work of the *Example* be well underftood, I prefume it will be found very eafy to raife any Power from a Binomial or Refidual Root, to what Height you pleafe; without the Trouble of a continued Involution; and without the Help of fuch a Table of Powers as is propofed by Mr *Oughtred* in his Key to the *Mathematicks*, Page 40, and fince by others.

Now from thefe Confiderations it was, that I propofed this thod of raifing Powers in my Compendium of *Algebra*, Page 57, as wholly New (*viz.* fo much of it as was there ufeful) having then (I profefs) neither feen the Way of doing it, nor fo much as heard of it's being done. But fince the writing of that Tract, I find in Dr *Wallis*'s *Hiftory of Algebra*, Page 319 and 331, that the Learned Sir *Ifaac Newton* had difcovered it long before: which the Doctor fets down in this manner.

Let $m$ be the Exponent of the Power.

$$\text{Then}\left\{ 1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \ \&c. \right.$$

Will be the Series of the Unciæ required; but he doth not tell us how they firft came to be found out, nor have I ever met with the leaft Hint of it in any Author.

## Sect. 6. Evolution *of whole* Quantities.

*Evolution* is the extracting of Roots from any given Power. That is, it is the Converfe Work to that of Involution, and in fingle Quantities it is eafy, if the given Power have fuch a Root as is required, which may be thus known.

If the given Power have no Numbers prefixed to it, and it's Index can be divided by the Index of the Root required, the Quotient will be the Index of the Root fought. Thus, if the Cube Root of $aaaaaa$, viz. $a^6$ were required (the Index of the Cube is 3) then 3) 6 (2. That is, $a^{\frac{6}{3}} = a^2$ the Root required. And fuch Operations are ufually fet down

Thus

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $a^6$ | $a^6\ b^6$ | $a^6\ b^6\ d^6$ |
| 1 $\omega^2$ | 2 | $a^3$ | $a^3\ b^3$ | $a^3\ b^3\ d^3$ |
| 1 $\omega^3$ | 3 | $a$ | $a^2\ b^2$ | $a^2\ b^2\ d^2$ |
| 3 $\omega^2$ | 4 | $a$ | $a\ b$ | $a\ b\ d$ |

*Note*, The Figures placed in the Margin after the Sign ($\omega$) of Evolution, denote the Index of the Root to be extracted.

If the given Powers have Co-ëfficients: (*viz.* Numbers prefixed to them;) then you muſt extract their reſpective Roots, as in Vulgar *Arithmetick*.

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $81\ a^4$ | $1296\ a^8\ b^8$ | $20736\ a^4\ b^4\ c^4$ |
| 1 $\omega^2$ | 2 | $9\ a^2$ | $36\ a^4\ b^4$ | $144\ a^2\ b^2\ c^2$ |
| 1 $\omega^4$ | 3 | $3\ a$ | $6\ a^2\ b^2$ | $12\ a\ b\ c$ |
| or 2 $\omega^2$ | 4 | $3\ a$ | $6\ a^2\ b^2$ | $12\ a\ b\ c$ |

But if the Root required cannot be truly extracted out of both the Co-ëfficients and Indices of the given Power; then it is a Surd, and muſt have the Sign of the Root required prefixed to it.

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $a^5$ | $67\ a^4$ | $216bbbddd$ |
| 1 $\omega^2$ | 2 | $\sqrt{a^5}$ | $\sqrt{67\ a^4}$ | $\sqrt{216bbbddd}$ |
| 1 $\omega^3$ | 3 | $\sqrt[3]{a^5}$ | $\sqrt[3]{67\ a^4}$ | $6bd$ |

Evolution of compound Quantities or Powers, is a little more troubleſome than that of Single Powers; and would require a great many Words to explain the Manner and Reaſon of forming the ſeveral Canons, that are commonly uſed in extracting the Roots of compound Quantities; eſpecially if the Powers be very high, &c. I ſhall therefore for Brevity's ſake omit them, and inſtead thereof propoſe an eaſy Method of diſcovering the Roots of all compound Powers in general. And in order to that, it will be neceſſary to premiſe; that if either the Sum or Difference of ſeveral Quantities be involved to any Power, there will ariſe ſo many ſingle Powers of the ſame height, as there are different Quantities.

As for inſtance, if $a+b+d$ be ſquared, that is, be involved to the ſecond Power, it will be $aa+2ab+2ad+bb+2bd+dd$, here you have $aa$, $bb$, and $dd$. Again, if $a+b+d$ were cubed, *viz.* involved to the third Power, then you will have $aaa$, $bbb$, $ddd$, in it, &c.

Y Whence

Whence it follows that in extracting the Roots of all compound Quantities, there muſt be conſidered,

1. How many different Letters (or Quantities) there are in the given Power.

2. Whether the ſingle Powers of each of thoſe Letters be of an equal Height, and have in them ſuch a ſingle Root as is required: which if they have, extract it as before.

3. Connect thoſe ſingle Roots together with the Sign $+$, and involve them to the ſame Height with the given Power; that being done, compare the new raiſed Power with the given Power; and if they are alike in all their reſpective Terms, then you have the Root required; or if they differ only in their Signs, the Root may be eaſily corrected with the Sign $-$ as occaſion requires.

*Example* 1. Let be required to extract the Square Root of $cc+2cb-2cd+bb-2bd+dd$. In this Compound Square, there are three diſtinct Powers, *viz.* $bb$, $cc$, $dd$, whoſe ſingle Roots are $b$, $c$, $d$, wherefore I ſuppoſe the Root ſought to be $b+c+d$, or rather $b+c-d$, becauſe in the given Power there is $-2cd$, and $-2bd$, therefore I conclude it is $-d$; then $b+c-d$, being ſquared, produces $bb+2bc-2bd+cc-2cd+dd$, which I find to be the ſame in all it's Terms with the given Power, although they ſtand in a different Poſition; conſequently $b+c-d$ is the true Root required.

*Example* 2. It is required to extract the Square Root of $a^4-2aabb+b^4$. Here are but two ſingle Powers, *viz.* $a^4$ and $b^4$, whoſe Square Roots are $aa$, and $bb$. And becauſe in the given Power there is $-2aabb$, therefore I conclude it muſt either be $aa-bb$, or $bb-aa$. Both which, being involved, will produce $a^4-2aabb+b^4$; conſequently the Root ſought may either be $aa-bb$, or $bb-aa$, according to the Nature or Deſign of the Queſtion from whence the given Power was produced.

*Example* 3. Let it be required to extract the Square Root of $36aaaa+108aa+81$. Here the two ſingle Powers are $36aaaa$, and $81$, whoſe Roots are $6aa$ and $9$. And becauſe the Signs are all $+$, therefore I ſuppoſe the Root to be $6aa+9$, the which being involved doth produce $36a^4+108aa+81$; conſequently $6aa+9$ is the true Root required.

*Example* 4. Suppoſe it were required to extract the Cube Root of $125aaa+300aae-450aa+250aee-720ae+64eee+540a-288ee+432e-216$. In this Example there are three diſtinct Powers, *viz.* $125aaa$, $64eee$, and $-216$. And the Cube Root of $125aaa$ is $5a$; of $64eee$ is $4e$; of $-216$ is $-6$. Wherefore I ſuppoſe the Root ſought to be $5a+4e-6$, which being involved to the third Power, does produce the

the ſame with the given Power; conſequently $5a + 4e - 6$ is the Cube Root required.

But if the new Power, raiſed from the ſuppoſed Root (being involved to it's due Height), ſhould not prove the ſame with the given Power, *viz.* if it hath either more or fewer Terms in it, *&c.* then you may conclude the given Power to be a Surd, which muſt have it's proper Sign prefixed to it, and cannot be otherwiſe expreſſed, until it come to be envolved in Numbers.

*Example* 5. Suppoſe it were required to extract the Cube Root of $27aaa + 54baa + 8bbb$. Here are two diſtinct and perfect Cubes, *viz.* $27aaa$, and $8bbb$, whoſe Cube Roots are $3a$ and $2b$. Wherefore one may ſuppoſe the Root ſought to be $3a + 2b$, which being involved to the third Power, is $27aaa + 54baa + 36bba + 8bb$. Now this new raiſed Power hath one Term (*viz.* $36bba$) more in it than the given Power hath; but this being a perfect Cube, one may therefore conclude the given Power is not ſo, *viz.* it is a Surd, and hath not ſuch a Root as was required, but muſt be expreſſed, or ſet down,

$$\text{Thus } \sqrt[3]{27aaa + 54baa + 8bbb}.$$

If theſe Examples be well underſtood, the Learner will find it very eaſy by this Method of proceeding to diſcover the true Root of any given Power whatſoever.

## CHAP. III.

## Of Algebraick Fractions, or Broken Quantities.

### Sect. 1. Notation of Fractional Quantities.

FRactional *Quantities* are expreſſed or ſet down like Vulgar Fractions in common *Arithmetick.*

$$\text{Thus} \left\{ \frac{a}{b},\ \frac{2bc}{d},\ \frac{5b - 4a}{4d + 7b} \right. \begin{matrix}\text{Numerators.}\\ \text{Denominators.}\end{matrix}$$

How they come to be ſo, ſee *Caſe* 4, in the laſt Chapter of *Diviſion.* Theſe Fractional Quantities are managed in all reſpects like Vulgar Fractions in Common *Arithmetick.*

## Sect. 2. *To* alter *or* change *different* Fractions *into one Denomination, retaining the same Value.*

### RULE.

*MULTIPLY all the Denominators into each other for a new Denominator, and each Numerator into all the Denominators but it's own for new Numerators.*

### *EXAMPLES.*

Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ into one Denomination.

First $a \times c$, and $d \times b$, will be the Numerators, and $b \times c$ will be the common Denominator, *viz.* $\frac{ca}{bc}$ and $\frac{bd}{bc}$ are the two Fractions required: that is, $\frac{ca}{bc} = \frac{a}{b}$, and $\frac{bd}{bc} = \frac{d}{c}$.

Again, let $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought in one Denomination, and they will be $\frac{bb+bc-bd-dc}{ba+bb-da-bd}$, and $\frac{ad-ac+bd-bc}{ba+bb-da-bd}$. &c.

---

## Sect. 3. *To* bring *whole* Quantities *into* Fractions *of a given Denomination.*

### RULE.

*MULTIPLY the whole Quantities into the given Denominator for a Numerator, under which subscribe the given Denominator, and you will have the Fraction required.*

### *EXAMPLES.*

Let it be required to bring $a+b$ into a Fraction, whose Denominator is $d-a$. First $\overline{a+b} \times \overline{d-a}$ is $da+bd-aa-ba$: Then $\frac{da+bd-aa-ba}{d-a}$ is the Fraction required.

Again $b+\frac{a}{d}$ will be $\frac{db+a}{d}$. And $\frac{aa}{d}-a$ will be $\frac{aa-da}{a}$.

Also $a+b+\frac{aa+bb}{a-b}$ will be $\frac{2aa}{a-b}$.

When

When whole Quantities are to be set down Fraction-wise, subscribe an Unit for the Denominator. Thus $ab$ is $\frac{ab}{1}$. And $aa - bb$, is $\frac{aa - bb}{1}$, &c.

## Sect. 4. *To* abbreviate, *or* reduce *Fractional Quantities into their lowest Denomination.*

RULE.

*DIVIDE both the Numerator and Denominator by their greatest common Divisor*, viz. *by such Quantities as are found in both; and their Quotients will be the Fraction in it's lowest Term.*

Thus $\frac{aac}{dc}$ is $\frac{aa}{d}$. $\frac{abbb}{abc}$ is $\frac{bb}{c}$. And $a + \frac{bdc}{bc} = a + d$.

In such single Fractions as these, the common Divisors (if there be any) are easily discovered by Inspection only; but in compound Fractions it often proves very troublesome, and must be done either by dividing the Numerator by the Denominator, until nothing remains, when that can be done: or else finding their common Measure, by dividing the Denominator by the Numerator, and the Numerator by the Remainder, and so on, as in Vulgar Fractions. (Sect. 4. Page 51.)

*EXAMPLES.*

Suppose $\frac{aac - aad}{cd - dd}$ were to be reduced lower.

Then $cd - dd)\ aac - aad\ \Big(\frac{aa}{d}$ the Fraction required.

$$\underline{aac - aad}$$

$$0 \quad 0$$

In this Example it so happens that the Numerator is divided just off by the Denominator; but in the next it is otherwise, and requires a double Division to find out the common Measure, *viz.*

Let it be required to reduce $\frac{aaa - abb}{aa + 2ab + bb}$ to it's lowest Terms.

First $aa + 2ab + bb)\ aaa - abb\ (a$

$$\underline{aaa + 2aab + abb}$$

$$-2aab - 2abb \text{ the Remainder.}$$

Then $-2aab - 2abb)\ aa + 2ab + bb\ \Big(-\frac{1}{2b} - \frac{1}{2a}$ ☞

$$\underline{aa + ab}$$

$$ab + bb$$

$$\underline{ab + bb}$$

$$0 \quad 0$$

Hence

Hence it appears that $-2aab-2abb$ is the common Meaſure; by which $aaa-abb$ being divided,

$$Viz.\ -2aab-2abb)\ aaa-abb\ \left(-\frac{a}{2b}+\frac{1}{2}\right.$$
$$\begin{array}{r} aaa+aab \\ \hline -aab-abb \\ -aab-abb \\ \hline 0 \quad\ 0 \end{array}$$

Then $-\frac{a}{2b}+\frac{1}{2}$ is the new Numerator; and ☞ $-\frac{1}{2b}-\frac{1}{2a}$ is the new Denominator. But $-\frac{a}{2b}+\frac{1}{2}=\frac{-2a+2b}{4b}=\frac{-a+b}{2b}$ the Numerator; and $-\frac{1}{2b}-\frac{1}{2a}=\frac{-2a-2b}{4ba}=\frac{-a-b}{2ba}$ the Denominator. Let both be multiplied with $2ba$, and you will have $\frac{-aa+ab}{-a-b}$ the Numerator. the Denominator. Or changing the Signs of all the Quantities, it will be $\frac{aa-ab}{a+b}$ the new Fraction required. That is, $\frac{aa-ab}{a+b}=\frac{aaa-abb}{aa+2ab+bb}$.

Again, let it be required to reduce $\frac{dd-bb}{ddd-bbb}$.

The common Meaſure of this Fraction will be the eaſieſt found (as appears from Trials) by dividing the Denominator by the Numerator, &c. Thus,

$$dd-bb)\ ddd-bbb\ (d$$
$$\begin{array}{r} ddd-bbd \\ \hline +bbd-bbb)\ dd-bb\ \left(\frac{d}{bb}\right. \\ dd-bd \\ \hline +bd-bb\ \left(bbd-b^3\ b\right. \\ bbd-b^3 \\ \hline 0 \quad\ 0 \end{array}$$

Hence it appears that $bd-bb$ is the common Meaſure that will divide both the Numerator and the Denominator.

Conſequently

Consequently $bd - bb)\ dd - bb\ \left(\frac{d}{b} + 1\right.$, is the new Numerator.

$$\begin{array}{r} dd - db \\ \hline + db - bb \\ db - bb \\ \hline 0 \quad 0 \end{array}$$

And $bd - bb)\ ddd - bbb\ \left(\frac{dd}{b} + d + b\right.$ the new Denominator.

$$\begin{array}{r} ddd - ddb \\ \hline + ddb - bbb \\ ddb - bbd \\ \hline + bbd - bbb \\ bbd - bbb \\ \hline 0 \quad 0 \end{array}$$

Let both be multiplied with $b$, and then you will have

$\frac{d + b}{dd + bd + bb}$ the Numerator, the Denominator, } of the Fraction required.

But if after all Means used (as above) there cannot be found one common Measure to both the Numerator and Denominator; then is that Fraction in it's least Terms already.

*Note*, These Operations will be understood by a Learner after he hath passed thro' *Multiplication*, and *Division* of Fractions.

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## Sect. 5. Addition *and* Substraction *of Fractional* Quantities.

THE given Fractions being of one Denomination, or if they are not, make them so, per Sect. 4. Then,

### RULE.

*Add or substract their Numerators, as Occasion requires, and to their Sum or Difference, subscribe the common Denominator: as in Vulgar Fractions.*

### *Examples in* Addition.

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 | $\frac{bb}{c}$ | $\frac{a+b}{d}$ | $\frac{2a-b}{d+c}$ | | $\frac{a-b+d}{d+a}$ |
| | 2 | $\frac{aa}{c}$ | $\frac{2a+c}{d}$ | $\frac{2b-a}{d+c}$ | | $\frac{a+b-d}{d+a}$ |
| 1+2 | 3 | $\frac{bb+aa}{c}$ | $\frac{3a+b+c}{d}$ | $\frac{a+b}{d+c}$ | | $\frac{2a}{d+a}$ |

*Examples*

### *Examples in* Subſtraction.

| | | | | | |
|---|---|---|---|---|---|
| | 1 | $\frac{bb+aa}{c}$ | $\frac{a+b}{d+c}$ | $\frac{3a+b+c}{d}$ | $\frac{2b}{d+a}$ |
| | 2 | $\frac{bb}{c}$ | $\frac{2b-a}{d+c}$ | $\frac{2a+c}{d}$ | $\frac{a+b-d}{d+a}$ |
| 1 − 2 | 3 | $\frac{aa}{c}$ | $\frac{2a-b}{d+c}$ | $\frac{a+b}{d}$ | $\frac{b-a+d}{d+a}$ |

## Sect. 6. Multiplication *of Fractional Quantities.*

FIRST prepare mixed Quantities (if there be any) by making them improper Fractions, and whole Quantities by ſubſcribing an Unit under them; as per Sect. 3. Then,

### RULE.

*Multiply the Numerators together for a new Numerator, and the Denominators together for a new Denominator; as in Vulgar Fractions.*

| | | | |
|---|---|---|---|
| Thus | 1 | $\frac{ab}{c}$ | $\frac{3a-2b}{2d+c}$ |
| | 2 | $\frac{d}{f}$ | $\frac{4a+2b}{d}$ |
| 1 × 2 | 3 | $\frac{abd}{cf}$ | $\frac{12aa-2ab-4bb}{2dd+dc}$ |

Suppoſe it were required to multiply $2a+\frac{b}{c}-25$, with $3b+4c$. Theſe prepared for the Work (per Sect. 3.) will ſtand

| | | |
|---|---|---|
| Thus | 1 | $\frac{2ac+b-25c}{c}$ |
| | 2 | $\frac{3b+4c}{1}$ |
| 2 × 2 | 3 | $\frac{6bac+3bb-75bc+8acc+4bc-100cc}{c}$ |
| or | 4 | $6ba-71b+8ac-100c+\frac{3bb}{c}$ per Sect. 4. |

N. B.

*N. B.* Any Fraction is multiplied with it's Denominator by casting off, or taking the Denominator away. Thus $\frac{b}{a} \times a$ gives $b$. For $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{1} = b$, &c.

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## Sect. 7. Division *of Fractional Quantities.*

THE Fractional Quantities being prepared, as directed in the last Section. Then,

### RULE.

*Multiply the Numerator of the Dividend, into the Denominator of the Divisor, for a new Numerator; and multiply the other two together for a new Denominator; as in Vulgar Fractions.*

### *EXAMPLES.*

Let $\frac{abd}{cf}$ be divided by $\frac{ab}{c}$, the Work may stand thus,

$\frac{ab}{c}\Big)\ \frac{abd}{cf}\ \Big(\frac{abdc}{abcf} = \frac{d}{f}$ per Sect. 4.

| Or thus | | | | |
|---|---|---|---|---|
| | 1 | $\frac{abd}{cf}$ | $\frac{a+b}{d}$ | $\frac{aaa-bbb}{a+b}$ |
| | 2 | $\frac{ab}{c}$ | $\frac{c-b}{a}$ | $\frac{aa-ab+bb}{c}$ |
| 1 ÷ 2 | 3 | $\frac{d}{f}$ | $\frac{aa+ba}{dc-db}$ | $\frac{aaac-bbbc}{aaa+bbb}$ |

Suppose it were required to divide $aa + \frac{3abb}{a+4b}$ by $a+b$. The Work prepared will stand thus,

$\frac{a+b}{1}\Big)\frac{aaa+4aab+3abb}{a+4b}\Big(\frac{aaa+4aab+3abb}{aa+5ba+4bb}$. But $\frac{aaa+4aab+3abb}{aa+5ba+4bb} = \frac{aa+3b}{a+4b}$ (per Sect. 4.)

When Fractions are of one Denomination, cast off the Denominators, and divide the Numerators. Thus, if $\frac{ab^3}{c}$ were to be divided by $\frac{bb}{c}$ it will be $bb)\ ab^3\ (ab$ the Quotient required.

For $\frac{bb}{c}\Big)\frac{ab^3}{c}\Big(\frac{ab^3c}{bbc}$. But $\frac{ab^3c}{bbc} = ab$ (per Sect. 4.)

Again, suppose it were required to divide $\frac{a^3 - abb}{c-d}$ by $\frac{aa + 2ab + bb}{c-d}$. Casting off $c - d$ in both, it will be $aa + 2ab + bb)\,aaa - abb\,\Big(\frac{aa - ba}{a+b}$, &c.

## Sect. 8. Involution *of Fractional Quantities.*

RULE.

*INVOLVE the Number into itself for a new Numerator, and the Denominator into itself for a new Denominator; each as often as the Power requires.*

| | | | | |
|---|---|---|---|---|
| Thus | 1 | $\frac{b}{a}$ | $\frac{3bc}{2ad}$ | $\frac{b+d}{a-c}$ |
| 1 ☉² | 2 | $\frac{bb}{aa}$ | $\frac{9bbcc}{4aadd}$ | $\frac{bb + 2bd + dd}{aa - 2ac + cc}$ |
| 1 ☉³ | 3 | $\frac{bbb}{aaa}$ | $\frac{27bbbccc}{8aaaddd}$ | $\frac{bbb + 3bbd + 3bdd + ddd}{aaa - 3aac + 3acc - ccc}$ |

## Sect. 9. Evolution *of Fractional Quantities.*

IF the Numerator and Denominator of the Fraction have each of them such a Root as is required (which very rarely happens) then evolve them; and their respective Roots will be the Numerator and Denominator of the new Fraction required.

| | | | |
|---|---|---|---|
| Thus | 1 | $\frac{9aabb}{4dd}$ | $\frac{aa + 2ab + bb}{aa - 2ab + bb}$ |
| 1 ω² | 2 | $\frac{3ab}{2d}$ | $\frac{a+b}{a-b}$ |

| | | | |
|---|---|---|---|
| Again | 1 | $\frac{27aaabbb}{8ddd}$ | $\frac{aaa + 3aab + 3abb + bbb}{aaa - 3aab + 3abb - bbb}$ |
| 1 ω³ | 2 | $\frac{3ab}{2d}$ | $\frac{a+b}{a-b}$ |

Sometimes it so falls out, that the Numerator may have such a Root as is required, when the Denominator hath not; or the Denominator may have such a Root, when the Numerator hath not. In those Cases the Operations may be set down

| | | | |
|---|---|---|---|
| Thus | 1 | $\frac{aabb}{ddd}$ | $\frac{aaa + 4bb - dd}{aa - 2ab + bb}$ |
| $1\ ww^{2}$ | 2 | $\frac{ab}{\sqrt{ddd}}$ | $\frac{\sqrt{aaa + 4bb - dd}}{a + b}$ |

But when neither the Numerator, nor the Denominator have just such a Root as is required, prefix the radical Sign of the Root to the Fraction; and then it becomes a Surd, as in the last Step; which brings me to the Business of managing Surds.

## CHAP. IV.

## Of Surd Quantities.

THE whole Doctrine of Surds (as they call it) were it fully handled, would require a very large Explanation (to render it but tolerably intelligible); even enough to fill a Treatise itself, if all the various Explanations that may be of Use to make it easy should be inserted; without which it is very intricate and troublesome for a Learner to understand. But now these tedious Reductions of Surds, which were heretofore thought useful to fit Equations for such a Solution, as was then understood, are wholly laid aside as useless: Since the new Methods of resolving all sorts of Equations render their Solutions equally easy, although their Powers are never so high. Nay, even since the true Use of Decimal *Arithmetick* hath been well understood, the Business of Surd Numbers has been managed that Way; as appears by several Instances of that Kind in Dr *Wallis*'s *History of Algebra*, from *Page* 23, to 29.

I shall therefore, for Brevity sake, pass over those tedious Reductions, and only shew the young *Algebraist* how to deal with such Surd Quantities as may arise in the Solution of hard Questions.

### Sect. 1. Addition and Subtraction of Surd Quantities.

*Case* 1. WHEN the Surd Quantities are Homogeneal, (*viz.* are alike) add or subtract the rational Part, if they are joined to any, and to their Sum, or Difference, adjoin the irrational or Surd.

### *Examples in* Addition.

| | | | | |
|---|---|---|---|---|
| | 1 | $5\sqrt{bc}$ | $6b\sqrt{ac}$ | $b\sqrt{aa+cc}$ |
| | 2 | $7\sqrt{bc}$ | $4b\sqrt{ac}$ | $3b\sqrt{aa+cc}$ |
| 1+2 | 3 | $12\sqrt{bc}$ | $10b\sqrt{ac}$ | $4b\sqrt{aa+cc}$ |

| | | | | |
|---|---|---|---|---|
| | 1 | $4d\sqrt[3]{aa}$ | $b+\sqrt[3]{aa-cc}$ | $bc:\sqrt[5]{aa+d}$ |
| | 2 | $d\sqrt[3]{aa}$ | $c-\sqrt[3]{aa-cc}$ | $3bc:\sqrt[5]{aa+d}$ |
| 1+2 | 3 | $5d\sqrt[3]{aa}$ | $b+c$ | $4bc:\sqrt[5]{aa+d}$ |

### *Examples in* Substraction.

| | | | | |
|---|---|---|---|---|
| | 1 | $12\sqrt{bc}$ | $10b\sqrt{ac}$ | $4b\sqrt{aa+cc}$ |
| | 2 | $7\sqrt{bc}$ | $4b\sqrt{ac}$ | $3b\sqrt{aa+cc}$ |
| 1—2 | 3 | $5\sqrt{bc}$ | $6b\sqrt{ac}$ | $b\sqrt{aa+cc}$ |

| | | | | |
|---|---|---|---|---|
| | 1 | $5d:\sqrt[3]{aa}$ | $b+c$ | $4bc:\sqrt[5]{aa+d}$ |
| | 2 | $4d:\sqrt[3]{aa}$ | $c-\sqrt[3]{aa-cc}$ | $3bc:\sqrt[5]{aa+d}$ |
| 1—2 | 3 | $d:\sqrt[3]{aa}$ | $b+\sqrt[3]{aa-cc}$ | $bc:\sqrt[5]{aa+d}$ |

*Case* 2. When the Surd Quantities are Heterogeneal, (*viz.* their Indices are unlike) they are only to be added, or substracted by their Signs, *viz.* + or —. And from thence will arise Surds either Binomial, or Residual.

### *Examples in* Addition.

| | | | | |
|---|---|---|---|---|
| | 1 | $\sqrt{bc}$ | $4d\sqrt{a}$ | $\sqrt[3]{ac-ba}$ |
| | 2 | $\sqrt{ba}$ | $3b\sqrt{ac}$ | $\sqrt{ac+ba}$ |
| 1+2 | 3 | $\sqrt{bc}:+\sqrt{ba}$ | $4d\sqrt{a}:+3b\sqrt{ac}$ | $\sqrt[3]{ac-ba}:+\sqrt{ac+ba}$ |

### *Examples in* Substraction.

| | | | |
|---|---|---|---|
| | 1 | $\sqrt{bc}$ | $b-d\sqrt{aaa+ca}$ |
| | 2 | $\sqrt{ba}$ | $d-2a\sqrt{bd+dd}$ |
| 1—2 | 3 | $\sqrt{bc}-\sqrt{ba}$ | $b-d\sqrt{aaa+ca}:-d+2a\sqrt{bd+dd}$ |

## Sect. 2. Multiplication *of Surd Quantities.*

*Case* 1. WHEN the Quantities are pure Surds of the same Kind; multiply them together, and to their Product prefix their radical Sign.

*EXAMPLES.*

| | | | | |
|---|---|---|---|---|
| | 1 | $\sqrt{b}$ | $\sqrt{ba+da}$ | $\sqrt{aa+bb}$ |
| | 2 | $\sqrt{a}$ | $\sqrt{ca}$ | $\sqrt{aa-bb}$ |
| 1 × 2 | 3 | $\sqrt{ba}$ | $\sqrt{bcaa+dcaa}$ | $\sqrt{aaaa-bbbb}$ |

*Case* 2. If Surd Quantities of the same Kind (as before) are joined to rational Quantities, then multiply the rational into the rational; and the Surd into the Surd, and join their Products together.

*EXAMPLES.*

| | | | | |
|---|---|---|---|---|
| | 1 | $d\sqrt{bc}$ | $5cd\sqrt{ba+da}$ | $15\sqrt{ab}$ |
| | 2 | $3b\sqrt{a}$ | $3a\sqrt{ca}$ | $5\sqrt{a}$ |
| 1 × 2 | 3 | $3db\sqrt{bca}$ | $15cda\sqrt{bcaa-dcaa}$ | $75\sqrt{abd}$ |

## Sect. 6. Division *of Surd Quantities.*

*Case* 1. WHEN the Quantities are pure Surds of the same Kind, and can be divided off, (*viz.* without leaving a Remainder) divide them, and to their Quotient prefix their radical Sign.

*EXAMPLES.*

| | | | | |
|---|---|---|---|---|
| | 1 | $\sqrt{ba}$ | $\sqrt{bcaa+dcaa}$ | $\sqrt{aaaa-bbbb}$ |
| | 2 | $\sqrt{b}$ | $\sqrt{ca}$ | $\sqrt{aa-bb}$ |
| 1 ÷ 2 | 3 | $\sqrt{a}$ | $\sqrt{ba+da}$ | $\sqrt{aa+bb}$ |

*Case* 2. If Surd Quantities, of the same Kind, are joined to rational Quantities; then divide the rational by the rational, if it can be, and to their Quotient join the Quotient of the Surd divided by the Surd with it's first radical Sign.

*EXAMPLES.*

| | | | | |
|---|---|---|---|---|
| | 1 | $3db\sqrt{bca}$ | $15cda\sqrt{bcaa+dcaa}$ | $75\sqrt{abd}$ |
| | 2 | $3b\sqrt{a}$ | $3a\sqrt{ca}$ | $5\sqrt{d}$ |
| 1 ÷ 2 | 3 | $d\sqrt{bc}$ | $5cd\sqrt{ba+da}$ | $15\sqrt{ab}$ |

*Note,*

*Note*, If any Square be divided by it's Root, the Quotient will be it's Root.

*EXAMPLES.*

| | | | | |
|---|---|---|---|---|
| | 1 | $a$ | $bb + 2bc + cc$ | $aaaa - 2bbaa + bbbb$ |
| | 2 | $\sqrt{a}$ | $\sqrt{bb + 2bc + cc}$ | $\sqrt{a^4 - 2bbaa + b^4}$ |
| $1 \div 2$ | 3 | $\sqrt{a}$ | $\sqrt{bb + 2bc + cc}$ | $\sqrt{a^4 - 2bbaa + b^4}$ |

## Sect. 4. Involution *of Surd Quantities.*

*Case* 1. WHEN the Surds are not joined to rational Quantities; they are involved to the same Height as their Index denotes, by only taking away their radical Sign.

*EXAMPLES.*

| | | | | | |
|---|---|---|---|---|---|
| | 1 | $\sqrt{a}$ | $\sqrt{bca}$ | $\sqrt{aa - bb}$ | $\sqrt{5a - da}$ |
| $1 \odot^2$ | 2 | $a$ | $bca$ | $aa - bb$ | $5a - da$ |

*Case* 2. When the Surds are joined to rational Quantities; involve the rational Quantities to the same Height as the Index of the Surd denotes; then multiply those involved Quantities into the Surd Quantities, after their radical Sign is taken away, as before.

*EXAMPLES.*

| | | | | |
|---|---|---|---|---|
| | 1 | $b\sqrt{a}$ | $5d\sqrt{ca}$ | $3b\sqrt{aa - dd}$ |
| $1 \odot^2$ | 2 | $bba$ | $25ddca$ | $9bbaa - 9bbdd$ |
| | 1 | $a : \sqrt[3]{bc}$ | $3d : \sqrt[3]{aa + bb}$ | $da : \sqrt[3]{b}$ |
| $1 \odot^3$ | 2 | $aaabc$ | $27dddaa + 27dddbb$ | $dddaaab$ |

The Reason of only taking away the radical Sign, as in *Case* 1. is easily conceived, if you consider that any Root being involved into itself, produces a Square, &c. And from thence the Reason of those Operations performed by the second Case may be thus stated.

Suppose $b\sqrt{a} = x$. Then $\sqrt{} = \frac{x}{b}$ *per Axiom* 4. and both Sides of the Equation being equally involved, it will be $a = \frac{xx}{bb}$. Then multiplying both Sides of the Equation into $bb$, it becomes $bba = xx$ *per Axiom* 3. Which was to be proved.

Again,

Again, Let $5d\sqrt{ca} = x$: Then $\sqrt{ca} = x\,\frac{x}{5d}$, and $ca = \frac{xx}{25dd}$.

Alſo from hence it will be eaſy to deduce the Reaſon of multiplying Surd Quantities, according to both the *Caſes*. For

| | | |
|---|---|---|
| Suppoſe { | 1 | $\sqrt{b} = z$ } *Example* 1. *Caſe* 1. |
| | 2 | $\sqrt{a} = x$ |
| $1 \circledcirc^{2}$ | 3 | $b = zz$ |
| $2 \circledcirc^{2}$ | 4 | $a = xx$ |
| $3 \times 4$ | 5 | $ba = zzxx$. *per Axiom* 2. |
| $5\,\omega^{2}$ | 6 | $\sqrt{ba} = zx$. which was to be proved. |

| | | |
|---|---|---|
| Let { | 1 | $d\sqrt{bc} = z$ } *Example* 1. *Caſe* 2. |
| | 2 | $3b\sqrt{a} = x$ |
| $1 \div d$ | 3 | $\sqrt{bc} = \frac{z}{d}$ |
| $2 \div 3b$ | 4 | $\sqrt{a} = \frac{x}{3b}$ |
| $4 \times 3$ | 5 | $\sqrt{abc} = \frac{zx}{3bd}$, from what is proved above. |
| $5 \times 3bd$ | 6 | $3bd\sqrt{bca} = zx$, &c. for the reſt. |

*Diviſion* being the Converſe to *Multiplication*, needs no other Proof.

## CHAP. V.

### *Concerning the Nature of* Equations *and how to prepare them for a* Solution.

WHEN any Problem or Queſtion is propoſed to be analytically reſolved; it is very requiſite that the true Deſign or Meaning thereof, be fully and clearly comprehended (in all it's Parts) that ſo it may be truly abſtracted from ſuch ambiguous Words as Queſtions of this Kind are often diſguiſed with; otherwiſe it will be very difficult, if not impoſſible, to ſtate the Queſtion right in it's ſubſtituted Letters, and ever to bring it to an Equation by ſuch various Methods of ordering thoſe Letters as the Nature of the Queſtions may require.

Now

Now the Knowledge of this difficult Part of the Work is only to be obtained by Practice, and a careful minding the Solution of fuch leading Queftions as are in themfelves very eafy. And for that Reafon I have inferted a Collection of feveral Queftions; wherein there is great Variety.

Having got fo clear an Underftanding of the Queftion propofed, as to place down all the Quantities concerned in their due Order, *viz.* all the fubftituted Letters, in fuch Order as their Nature requires; the next thing muft be to confider whether it be limited or not. That is, whether it admits of more Anfwers than one. And to difcover that, obferve the two following Rules.

## RULE 1.

*When the Number of the Quantities fought exceed the Number of the given Equations, the Queftion is capable of innumerable Anfwers.*

## *EXAMPLE.*

Suppofe a Queftion were propofed thus; there are three fuch Numbers, that if the firft be added to the fecond, their Sum will be 22. And if the fecond be added to the third, their Sum will be 46. What are thofe Numbers?

Let the three Numbers be reprefented by three Letters, thus, call the firft $a$, the fecond $e$, and the third $y$.

Then $\left\{ \begin{array}{l} a + e = 22 \\ e + y = 46 \end{array} \right\}$ according to the Queftion.

Here the Number of Quantities fought are three, $a$, $e$, $y$, and the Number of the given Equations are but two. Therefore this Queftion is not limited, but admits of various Anfwers; becaufe for any one of thofe three Letters you may take any Number at Pleafure, that is lefs than 22. Which with a little Confideration will be very eafy to conceive.

## RULE 2.

*When the Number of the given Equations (not depending upon one another) are juft as many as the Number of the Quantities fought; then is the Queftion truly limited,* viz. *each Quantity fought hath but one fingle Value.*

As for Inftance, let the aforefaid Queftion be propofed thus. There are three Numbers ($a$, $e$, and $y$, as before); if the firft be added to the fecond, their Sum will be 22; if the fecond be added

to

to the third, their Sum will be 46; and if the firſt be added to the third, their Sum will be 36. What are the Numbers? That is, $a + e = 22$. $e + y = 46$. and $a + y = 36$. Now the Queſtion is perfectly limited, each ſingle Quantity having but one ſingle Value, to wit $a = 6$, $e = 16$, and $y = 30$.

*N. B.* If the Number of the given Equations exceeds the Number of the Quantities ſought; they not only limit the Queſtion, but oftentimes render it impoſſible, by being propoſed inconſiſtent one to another.

Having truly ſtated the Queſtion in it's ſubſtituted Letters, and found it limited to one Anſwer (or at leaſt ſo bounded as to have a certain determinate Number of Anſwers), then let all thoſe ſubſtituted Letters be ſo ordered or compared together, either by adding, ſubſtracting, multiplying, or dividing them, &c. according as the Nature of the Queſtion requires, until all the unknown Quantities except one, are caſt off or vaniſhed; but therein great Care muſt be taken to keep them to an exact Equality; and when that unknown Quantity, or ſome Power of it (as Square, Cube, &c.) is found equal to thoſe that are known; then the Queſtion is ſaid to be brought to an Equation, and conſequently to a Solution, *viz.* fitted for an Anſwer.

But no particular Rules can be preſcribed for the caſting off, or getting away Quantities out of an Equation; that Part of the Art is only to be obtained by Care and Practice. And when that is done, it generally happens ſo, that the unknown Quantity which is retained in the Equation, is ſo mixed and entangled with thoſe that are known, that it often requires ſome Trouble and Skill to bring it (or it's Powers, &c.) to one Side of the Equation, and thoſe that are known to the other ſide; (ſtill keeping them to a juſt Equality) which the ingenious Mr *Scooten* in his *Principia Matheſeos Univerſalis*, calls Reduction of Equations.

The Buſineſs of reducing Equations (*as of moſt, if not all Algebraick Operations*) is grounded and depends upon a right Application of the five *Axioms* propoſed in Page 146, and therefore, if thoſe *Axioms* be well underſtood, the Reaſon of ſuch Operations muſt needs appear very plain, and the Work be eaſily performed; as in the following *Sections*.

## Sect. 1. *Of Reduction by* Addition.

REDUCTION by *Addition* is grounded upon *Axiom* 1. and is only the tranſpoſing (*viz.* the removing) of any Negative Quantity from either Side of an Equation to the other Side, with the Sign + before it; as in theſe

*EXAMPLES.*

| | | | | | |
|---|---|---|---|---|---|
| Suppoſe | 1 | $a - b = d$ | Again, | | |
| Then | 2 | $a = d + b$ | Let | 1 | $aa - d = c - aa$ |
| For | 3 | $b = d$ | $1 + d$ | 2 | $aa = c - aa + d$ |
| $1 + 3$ | 4 | $a = d + b$ | $2 + aa$ | 3 | $2aa = c + d$ |

| | | |
|---|---|---|
| Let | 1 | $3a - 4 = 6 - a$ |
| $1 + \bar{4}$ | 2 | $3a = 6 + 4 - a$ |
| $2 + a$ | 3 | $4a = 6 + 4 = 10$ |

*Note*, When any abſolute Number is regiſtered in the Margin, you muſt draw a Line over it, to diſtinguiſh it from the other Numbers. As $\bar{4}$ in the 2d Step of this Example.

| | | |
|---|---|---|
| Let | 1 | $aa - dc - b = dd - 2ba$ |
| $1 + b$ | 2 | $aa - dc = dd - 2ba + b$ |
| $2 + dc$ | 3 | $aa = dd - 2ba + b + dc$ |
| $3 + 2ba$ | 4 | $aa + 2ba = dd + b + dc$ |

| | | |
|---|---|---|
| Suppoſe | 1 | $2da - d = cc - 3baa - aaa$ |
| $1 + aaa$ | 2 | $aaa + 2da - d = cc - 3baa$ |
| $2 + 3baa$ | 3 | $aaa + 3baa + 2da - d = cc$ |
| $3 + d$ | 4 | $aaa + 3baa + 2da = cc + d$, &c. |

## Sect. 2. *Of Reduction by* Subſtraction.

REDUCTION by *Subſtraction* is grounded upon *Axiom* 2, and is performed by tranſpoſing (or removing) any Affirmative Quantity from either Side of the Equation, to the other Side, with the Sign — before it; as in theſe

*EXAMPLES.*

| | | | | | |
|---|---|---|---|---|---|
| Suppoſe | 1 | $a + b - d$ | Let | 1 | $3a + 4 = 6 + a$ |
| And | 2 | $b = b$ | $1 - a$ | 2 | $2a + 4 = 6$ |
| $1 - 2$ | 3 | $a = d - b$ | $2 - \bar{4}$ | 3 | $2a = 6 - 4 = 2$ |

| | | |
|---|---|---|
| Suppoſe | 1 | $aa + dc + b = dd + 2ba$ |
| $1 - 2ba$ | 2 | $aa - 2ba + dc + b = dd$ |
| $2 - dc$ | 3 | $aa - 2ba + b = dd - dc$ |
| $3 - b$ | 4 | $aa - 2ba = dd - dc - b$ |

| | | |
|---|---|---|
| Let | 1 | $aaa + d = cc + 3baa + 2da$ |
| $1 - 3baa$ | 2 | $aaa - 3baa + d = cc + 2da$ |
| $2 - 2da$ | 3 | $aaa - 3baa - 2da + d = cc$ |
| $3 - d$ | 4 | $aaa - 3baa - 2da = cc - d$ |

## Sect. 3. *Of Reduction by* Multiplication.

FRACTIONAL Quantities, in any Equation, are brought into whole Quantities by multiplying every Term in the Equation with the Denominators of the Fractions, *per Axiom* 3; as in these

### *EXAMPLES.*

| | | |
|---|---|---|
| Suppose | 1 | $\frac{a}{5} = 6$ |
| Then | 2 | $a = 6 \times 5 = 30$. For $\frac{a}{5} \times 5 = \frac{5a}{5} = a$. |

| | | | | | |
|---|---|---|---|---|---|
| Let | 1 | $3a = \frac{dc}{2b}$ | Suppose | 1 | $a = \frac{dd}{a - b}$ |
| $1 \times 2b$ | 2 | $6ba = dc$ | $1 \times \overline{a - b}$ | 2 | $aa - ba = dd$ |

| | | |
|---|---|---|
| Suppose | 1 | $\frac{aa}{b} + c + f = \frac{dx}{a}$ |
| $1 \times b$ | 2 | $aa + bc + bf = \frac{axb}{a}$ |
| $2 \times a$ | 3 | $aaa + bca + bfa = dxb$ |

| | | |
|---|---|---|
| Suppose | 1 | $\frac{aaa}{aa - bb} = \frac{ba - bb}{a + b}$ |
| $1 \times \overline{aa - bb}$ | 2 | $aaa = \frac{baaa - bbaa - bbba + bbbb}{a + b}$ |
| $1 \times \overline{a + b}$ | 3 | $aaaa + baaa = baaa - bbaa - bbba + bbbb$ |

## Sect. 4. *Of Reduction by* Division.

WHEN any Quantity (either known or unknown) is in every Term of an Equation, if the whole Equation be divided by that Quantity, it will be reduced into lower Terms, *per Axiom* 4, as in these following *Examples*.

*EXAMPLES.*

| | | |
|---|---|---|
| Suppose | 1 | $baa + bca = bcd$ |
| $1 \div b$ | 2 | $aa + ca = cd$ |

| | | |
|---|---|---|
| Let | 1 | $aa = 7a$ |
| $1 \div 1a$ | 2 | $a = 7$ |

| | | |
|---|---|---|
| Let | 1 | $ffaa + ffcaa - ffa = ffda + ffdda$ |
| $1 \div ff$ | 2 | $aa + caa - a = da + dda$ |
| $2 \div a$ | 3 | $a + ca - 1 = d + dd$ |

Or when the unknown Quantity is multiplied (*viz.* joined) with any that is known; let the whole Equation be divided by the known Quantity, that so the unknown may be cleared; as in these

*EXAMPLES.*

| | | |
|---|---|---|
| Suppose | 1 | $ba - ca = d$ |
| $1 \div \overline{b - c}$ | 2 | $a = \frac{d}{b - c}$ |

| | | |
|---|---|---|
| Let | 1 | $caa - daa = cd - dd$ |
| $1 \div \overline{c - d}$ | 2 | $aa = \frac{cd - dd}{c - d} = d.$ |

| | | |
|---|---|---|
| Suppose | 1 | $bbaaa - 2bbaa = bda + cba$ |
| $1 \div ba$ | 2 | $baa - 2ba = d + c$ |
| $2 \div b$ | 3 | $aa - 2a = \frac{d + c}{b}$ |

| | | |
|---|---|---|
| Let | 1 | $49daa + 42aa = 7bca + 21ca$ |
| $1 \div \overline{7}$ | 2 | $7daa + 6aa = bca + 3ca$ |
| $2 \div a$ | 3 | $7da + 6a = bc + 3c$ |
| $3 \div a$ | 4 | $a = \frac{bc + 3c}{7d + 6}$ |

## Sect. 5. *Of Reduction by* Involution.

WHEN there happens to be an Equation, between any homogeneal or like Surds, take away the radical Signs from the Quantities, and they will become rational; as in these

*EXAMPLES.*

| | | |
|---|---|---|
| Suppose | 1 | $\sqrt{a} = \sqrt{d + c}$ |
| $1 \circledast^{2}$ | 2 | $a = d + c$ |

| | | |
|---|---|---|
| Let | 1 | $\sqrt[3]{aa} = \sqrt[3]{db + bc}$ } *per Sect.* 4. |
| $1 \circledast^{3}$ | 2 | $aa = db + bc$ } *Chap* 3. |

Or if one Side of the Equation consists of Surd Quantities, and the other Side be rational, then involve the rational Quantities to the

the fame Power (or Height) with the Index of the Surd, and take away the radical Sign; as in thefe

*EXAMPLES.*

| | | | | | |
|---|---|---|---|---|---|
| Let | 1 | $\sqrt{a}=6$ | Suppofe | 1 | $\sqrt{a}=b+c$ |
| 1 ⓔ$^2$ | 2 | $a=36$ | 1 ⓔ$^2$ | 2 | $a=bb+2bc+cc$ |
| Suppofe | 1 | $\sqrt[3]{aa-ba}=d$ | Let | 1 | $\sqrt[5]{aa}=7$ |
| 1 ⓔ$^3$ | 2 | $aa-ba=ddd$ | 1 ⓔ$^5$ | 2 | $aa=16807.$ |

## Sect. 6. *Of Reduction by* Evolution.

WHEN any fingle Power of the unknown Quantity is on one Side of an Equation; evolve both Sides of the Equation, according as the Index of that Power denotes, and their Roots will be equal; as in thefe

*EXAMPLES.*

| | | | | | |
|---|---|---|---|---|---|
| Suppofe | 1 | $aa=36$ | Let | 1 | $aaa=27$ |
| 1 ɯ$^2$ | 2 | $a=\sqrt{36}=6$ | 1 ɯ$^3$ | 2 | $a=\sqrt[3]{27}=3$, &c. |
| Suppofe | 1 | $aa=bb-dd$ | Let | | $aaa=b^3+3bbc+3bcc+c^3$ |
| 1 ɯ$^2$ | 2 | $a=\sqrt{bb-dd}$ | 1 ɯ$^3$ | | $a=b+c$ |

Or if any compound Power of the unknown Quantity be on one Side of the Equation (that hath a true Root of it's kind) evolve both Sides of the Equation, and it will be depreffed into lower Terms; as in thefe

*EXAMPLES.*

| | | | |
|---|---|---|---|
| Suppofe | 1 | $aa+2ba+bb=dd$ | $aa-2ba+bb=ddcc$ |
| 1 ɯ$^2$ | 2 | $a+b=d$ | $a-b=dc$ |

Here follow a few Examples of clearing Equations, wherein all the foregoing Reductions are promifcuoufly ufed, as Occafion requires.

*EXAMPLE* 1.

| | | |
|---|---|---|
| Suppofe | 1 | $\frac{aa+c-d}{4}=\frac{g-aa}{b}$, what is $a=$ to? |
| $1\times4$ | 2 | $aa+c-d=\frac{4g-4aa}{b}$ |

| | | |
|---|---|---|
| $2 \times b$ | 3 | $baa + bc - bd = 4g - 4aa$ |
| $3 + 4aa$ | 4 | $baa + 4aa + bc - bd = 4g$ |
| $4 + bd$ | 5 | $baa + 4aa + bc = 4g + bd$ |
| $5 - bc$ | 6 | $baa + 4aa = 4g + bd - bc$ |
| $6 \div \overline{b+4}$ | 7 | $aa = \frac{4g + bd - bc}{b+4}$ |
| 7 ɯ$^{2}$ | 8 | $a = \sqrt{\frac{4g + bd - bc}{b+4}}$ as was required. |

## EXAMPLE 2.

| | | |
|---|---|---|
| Suppose | 1 | $\frac{a+354}{a} = \frac{3a}{354-a}$ what is the Value of $a$? |
| $1 \times a$ | 2 | $a + 354 = \frac{3aa}{354-a}$ |
| $2 \times \overline{353 - a}$ | 3 | $125316 - aa = 3aa$ |
| $2 + aa$ | 4 | $4aa = 125316$ |
| $4 \div 4$ | 5 | $aa = 31329$ |
| 5 ɯ$^{2}$ | 6 | $a = \sqrt{31329} = 177$, the Value of $a$ required. |

## EXAMPLE 3.

| | | |
|---|---|---|
| Suppose | 1 | $\sqrt{\frac{aa+3bb}{4}} - \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}} : a = ?$ |
| 1 ⊙$^{2}$ | 2 | $\left\{ \frac{aa+3bb}{4} - 2\sqrt{\frac{aa+3bb}{4}} \times \sqrt{\frac{aa-3bb}{4}} : + \frac{aa-3bb}{4} = \frac{baa}{c} \right.$ |
| That is | 3 | $\frac{aa}{2} - \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}$ |
| For | | $\frac{aa+3bb}{4} + \frac{aa-3bb}{4} = \frac{2aa}{4} = \frac{aa}{2}$ |
| And | | $2\sqrt{\frac{aa+3bb}{4}} = \sqrt{\frac{4aa+12bb}{4}} = \sqrt{aa+3bb}$ |
| Then | | $\sqrt{aa+3bb} \times \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{a^4-9b^4}{4}}$ |
| $3 + \sqrt{}$ &c. | 4 | $\frac{aa}{2} = \frac{baa}{c} + \sqrt{\frac{a^4-9b^4}{4}}$ |

4 —

| | | |
|---|---|---|
| $4 - \frac{baa}{c}$ | 5 | $\frac{aa}{2} - \frac{baa}{c} = \sqrt{\frac{a^4 - 9b^4}{4}}$ |
| $5 \odot^2$ | 6 | $\frac{a^4}{4} - \frac{ba^4}{c} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4}$ |
| $6 + \frac{ba^4}{c}$ | 7 | $\frac{a^4}{4} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4} + \frac{ba^4}{c}$ |
| $7 \pm$ | 8 | $\frac{bba^4}{cc} + \frac{9b^4}{4} = \frac{ba^4}{c}$ |
| $8 \div b$ | 9 | $\frac{ba^4}{cc} + \frac{9b^3}{4} = \frac{a^4}{c}$ |
| $9 \times cc$ | 10 | $ba^4 + \frac{9ccb^3}{4} = ca^4$ |
| $10 \times 4$ | 11 | $4ba^4 + 9ccb^3 = 4ca^4$ |
| $11 - 4ba^4$ | 12 | $9ccb^3 = 4ca^4 - 4ba^4$ |
| $12 \div$ | 13 | $aaaa = \frac{9ccb^3}{4c - 4b}$ |
| For | | $\overline{4c - 4b} \times a^4 = 4ca^4 - 4ba^4$ |
| $13\ \omega^2$ | 14 | $aa = \sqrt{\frac{9ccb^3}{4c - 4b}}$ |
| $14\ \omega^2$ | 15 | $a = \sqrt{:}\sqrt{\frac{9ccb^3}{4c - 4b}}$, as was required. |

By Help of these Reductions (properly applied) the unknown Quantity ($a$) or it's Powers, are cleared and brought to one Side of an Equation; and if the unknown Quantity ($a$) chance to be equal to those that are known, the Question is answered: as in the first *Example* of *Sect.* 1, and 2. Or if any single Power of the unknown Quantity ($a$) is found equal to those that are known, then the respective Root of the known Quantities is the Answer; as in the first four *Examples* of *Sect.* 6, &c.

But when the Powers of the unknown Quantities are either mixed with their Root, as $aa + ba = dd$, &c; or do consist of different Powers, as $aaa + baa = dd$, &c: Then they are called Affected, or Adfected Equations, which require other Methods to resolve them; *viz.* to find out the Value of ($a$) as shall be shewed further on.

CHAP.

## CHAP. VI.

## Of Proportional Quantities; both Arithmetical, Geometrical, and Musical.

WHAT hath been ſaid of Numbers in *Arithmetical Progreſſion*, Chap. 6. Part I. may be eaſily applied to any Series of Homogeneal or like Quantities.

### Sect. I. Of Quantities in Arithmetical Progreſsion.

THOSE Quantities are ſaid to be in the moſt ſimple or natural Progreſſion, that begin their Series of increaſe or decreaſe with a Cypher:

Thus $\begin{cases} 0 : a : 2a : 3a : 4a : 5a : 6a : \text{\&c. increaſing.} \\ 0 : -a : -2a : -3a : -4a : -5a : -6a : \text{\&c. decreaſing.} \end{cases}$

Or Univerſally, putting $a$ the firſt Term in the Progreſſion, and $e$ the common Exceſs or Difference.

Then $\begin{cases} a : a+e : a+2e : a+3e : a+4e : a+5e : a+6e : \text{\&c.} \\ a : a-e : a-2e : a-3e : a-4e : a-5e : a-6e : \text{\&c.} \end{cases}$

In the firſt of theſe Series it is evident, that if there be but three Terms; the Sum of the Extreams will be double to the Mean.

As in theſe, $0 : a : 2a$ : or, $a : 2a : 3a$ : or, $2a : 3a : 4a$, &c. *viz.* $2a : +0 = a + a$ : or, $a + 3a = 2a + 2a$, &c.

Alſo, in the ſecond Series, either increaſing or decreaſing, it is evident, that if the Terms be $a : a+e : a+2e$, &c. increaſing; then $a + a + 2e$, *viz.* $2a + 2e$ the Sum of the Extreams, is double to $a+e$ the Mean, or if they be $a : a-e : a-2e$, &c. decreaſing; then $a + a - 2e$ : *viz.* $2a - 2e$, the Sum of the Extreams, is double to $a - e$ the Mean. And ſo it will be in any other three of the Terms. Secondly, if there are four Terms; then the Sum of the two Extreams, will be equal to the Sum of the two Means; as in theſe, $a : a+e : a+2e : a+3e$, in the Series increaſing; here $a + a + 3e = a + e + a + 2e$.

Alſo in theſe, $a : a-e : a-2e : a-3e$, in the Series decreaſing; here $a + a - 3e = a - e + a - 2e$, &c. in any other four Terms.

*Conſequently*, If there are never ſo many Terms in the Series, the Sum of the two Extreams will always be equal to the Sum of of any two Means, that are equally diſtant from thoſe Extreams. As in theſe, $a : a + e : a + 2e : a + 3e : a + 4e : a + 5e :$ &c. Here $a + a + 5e = a + e + a + 4e = a + 2e + a + 3e$, &c. And if the Number of Terms be odd, the Sum of the two Extreams will be double to the middle Term, &c. as in Corol. 1. Chap. 6. before-mentioned.

## CONSECTARY 1.

*Whence it follows, (and is very eaſy to conceive) that if the Sum of the two Extreams be multiplied into the Number of all the Terms in the Series, the Product will be double the Sum of all the Series.*

*Now for the eaſier reſolving ſuch Queſtions as depend upon theſe Progreſſional Quantities.*

Let
- $a$ = the firſt Term, as before.
- $y$ = the laſt Term.
- $e$ = the common Exceſs, &c. as before.
- $N$ = the Number of all the Terms.
- $S$ = the Sum of all the Series, *viz.* of all the Terms.

Then will $\overline{a + y} \times N = 2S$, by the precedent Conſectary: that is, $Na + Ny = 2S$. Conſequently $\frac{Na + Ny}{2} = S$, the Sum of all the Series, be the Terms never ſo many. Thirdly, In theſe Series it is eaſy to perceive, that the common Difference $(e)$ is ſo often added to the laſt Term of the Series; as are the Number of Terms, except the firſt; that is, the firſt Term $(a)$ hath no Difference added to it, but the laſt Term hath ſo many times $(e)$ added to it, as it is diſtant from the firſt.

Conſequently, the Difference betwixt the two Extreams, is only the common Difference $(e)$ multiplied into the Number of all the Terms leſs Unity or 1. That is, $\overline{N - 1} \times e = y - a$, the Difference betwixt the two Extreams, *viz.* $Ne - e = y - a$.

## CONSECTARY 2.

*Whence it follows, that if the Difference betwixt the two Extreams be divided by the Number of Terms leſs* 1, *the Quotient will be the common Difference of the Series.*

To wit, $\frac{y - a}{N - 1} = e$.

Now by the Help of these two Consectaries, if any three of the aforesaid five Parts (*viz.* $a.$ $y.$ $e.$ $N.$ $S.$) be given; the other two may be easily found.

| | | |
|---|---|---|
| Thus, | 1 | $\frac{Na+Ny}{2}=S$ } as before. |
| And | 2 | $\frac{y-a}{N-1}=e$ } |
| $2\times\overline{N-1}$ | 3 | $y-a=Ne-e$ |
| $3+e$ | 4 | $y-a+e=Ne$ |
| $4\div e$ | 5 | $\frac{y-a+e}{e}=N$, the Number of Terms. |
| $1\times 2$ | 6 | $Na+Ny=2S$ |
| $6-Na$ | 7 | $Ny=2S-Na$ |
| $7\div N$ | 8 | $\frac{2S-Na}{N}=y$, the last Term. |
| $6-yN$ | 9 | $Na=2S-Ny$ |
| $9\div N$ | 10 | $\frac{2S-Ny}{N}=a$, the first Term. |
| $6\div\overline{a+y}$ | 11 | $\frac{2S}{a+y}=N$, the Number of Terms. |
| 5, and 11 | 12 | $\frac{y-a+e}{e}=\frac{2S}{a+y}$, *per Axiom* 5. |
| $12\times\overline{a+y}$ | 13 | $\frac{yy-aa}{e}+a+y=2S$ |
| $13\div 2$ | 14 | $\frac{yy-aa}{2e}+\frac{a+y}{2}=2S$, the Sum of all the Series. |
| $14\times 2e$ | 15 | $yy-aa+ae+ye=2Se$ |
| $15-ae$ | 16 | $yy-aa+ye=2Se-ae$ |
| $16-ye$ | 17 | $yy-aa=2Se-ae-ye$ |
| $17\div$ | 18 | $\frac{yy-aa}{2S-a-y}=e$, the common Difference. |
| $3+a$ | 19 | $Ne-e+a=y$, the last Term. |
| $19+e$ | 20 | $Ne+a=y+e$ |
| $20-Ne$ | 21 | $y+e-Ne=a$, the first Term. |
| | | &c. |

In like Manner you may proceed to find out any of the five Quantities ($a.$ $e.$ $y.$ $N.$ $S.$) otherwise, *viz.* by varying or comparing those Equations one with another, you may produce new

Equation

Equations with other Data in them; the which I shall here omit pursuing, and leave them for the Learner's Practice.

## Sect. 2. *Of* Quantities *in Geometrical Proportion.*

GEOMETRICAL Proportion continued has been already defined in Sect. 2. Chap. 6. Part 1. And what is there said concerning Numbers in ∺, may easily be applied to any sort of Homogeneal Quantities that are in ∺.

The most natural and simple Series of Geometrical Proportionals, is when it begins with Unity or 1.

As $1 . a . aa . aaa . aaaa . a^5 . a^6$, &c. in ∺

For $1 : a :: a : aa :: aa : aaa :: aaa : aaaa$, &c.

Or $a . b . \frac{bb}{a} . \frac{bbb}{aa} . \frac{bbbb}{aaa} . \frac{b^5}{a^4}$, &c. are Terms in ∺

For $a : b :: b : \frac{bb}{a} :: \frac{bb}{a} : \frac{bbb}{aa} :: \frac{bbb}{aa} : \frac{b^4}{a^3} :: \frac{b^4}{a^3} : \frac{b^5}{a^4}$, &c.

That is, when all the middle Terms betwixt the two Extreams are both Consequents and Antecedents, that Series is in Geometrical Proportion continued. Therefore in every Series of Quantities in ∺ all the Terms except the last are Antecedents; and all the Terms except the first are Consequents. But universally putting $a$ the first Term in the Series, and $e$ the Ratio, *viz.* the common Multiplier, or Divisor; then it will be

$a . ae . aee . aeee . aeeee . ae^5 . ae^6$. &c. in ∺

Or $a . \frac{a}{e} . \frac{a}{ee} . \frac{a}{eee} . \frac{a}{eeee} . \frac{a}{e^5}$. &c. are in ∺ decreas.

For $a : ae :: ae : \frac{aaee}{a} = aee$, &c.

And $a : \frac{a}{e} :: \frac{a}{e} : \frac{aa}{aee} = \frac{a}{ee}\ a : \frac{a}{e} :: \frac{a}{ee} : \frac{a}{eee}$, &c.

I. In any of these Series it is evident, that if three Quantities are in ∺, the Rectangle of the two Extreams will be equal to the Square of the Mean; as in these, $a : ae . aee$, here $a \times aee = ae \times ae, = aaee$, &c.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee}$; here alſo $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$, &c.

II. If four Quantities are in $\div\div$ the Rectangle of the Extreams will be equal to the Rectangle of the Means.

As in theſe, $a \cdot ae \cdot aee \cdot aeee$; here $a \times ae^3 = ae \times aee$.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee}$; here alſo $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{aa}{eee}$, &c.

Conſequently, If there are never ſo many Terms in the Series of $\div\div$, the Rectangle of the Extreams will be equal to the Rectangle of any two Means that are equally diſtant from thoſe Extreams.

As in theſe, $a \cdot ae \cdot aee \cdot aeee \cdot ae^4 \cdot ae^5$

viz. $ae^5 \times a = ae^4 \times ae$. Or $ae^5 \times a = aeee \times aee = aae^5$

III. If never ſo many Quantities are in $\div\div$ it will be, as any one of the Antecedents is to it's Conſequents; ſo is the Sum of all the Antecedents, to the Sum of all the Conſequents.

As in theſe, $\begin{cases} a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5, \text{ \&c. increaſing.} \\ a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee} \cdot \frac{a}{eeee} \cdot \frac{a}{e^5}, \text{ \&c. decreaſing.} \end{cases}$

$a : ae :: a + ae + aee + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + ae^5$

Or $a : \frac{a}{e} :: a + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} : \frac{a}{e} + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e^5}$, viz. $a \times \overline{ae + aee + ae^3 + ae^4 + ae^5} = ae \times \overline{a + ae + aee + ae^3 + ae^4}$.

That is, the Rectangle of the Extreams is equal to the Rectangle of the Means; *per* Second of this *Sect.*

*Note*, The Ratio of any Series in $\div\div$ increaſing, is found by dividing any of the Conſequents by it's Antecedent.

Thus, $a)\ ae\ (e$ Or $ae)\ aee\ (e$, &c.

But if the Series be decreaſing, then the Ratio is found by dividing any of the Antecedents by it's Conſequent.

Thus, $\frac{a}{e} \Big)\ a\ \Big(e$ Or $\frac{a}{ee} \Big)\ \frac{a}{e}\ \Big(e$, &c.

CON-

## CONSECTARY.

*These Things being premised, such Equations may be deduced from them, as will solve all such Questions as are usually proposed about Quantities in Geometrical Proportion. In order to that,*

let $a$ = the first Term.
$e$ = the common Ratio. } as before.
$y$ = the last Term.
$S$ = the Sum of all the Terms.

Then $S - y =$ the Sum of all the Antecedents.
And $S - a =$ the Sum of all the Consequents.

| Analogy. | | |
|---|---|---|
| | 1 | $a : ae :: S - y : S - a$ *per* III. of this *Sect.* |
| 1 ∵ | 2 | $Sa - aa = aeS - aey$ |
| $2 \div a$ | 3 | $S - a = eS - ey$ |
| $3 + ey$ | 4 | $S + ey - a = eS$ |
| $4 - S$ | 5 | $ey - a = eS - S$ |
| $5 \div \overline{e - 1}$ | 6 | $\frac{ye - a}{e - 1} = S$, the Sum of all the Series. |
| $3 \div \overline{S - y}$ | 7 | $\frac{S - a}{S - y} = e$, the common Ratio. |
| $5 + a$ | 8 | $ey = eS + a - S$ |
| $8 \div e$ | 9 | $\frac{eS + a - S}{e} = y$, the last Term. |
| $4 + a$ | 10 | $S + ey = eS + a$ |
| $10 - eS$ | 11 | $S + ey - eS = a$, the first Term. |

*Note*, The ∵ set in the Margin at the second Step, is instead of *ergo*; and imports that the Rectangle of the two Extreams in the first Step, is equal to the Rectangle of the Means. And so for any other Proportion.

## Sect. 3. *Of* Harmonical *Proportion.*

HARMONICAL or Musical Proportion is, when of three Quantities (or rather Numbers) the first hath the same Ratio to the third, as the Difference between the first and second, hath to the Difference between the second and third. As in these following.

Suppose $a$, $b$, $c$, in Musical Proportion.

| | | |
|---|---|---|
| Then | 1 | $a : c :: b - a : c - b$ |
| 1 ∵ | 2 | $cb - ca = ac - ba$ |

| | | |
|---|---|---|
| $2 + ca$ | 3 | $cb = 2ac - ba$ |
| $3 \div \overline{2c - b}$ | 4 | $\frac{cb}{2c - b} = a$, the firſt Term. |
| $3 + ba$ | 5 | $2ac = cb + ba$ |
| $5 \div \overline{c + a}$ | 6 | $\frac{2ac}{c + a} = b$, the ſecond Term. |
| $5 - cb$ | 7 | $2ac - cb = ba$ |
| $7 \div \overline{2a - c}$ | 8 | $\frac{ba}{2a - b} = c$, the third Term. |

If there are four Terms in Muſical Proportion, the firſt hath the ſame Ratio to the fourth, as the Difference between the firſt and ſecond hath to the Difference between the third and fourth.

That is, let $a$, $b$, $c$, $d$, be the four Terms, &c.

| | | |
|---|---|---|
| Then | 1 | $a : d :: b - a : d - c$ |
| 1 ∵ | 2 | $db - da = da - ca$ |
| $2 - da$ | 3 | $db = 2da - ca$ |
| $3 \div \overline{2d - c}$ | 4 | $\frac{db}{2d - c} = a$ |
| $3 \div d$ | 5 | $b = \frac{2da - ca}{d}$ |
| $3 + ca$ | 6 | $db + ca = 2da$ |
| $6 - db$ | 7 | $ca = 2da - db$ |
| $7 \div a$ | 8 | $c = \frac{2da - db}{a}$ |
| $7 \div \overline{2a - b}$ | 9 | $\frac{ca}{2a - b} = d.$ |

## CHAP. VII.

### *Of Proportion* Disjunct, *and how to turn Equations into* Analogies, &c.

PROPORTION Disjunct, or the Rule of Three in Numbers, is already explained in *Chap.* 7. *Part* 1. And what hath been there ſaid, is applicable to all Homogeneous Quantities, *viz.* of Lines to Lines, &c.

Sect.

## SECT. I.

IF four Quantities, (*viz.* either Lines, Superficies, or Solids) be proportional: the Rectangle comprehended under the Extreams, is equal to the Rectangle comprehended under the two Means. (16 *Euclid* 6.)

For Inſtance, Suppoſe, $a \,.\, b \,.\, c \,.\, d$. to repreſent the four Homogeneal Quantities in Proportion, *viz.* $a : b :: c : d$; then will $ad = bc$. For ſuppoſe $b = 2a$, then will $d = 2c$, and it will be $a : 2a :: c : 2c$. Here the Ratio is 2. But $a \times 2c = 2a \times c$. *viz.* $2ca = 2ac$. Or ſuppoſe $b = 3a$ then will $d = 3c$, and it will be $a : 3a :: c : 3c$. Here the Ratio is 3. But $a \times 3c = 3a \times c$. *viz.* $3ca = 3ae$. Or univerſally putting $e$ for the Ratio of the Proportion, *viz.* making $b = ae$, then will $d = ce$, and it will be $a : ae :: c : ce$. But $a \times ce = ae \times c$, viz. $ace = aec$. Conſequently, $ad = bc$ which was to be proved.

Whence it follows, that if any three of the four proportional Quantities be given, the fourth may be eaſily found; thus,

| | | | |
|---|---|---|---|
| Let | 1 | $a : b :: c : d$ | |
| $1 \because$ | 2 | $ad = bc$ as before | |
| $2 \div d$ | 3 | $a = \frac{bc}{d}$ | |
| $2 \div c$ | 4 | $b = \frac{ad}{c}$ | |
| $2 \div b$ | 5 | $c = \frac{ad}{b}$ | |
| $2 \div a$ | 6 | $d = \frac{bc}{a}$ | |
| $2 \div bd$ | 7 | $\frac{a}{b} = \frac{c}{d}$ | *Note*, In this Manner *Euclid*, in his 5th Book, expreſſes the Ratio of Proportionals, *viz.* the Ratio of |
| Or $2 \div ac$ | 8 | $\frac{b}{a} = \frac{d}{c}$ | $a$ to $b$ is $\frac{a}{b}$ |

If four Quantities are Proportionals, they will alſo be Proportionals in Alternation, Inverſion, Compoſition, Diviſion, Converſion, and Mixtly. *Euclid* 5. *Def.* 12, 13, 14, 15, 16.

That

| | | |
|---|---|---|
| That is, if | 1 | $a : b :: c : d$ be in direct Proportion, as before. |
| Then | 2 | $a : c :: b : d$, alternate. For $ad = bc$. |
| And | 3 | $b : a :: d : c$, inverted. For $ad = bc$. |
| Also | 4 | $a + b : b :: c + d : d$; compounded. |
| 4 ∵ | 5 | $da + bd = bc + bd$, that is, $ad = bc$, as before. |
| Or | 6 | $a + c : c :: b + d : d$; alternately compounded. |
| 6 ∵ | 7 | $ad + cd = bd + cd$, that is, $ad = bc$. |
| Again, | 8 | $a - b : b :: c - d : d$, divided. |
| 8 ∵ | 9 | $ad - bd = bc - bd$, that is, $ad = bc$. |
| Or | 10 | $a - c : c :: b - d : d$, divided. |
| 10 ∵ | 11 | $ad - cd = bc - cd$, that is, $ad = bc$. |
| And | 12 | $a : b \pm a :: c : d \pm c$, converted. |
| 12 ∵ | 13 | $ad \pm ac = bc \pm ac$, that is, $ad = bc$. |
| Lastly | 14 | $a + b : a - b :: c + d : c - d$, mixtly. |
| 14 ∵ | 15 | $ac - ad + bc - bd = ac + ad - bc - bd$. |
| 15 $\pm$ | 16 | $2bc = 2ad$, that is, $ad = bc$; as at first. |

*Note*, What has been here done about whole Quantities in Simple Proportion, may be easily perform'd in Fractional Quantities, and Surds, &c.

For Instance, If $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{c}$, and if it be required to find the fourth Term, it will be $\frac{dd-cc}{fc}$ the Rectangle of the Means; which being divided by the first Extream $\frac{ab}{c}$ will become $\frac{ab}{c}\Big) \frac{dd-cc}{fc} \Big( \frac{ddc-ccc}{abfc} = \frac{dd-cc}{abf}$ the fourth Term.

Or if $b : \sqrt{bd+bc} :: \sqrt{bc+bc}$ : to a fourth Term. Then is, $\sqrt{bd+bc} \times \sqrt{bd+bc} = bd+bc$ the Rectangle of the Means; and $b) \, bd+bc \, (d+c$ the fourth Term. That is, $b : \sqrt{bd+bc} :: \sqrt{bd+bc} : d+c$, &c.

## Sect. 2. *Of* Duplicate *and* Triplicate Proportion.

THE Proportions treated of in the last Section, are to be understood when Lines are compared to Lines, and Superficies to Superficies; or Solids to Solids, *viz.* when each is compared to that of it's like Kind, which is only called Simple Proportion.

But

But when Lines are compared to Superficies, or Lines are compared to Solids, ſuch Compariſons are diſtinguiſhed from the former, by the Names of Duplicate, and Triplicate, &c. Proportions; ſo that Simple, Duplicate, and Triplicate, *&c.* Proportions are to be underſtood in a different Senſe from Simple, Double, Treble, *&c.* Proportions, which are only as 1, 2, 3, *&c.* to 1; but thoſe of Simple, Duplicate, Triplicate, *&c.* Proportions, are thoſe of $a \,.\, aa \,.\, aaa$., &c. to 1. Or if the Simple Proportions be that of $a$ to $b$, whoſe Ratio or Exponent is $\frac{a}{b}$ or $\frac{b}{a}$.

Then $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$ is the Exponent of the Duplicate.

And $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$ is the Exponent of the Triplicate Proportions, *&c.*

} of $\frac{a}{b}$

And if there are three, four, or more Quantities in ∺, as $1 \,.\, a \,.\, aa \,.\, aaa \,.\, a^4 \,.\, a^5$, &c. (as in the firſt Series, Sect. 2. of the laſt Chapter.) Then, that of the firſt to the third, fourth, and fifth, *&c.* (*viz.* 1 to $aa \,.\, aaa \,.\, a^4 \,.\, a^5$) is Duplicate, Triplicate, Quadruplicate, *&c.* of the firſt to the ſecond (*viz.* of 1 to $a$); and by Inverſion, that of the third, fourth, fifth, is Duplicate, Triplicate, *&c.* of that of the ſecond to the firſt ($a$ to 1) *per Def.* 10. *Eucl.* 5. But the Name of theſe Proportions will appear more evident, and be eaſier underſtood when they are applied to Practice, and illuſtrated by Geometrical Figures, further on.

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## Sect. 3. *How to turn Equations into* Analogies.

FROM the firſt Section of this Chapter, it will be eaſy to conceive how to turn or diſſolve Equations into Analogies or Proportions. For if the Rectangle of two (or more) Quantities, be equal to the Rectangle of two (or more) Quantities; then are thoſe four (or more) Quantities Proportional. By the 16 *Eucl.* 6. That is, if $ab = cd$, then is $a : c :: d : b$, or $c : a :: b : d$, &c. From whence there ariſes this general Rule for turning Equations into Analogies.

### RULE.

*Divide either Side of the given Equation (if it can be done) into two ſuch Parts, or Factors, as being multiplied together will produce that Side again; and make theſe two Parts the two Extreams. Then divide the other Side of the Equation (if it can be done) in the ſame Manner as the firſt was, and let thoſe two Parts or Factors be the two Means.*

For Inſtance, Suppoſe $ab + ad = bd$. Then $a : b :: d : b + d$, or $b : a :: b + d : d$, &c. Or taking $ad$ from both Sides of the Equation, and it will be $ab = bd - ad$; then $a : d :: b - a : b$, or, $b : d :: b - a : a$, &c.

Again, ſuppoſe $aa + 2ae = 2by + yy$. Here $a$ and $a + 2e$ are the two Factors of the firſt Side in this Equation; for $\overline{a + 2e} \times a = aa + 2ae$.

Again, $y$ and $2b + y$ are the two Factors of the other Side; therefore, $a : y :: 2b + y : a + 2e$, or $2b + y : a + 2e :: a : y$, &c.

When one Side of any Equation can be divided into two Factors, as before, and the other Side cannot be ſo divided, then make the Square Root of that Side either the two Extreams or the two Means. For Inſtance, Suppoſe $bc + bd = da + g$, then $b : \sqrt{da + g} :: \sqrt{da + g} : c + d$, or $\sqrt{da + g} : b :: c + d : \sqrt{da + g}$, &c.

## CHAP. VIII.

## Of Subſtitution, and the Solution of Quadratick Equations.

### Sect. 1. Of Subſtitution.

WHEN new Quantities not concerned in the firſt ſtating of any Queſtion, are put inſtead of ſome that are engaged in it, that is called *Subſtitution*. For Inſtance, If inſtead of $\sqrt{bc - dc}$ you put $z$, or any other Letter; that is, make $z = \sqrt{bc - dc}$. Or ſuppoſe $aa + ba - ca + da = dc$, inſtead of $b - c + d$ put $s$, or any other Letter not engaged with the Queſtion, viz. $s = b - c + d$, then $aa \mp sa = dc$. That is, if $c$ be greater than

than $b+d$, it is $aa - sa = dc$; but if $b+d$ be greater than $c$, then it is $aa + sa = dc$.

And this way of ſubſtituting or putting of new Quantities inſtead of others, may be found very uſeful upon ſeveral Occaſions; *viz.* in Order to make ſome following Operations in the Queſtion more eaſy, and perhaps much ſhorter than they would be without it, as you may obſerve in ſome Queſtions hereafter propoſed in this Tract.

And when thoſe Operations, in which the ſubſtituted Quantities were aſſiſting or uſeful, are performed according as the Nature of the Queſtion required, you may then (if there be Occaſion) bring the original or firſt Quantities into the Equation, in the Place (or Places) of thoſe ſubſtituted Quantities, which is called Reſtitution, as you may ſee further on.

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## Sect. 2. *The Solution of* Quadratick Equations.

WHEN the Quantity ſought is brought to an Equality with thoſe that are known, and is on one Side of the Equation, in no more than two different Powers whoſe Indices are double one to another, thoſe Equations are called Quadratick Equations Adfected; and do fall under the Conſideration of three Forms or Caſes.

$$\left.\begin{array}{l}\text{Caſe 1. } aa + 2ba = dc.\\ \text{Caſe 2. } aa - 2ba = dc.\\ \text{Caſe 3. } 2ba - aa = dc.\end{array}\right\} \text{ And } \left\{\begin{array}{l} a^4 + 2ba^2 = dc.\\ a^4 - 2ba^2 = dc.\\ 2ba^2 - a^4 = dc.\end{array}\right.$$

$$\text{Alſo} \left\{\begin{array}{l} a^6 + 2ba^3 = dc.\\ a^6 - 2ba^3 = dc.\\ 2ba^3 - a^6 = dc.\end{array}\right\} \text{ And } \left\{\begin{array}{l} a^8 + 2ba^4 = dc.\\ a^8 - 2ba^4 = dc.\\ 2ba^4 - a^8 = dc.\end{array}\right\} \text{ \&c.}$$

When there happens to be more Terms in one of theſe Kind of Equations than two, and the higheſt Power of the unknown Quantity is multiplied into ſome known Co-efficients; you muſt reduce them by *Diviſion*; as in Sect. 4. of Chap. 5. and for the Fractional Quantities that may ariſe by thoſe Diviſions, ſubſtitute another Quantity doubled.

For Inſtance, let $baa + caa - ca - da = dc + cb$, then $aa - \frac{ca - da}{b + c} = \frac{dc + cb}{b + c}$. Make $\frac{c - d}{b + c} = 2x$, and if you pleaſe,

 for

for $\frac{dc + cb}{b + c}$ put $z$. Then will $aa - 2xa = z$ be the new Equation, equal to the other, being now fitted for a Solution.

Now any of these three Forms of Equations being thus prepared for a Solution, may be reduced to simple Powers by casting off the second or lowest Term of the unknown Quantity; which is done by Substitution; thus, always take half the known Co-efficient, and add it to (Case 1.) or substract it from (Case 2.) it's fellow Factor; and for their Sum, or Difference, Substitute another Letter; as in these.

| | | |
|---|---|---|
| Let | 1 | $aa + 2ba = dc$ Case 1. |
| Put | 2 | $a + b = e$ |
| 2 ℘² | 3 | $aa + 2ba + bb = ee$ |
| 3 — 1 | 4 | $bb = ee - dc$ |
| 4 + $dc$ | 5 | $ee = bb + dc$ |
| 5 $\omega^2$ | 6 | $e = \sqrt{bb + dc}$ |
| 2 and 6 | 7 | $a + b = \sqrt{bb + dc}$, *per Axiom* 5. |
| 7 — $b$ | 8 | $a = \sqrt{bb + dc} : - b$ |

Again,

| | | |
|---|---|---|
| Let | 1 | $aa - 2ba = dc$ Case 2. |
| Put | 2 | $a - b = e$ |
| 2 ℘² | 3 | $aa - 2ba + bb = ee$ |
| 3 — 1 | 4 | $bb = ee - dc$ |
| 4 + $dc$ | 5 | $ee = dc + bb$ |
| 5 $\omega^2$ | 6 | $e = \sqrt{dc + bb}$ |
| 2 and 6 | 7 | $a - b = \sqrt{dc + bb}$ |
| 7 + $b$ | 8 | $a = b + \sqrt{dc + bb}$ |

In Case 3. From Half the known Co-efficient substract it's fellow Factor.

| | | |
|---|---|---|
| Thus, Let | 1 | $2ba - aa = dc$ |
| Put | 2 | $b - a = e$ |
| 2 ℘² | 3 | $bb - 2ba + aa = ee$ |
| 1 + 3 | 4 | $bb = dc + ee$ |
| 4 — $dc$ | 5 | $ee = bb - dc$ |

5 $\omega^2$

| | | |
|---|---|---|
| $5\ ww^2$ | 6 | $e = \sqrt{bb - dc}$ |
| 2 and 6 | 7 | $b - a = \sqrt{bb - dc}$ |
| $7 + a$ | 8 | $b = a + \sqrt{bb - dc}$ |
| $8 - \sqrt{}$, &c. | 9 | $a = b - \sqrt{bb - dc}$ |

And this Method holds good in thoſe other Equations, wherein the higheſt Powers are $a^4$, $a^6$, $a^8$, &c. As, for inſtance,

| | | |
|---|---|---|
| Let | 1 | $a^6 + 2b^3 = dc$ Caſe 1. |
| Put | 2 | $a^3 + b = e$ |
| $2\ \circledast^2$ | 3 | $a^6 + 2ba^3 + bb = ee$ |
| $3 - 1$ | 4 | $bb = ee - dc$ |
| $4 + cd$ | 5 | $ee = bb + dc$ |
| $5\ ww^2$ | 6 | $e = \sqrt{bb + dc}$ |
| 2 and 6 | 7 | $a^3 + b = \sqrt{bb + dc}$ |
| $7 - b$ | 8 | $a^3 = \sqrt{bb + dc} : - b$ |
| $8\ ww^3$ | 9 | $a = \sqrt[3]{: \sqrt{bb + dc} : - b}$ |

The ſame may be done with all the reſt, Care being taken to add, or ſubſtract, according as the Caſe requires.

But all Quadratick Equations may be more eaſily reſolved by compleating the Square, which is grounded upon the Conſideration of raiſing a Square from any Binomial, or Reſidual Root. (See Sect. 5. Chap. 1.) *Viz.* if $a + b$ be involved to a Square, it will be $aa + 2ba + bb$; and if $a - b$ be ſo involved, it will be $aa - 2ba + bb$. Whence it is eaſy to obſerve, that $aa + 2ba = dc$ (Caſe 1.), and $aa - 2ba = dc$ (Caſe 2.), are imperfect Squares, wanting only $bb$ to make them compleat. And therefore it is, that if half the known Co-efficient be involved to the ſecond Power, and the Square be added to both Sides of the Equation, the unknown Side will become a compleat Square.

| | | |
|---|---|---|
| Thus Let | 1 | $aa + 2ba = dc$ |
| But | 2 | $bb = bb$ |
| $1 + 2$ | 3 | $aa + 2ba + bb = dc + bb$ Caſe 1. |
| $3\ ww^2$ | 4 | $a + b = \sqrt{dc + bb}$, as before |

Here half the Co-efficient $2b$ is $b$, which being ſquared, is $bb$.

Again.

Again.

| | | |
|---|---|---|
| Let | 1 | $aa - 2ba = dc$ Case 2. |
| But | 2 | $bb = bb$ |
| 1 + 3 | 3 | $aa - 2ba + bb = dc + bb$ |
| $3\ ww^2$ | 4 | $a - b = \sqrt{dc + bb}$, &c. as before. |

But in Case 3. you must change the Signs of all the Terms in the Equation,

| | | |
|---|---|---|
| Thus | 1 | $2ba - aa = dc$ Case 3. |
| 1 ± | 2 | $aa - 2ba = -dc$ |
| Then | 3 | $aa - 2ba + bb = bb - dc$, &c. |

And this Method of compleating the Square, holds true in those other Equations.

| | | |
|---|---|---|
| *Viz.* | 1 | $aaaa + 2baa = dc$ Case 1. |
| For | 2 | $bb = bb$, as before. |
| 1 + 2 | 3 | $aaaa + 2baa + bb = dc + bb$ |
| $3\ ww^2$ | 4 | $aa + b = \sqrt{dc + bb}$ |
| $4 - b$ | 5 | $aa = \sqrt{dc + bb} : - b$ |
| $5\ ww^2$ | 6 | $a = \sqrt{: \sqrt{dc + bb} : - b}$, and so on for the rest. |

| | | |
|---|---|---|
| Or let | 1 | $a^6 + 2baaa = dc$, as before, Case 1. |
| And | 2 | $bb = bb$ |
| 1 + 2 | 3 | $a^6 + 2baaa + bb = dc + bb$ |
| $1\ ww^2$ | 4 | $aaa + b = \sqrt{dc + bb}$ |
| $4 - b$ | 5 | $aaa = \sqrt{dc + bb} : - b$ |
| $5\ ww^3$ | 6 | $a = \sqrt[3]{: \sqrt{dc + bb} : - b}$, &c. |

## COROLLARY.

*Hence it is evident, that whatsoever Method is used in solving these (or indeed any other) Equations, the Result will still be the same, if the Work be true; as you may observe from the Operations of this Section: for both these Methods here proposed, give the same Theorems in their respective Cases for the Value of* $(a)$.

Thus

Thus, when $aa + 2ba = dc$, then
*Theorem* 1. $a = \sqrt{dc + bb} : - b$

And when $aa - 2ba = dc$, then
*Theorem* 2. $a = b + \sqrt{dc + bb}$

Again, when $2ba - aa = dc$, then
*Theorem* 3. $a = b - \sqrt{bb - dc}$

The like *Theorems* may be eaſily raiſed for the reſt.

If the known Co-efficients (of the ſecond or loweſt Term) be any ſingle Quantity, as $aa + ba = dc$, &c. then is $\frac{1}{2}b$ it's Half, and $\frac{1}{4}bb$ will be the Square of that Half; that is, $\frac{1}{2}b \times \frac{1}{2}b = \frac{1}{4}bb$, and then the Work will ſtand

| | | |
|---|---|---|
| Thus | 1 | $aa + ba = dc$ |
| 1 C □ | 2 | $aa + ba + \frac{1}{4}bb = dc + \frac{1}{4}bb$ |
| $2\ \omega^2$ | 3 | $a + \frac{1}{2}b = \sqrt{dc + \frac{1}{4}bb}$ |
| $3 - \frac{1}{2}b$ | 4 | $a = \sqrt{dc + \frac{1}{4}bb} : - \frac{1}{2}b$, and ſo for the reſt. |

*Note*, *C* □ placed in the Margin againſt the ſecond Step, ſignifies that the imperfect Square $aa + ba$ in the firſt Step, is there compleated, *viz.* in the ſecond Step.

Now by the help of theſe *Theorems*, it will be eaſy to calculate or find the Value of the unknown Quantity ($a$) in Numbers.

## EXAMPLE 1.

Suppoſe $aa + 2ba = z$. Let $b = 16$, and $z = 4644$.
then $a = \sqrt{z + bb} : - b$ per *Theorem* 1.
But $z + bb = 4644 + 256 = 4900$, and $\sqrt{4900} = 70$
Conſequently $a = 70 - 16$. viz. $a = 54$.

But every Adfected Equation, hath as many Roots (or rather Values of the unknown Quantity) either real or imaginary, as are the Dimenſions (*viz.* the Index) of it's higheſt Power; and therefore the Quantity $a$, in this Equation, hath another Value either Affirmative or Negative; which may be thus found.

The given Equation is $aa + 32a = 4644$, and it's Root $a = 54$.

Let theſe two Equations be made equal or equated to 0, *viz.* to Nothing.

Thus,

Thus, $aa + 32a - 4644 = 0$, and $a - 54 = 0$.

Then divide the given Equation by it's firſt Root, and the Quotient will ſhew the ſecond Value of $a$.

$$\text{Thus, } a - 54 = 0)\ aa + 32a - 4644 = 0\ (a + 86 = 0$$

$$\begin{array}{r} aa - 54a \\ \hline + 86a - 4644 \\ 86a - 4644 \\ \hline (0) \end{array}$$

Hence the ſecond Value of $a$ is $= -86$, or $86 = -a$, which ſeems impoſſible, *viz.* that an Affirmative Quantity ſhould be equal to a Negative Quantity; yet even by this ſecond Value of $a$, and the ſame Co-efficient, the true (or firſt) Equation may be formed

| | | |
|---|---|---|
| Thus, Let | 1 | $a = -86$ |
| 1 $\circledcirc^{2}$ | 2 | $aa = +7396$, viz. $-86 \times -86 = +7396$ |
| $1 \times 32$ | 3 | $32a = -2752$ |
| $2 + 3$ | 4 | $aa + 32a = 4644$, as at firſt. |

## *EXAMPLE* 2.

| | | |
|---|---|---|
| Suppoſe | 1 | $aa - 7a = 948{,}75$, then *per Theorem* 2. |
| 1 *C* $\square$ | 2 | $aa - 7a + \frac{49}{4} = 948{,}75 + \frac{49}{4} = 961$ |
| 2 $\underline{\omega^{2}}$ | 3 | $a - \frac{7}{2}$ (or 3,5) $= \sqrt{961} = 31$ |
| $3 + 3{,}5$ | 4 | $a = 31 + 3{,}5 = 34{,}5$ |

Again, for the ſecond Value of $a$, let $aa - 7a - 948{,}75 = 0$, and $a - 34{,}5 = 0$. Then

$$a - 34{,}5 = 0)\ aa - 7a - 948{,}75 = 0\ (a + 27{,}5 = 0.$$

Conſequently this ſecond Value is $a = -27{,}5$ which will form the original Equation, $aa - 7a = 948{,}75$ if it be ordered as the laſt was.

## *EXAMPLE* 3.

Suppoſe $36a - aa = 243$, then *per Theorem* 3. $a = 18 - \sqrt{324 - 243}$, *viz.* half 36 ſquared is 324, &c. that is, $a = 18 - \sqrt{81}$; but $\sqrt{81} = 9$, therefore $a = 18 - 9 = 9$. Now this third Form is called an ambiguous Equation, becauſe it hath two Affirmative Values of the unknown Quantity ($a$), both which may be found without ſuch Diviſion as was uſed before.

before. For in this Case, $a = 18 + \sqrt{81}$, viz. $a = 18 + 9 = 27$, or, $a = 18 - 9 = 9$, as before. And both these Values of $a$ are equally true, as to forming the given Equation; *viz.* $36a - aa = 243$. For if $a = 9$, then $aa = 81$, and $36a = 324$; but $324 - 81 = 243$, therefore $a = 9$.

Again, if $a = 27$, then will $aa = 729$, and $36a = 972$: But $972 - 729 = 243$, consequently it may be, $a = 27$. Now either of these Values of $a$ may be found by *Division*, as those were in the other two Cases, one of them being first found by the Theorem. Thus, let $36a - aa - 243 = 0$, and $9 - a = 0$, then $9 - a = 0)\ 36a - aa - 243 = 0\ (a - 27 = 0$

$$\begin{array}{l} 9a - aa \\ \hline 27a - 0 - 243 \\ 27 \quad\quad\quad 243 \\ \hline (0) \quad\quad\quad (0) \end{array}$$

Hence, if $a - 27 = 0$, then $a = 27$, as before.

Notwithstanding all Quadratick Equations of this third Form have two Affirmative Roots (as in this), yet but one of those Roots will give a true Answer to the Question, and that is to be chosen according to the Nature and Limits of the Question, as shall be shewed further on.

## SCHOLIUM.

*From the Work of the three last Examples, it may be observed; that the Sum of both the Roots will always be equal to the Co-efficient of their respective Equations, with a contrary Sign.*

Thus. In *Example* 1. $aa + 32a = 4644$

$$\left.\begin{array}{l} \text{Here } a = 54 \\ \text{And } a = -86 \end{array}\right\} \text{Add}$$
$$2a = -32$$

In *Example* 2. $aa - 7a = 948{,}75$

$$\left.\begin{array}{l} \text{Here } a = 34{,}5 \\ \text{And } a = -27{,}5 \end{array}\right\} \text{Add}$$
$$2a = +7$$

In the last *Example* $36a - aa = 243$

Which was changed into $aa - 36a = -243$

$$\left.\begin{array}{l} \text{Here } a = 9 \\ \text{And } a = 27 \end{array}\right\} \text{Add}$$
$$2a = 36$$

Hence it is evident, that if either of the Roots be found, the other may be easily had without Divisions.

If the Contents of this Section be well understood, it will be easy to give a Numerical Solution to any Quadratick Equation, that happens to arise in resolving of Questions, *&c.* And as for giving a Geometrical Construction of them, I think it not proper in this Place; because I here suppose the Learner wholly ignorant of the first Principles of Geometry, therefore I shall refer that Work to the next Part.

## CHAP. IX.

### *Of* Analysis, *or the Method of resolving* Problems *exemplified by Variety of Numerical* Questions.

N. B. *HERE I advise the Learner to make use always of the same Letters, to represent the same* Data *in all Questions.*

*Viz.* { If $a$ represent any Number / And $e$ represent a less Number } or other Quantity,

Then let:

$a + e = s$ their Sum.
$a - e = d$ their Difference.
$ae = p$ their Product.
$\frac{a}{e} = q$ their Quotient.
$aa + ee = z$ the Sum of their Squares.
$aa - ee = x$ the Difference of their Squares.

Any two of these six ($s$, $d$, $p$, $q$, $z$, $x$) being given, thence to find the rest; which admits of fifteen Variations, or Questions.

*Question* 1. Suppose $s$ and $d$ were given, and it were required by them to find $a$ . $e$ . $p$ . $q$ . $z$ . and $x$ .

| | | |
|---|---|---|
| Let | 1 | $a + e = s$ } and suppose { $s = 240$ } Then |
| | 2 | $a - e = d$ } { $d = 192$ } |
| $1 + 2$ | 3 | $2a = s + d = 432$ |
| $3 \div 2$ | 4 | $a = \frac{s+d}{2} = 216$, here $a$ is found. |
| $1 - 2$ | 5 | $2e = s - d = 48$. |

5 ÷ 2

| | | |
|---|---|---|
| $5 \div \overline{2}$ | 6 | $e = \frac{s-d}{2} = 24$, here $e$ is found. |
| $4 \times 6$ | 7 | $ae = \frac{ss-dd}{4} = p = 5184$, here $p$ is found. |
| $4 \div 6$ | 8 | $\frac{a}{e} = \frac{s+d}{s-d} = q = 9$, here $q$ is found. |
| $4\ \circledS^2$ | 9 | $aa = \frac{ss+2sd+dd}{4} = 46656$ |
| $6\ \circledS^2$ | 10 | $ee = \frac{ss-2sd+dd}{4} = 576$ |
| $9 + 10$ | 11 | $aa+ee = \frac{ss+dd}{2} = z = 47232$, $z$ found. |
| $9 - 10$ | 12 | $aa-ee = sd = x = 46080$, $x$ found. |

*Question* 2. Let $s$ and $p$ be given, to find the rest.

| | | |
|---|---|---|
| That is { | 1 | $a+e = s = 240$ } Quære $a.e.d.q.z.x.$ |
| | 2 | $ae = p = 5184$ |
| $1\ \circledS^2$ | 3 | $aa+2ae+ee = ss = 57600$ |
| $2 \times 4$ | 4 | $4ae = 4p = 20736$ |
| $3 - 4$ | 5 | $aa-2ae+ee = ss-4p = 36864$ |
| $5\ \omega^2$ | 6 | $a-e = \sqrt{ss-4p} = d = 192$ |
| $1 + 6$ | 7 | $2a = s+\sqrt{ss-4p}$ |
| $7 \div \overline{2}$ | 8 | $a = \frac{s+\sqrt{ss-4p}}{2}$, hence $a = 216$ |
| $1 - 6$ | 9 | $2e = s-\sqrt{ss-4p}$ |
| $9 \div \overline{2}$ | 10 | $e = \frac{s-\sqrt{ss-4p}}{2}$, hence $e = 24$. |
| $8 \div 10$ | 11 | $\frac{a}{e} = \frac{s+\sqrt{ss-4p}}{s-\sqrt{ss-4p}} = q = 9.$ |
| $8\ \circledS^2$ | 12 | $aa = \frac{ss+s\sqrt{ss-4p}}{2} : -p$ |
| $10\ \circledS^2$ | 13 | $ee = \frac{ss-s\sqrt{ss-4p}}{2} : -p$ |
| $12 + 13$ | 14 | $aa+ee = ss-2p = z = 47232$ |
| $12 - 13$ | 15 | $aa-ee = s\sqrt{ss-4p} = x = 46080$ |

*Question* 3. Suppose $s$ and $q$ are given, to find the rest.

| | | |
|---|---|---|
| Viz. | 1 | $a+e=s.=240$ |
| | 2 | $\frac{a}{e}=q=9$ } Quære $a.e.d.p.z.x.$ |
| $2 \times e$ | 3 | $a=qe$ |
| $1-3$ | 4 | $e=s-qe$ |
| $4+qe$ | 5 | $qe+e=s$ |
| $5 \div \overline{q+1}$ | 6 | $e=\frac{s}{q+1}$, for $\overline{q+1}\times e=qe+e$ |
| $1-6$ | 7 | $a=s-\frac{s}{q+1}=\frac{qs}{q+1}$ |
| $6\times 7$ | 8 | $ae=\frac{qss}{qq+2q+1}=p$ |
| $7-6$ | 9 | $a-e=\frac{qs-s}{q+1}=d$ |
| $7\ \circledcirc^{2}$ | 10 | $aa=\frac{qqss}{qq+2q+1}$ |
| $6\ \circledcirc^{2}$ | 11 | $ee=\frac{ss}{qq+2q+1}$ |
| $10+11$ | 12 | $aa+ee=\frac{qqss+ss}{qq+2q+1}=z$ |
| $10-11$ | 13 | $aa-ee=\frac{qqss-ss}{qq+2q+1}=x$ |

*Question* 4. Let $s$ and $z$ be given, to find the rest.

| | | |
|---|---|---|
| Viz. | 1 | $a+e=s.=240$ |
| | 2 | $aa+ee=z.=47232$ } Quære $a.e.d.p.q.x.$ |
| $1\ \circledcirc^{2}$ | 3 | $aa+2ae+ee=ss$ |
| $3-2$ | 4 | $2ae=ss-z$ |
| $2-4$ | 5 | $aa-2ae+ee=2z-ss$ |
| $5\ \omega^{2}$ | 6 | $a-e=\sqrt{2z-ss}=d$ |

| | | |
|---|---|---|
| 1 + 6 | 7 | $2a = s + \sqrt{2z - ss}$ |
| 7 ÷ $\overline{2}$ | 8 | $a = \frac{s + \sqrt{2z - ss}}{2}$ |
| 1 − 6 | 9 | $2e = s - \sqrt{2z - ss}$ |
| 9 ÷ $\overline{2}$ | 10 | $e = \frac{s - \sqrt{2z - ss}}{2}$ |

The reſt are found juſt as in the 2d *Queſtion*; the 8 and 10 Steps here being the very ſame with the 8 and 10 Steps there.

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*Queſtion* 5. When $s$ and $x$ are given, to find the reſt.

| | | |
|---|---|---|
| *Viz.* | 1 | $a + e = s = 240$ } Quære $a.e.d.p.q.z.$ |
| | 2 | $aa - ee = x = 46080$ |
| 2 ÷ 1 | 3 | $a - e = \frac{x}{s} = d$, viz. $a + e)\ aa - ee\ (a - e$ |
| 1 + 3 | 4 | $2a = s + \frac{x}{s} = \frac{ss + x}{s}$ |
| 4 ÷ $\overline{2}$ | 5 | $a = \frac{ss + x}{2s}$ |
| 1 − 3 | 6 | $2e = s - \frac{x}{s} = \frac{ss - x}{s}$ |
| 6 ÷ $\overline{2}$ | 7 | $e = \frac{ss - x}{2s}$ |
| 5 × 7 | 8 | $ae = \frac{ssss - xx}{4ss} = p$ |
| 5 ÷ 7 | 9 | $\frac{a}{e} = \frac{ss + x}{ss - x} = q$ |
| 5 ℈² | 10 | $aa = \frac{s^4 + 2ssx + xx}{4ss}$ |
| 7 ℈² | 11 | $ee = \frac{s^4 - 2ssx + xx}{4ss}$ |
| 10 + 11 | 12 | $aa + ee = \frac{s^4 + xx}{2ss} = z$ |

*Queſtion*

*Question* 6. Suppose $d$ and $p$ are given, to find the rest.

| | | |
|---|---|---|
| Viz. | 1 | $a - e = d = 192$ |
| | 2 | $ae = p = 5184$ } Quære $a.e.s.q.z.x.$ |
| $1\ ⊚^2$ | 3 | $aa - 2ae + ee = dd$ |
| $2 \times 4$ | 4 | $4ae = 4p$ |
| $3 + 4$ | 5 | $aa + 2ae + ee = dd + 4p$ |
| $5\ \omega^2$ | 6 | $a + e = \sqrt{dd+4p} = s$ |
| $6 + 1$ | 7 | $2a = d + \sqrt{dd+4p}$ |
| $7 \div 2$ | 8 | $a = \dfrac{d + \sqrt{dd+4p}}{2}$ |
| $6 - 1$ | 9 | $2e = \sqrt{dd+4p}\,\surd - d$ |
| $9 \div 2$ | 10 | $e = \dfrac{\sqrt{dd+4p}\,\surd - d}{2}$ |
| $8 \div 10$ | 11 | $\dfrac{a}{e} = \dfrac{d + \sqrt{dd+4p}}{\sqrt{dd+4p}\,\surd - d} = q$ |
| $8\ ⊚^2$ | 12 | $aa = \dfrac{dd + 2p + d\sqrt{dd+4p}}{2}$ |
| $10\ ⊚^2$ | 13 | $ee = \dfrac{dd + 2p - d\sqrt{dd+4p}}{2}$ |
| $12 + 13$ | 14 | $aa + ee = dd + 2p = z$ |
| $12 - 13$ | 15 | $aa - ee = d\sqrt{dd+4p} = x$ |

*Question* 7. Let $d$ and $q$ be given, to find the rest.

| | | |
|---|---|---|
| Viz. | 1 | $a - e = d = 192$ |
| | 2 | $\dfrac{a}{e} = q = 9$ } Quære $a.e.s.p.z.x.$ |
| $2 \times e$ | 3 | $a = qe$ |
| $1 + e$ | 4 | $a = d + e$ |
| 3 and 4 | 5 | $qe = d + e$ |
| $5 - e$ | 6 | $qe - e = d$ |
| $6 \div \overline{q-1}$ | 7 | $e = \dfrac{d}{q-1}$, for $\overline{q-1} \times e = qe - e$ |
| $1 + 7$ | 8 | $a = d + \dfrac{d}{q-1} = \dfrac{qd}{q-1}$ |
| $7 + 8$ | 9 | $a + e = \dfrac{qd+d}{q-1} = s$ |

| | | |
|---|---|---|
| 7 × 8 | 10 | $ae = \frac{qdd}{qq - 2q + 1} = p$ |
| 8 ⊚$^2$ | 11 | $aa = \frac{qqdd}{qq - 2q + 1}$ |
| 7 ⊚$^2$ | 12 | $ee = \frac{dd}{qq - 2q + 1}$ |
| 11 + 12 | 13 | $aa + ee = \frac{qqdd + dd}{qq - 2q + 1} = z$ |
| 11 − 12 | 14 | $aa - ee = \frac{qqdd - dd}{qq - 2q + 1} = x$ |

*Question* 8. Suppose $d$ and $z$ given, to find the rest.

| | | |
|---|---|---|
| *Viz.* { | 1 | $a - e = d = 192$ } Quære $a.e.s.p.q.x.$ |
| | 2 | $aa + ee = z = 47232$ } |
| 1 ⊚$^2$ | 3 | $aa - 2ae + ee = dd$ |
| 2 − 3 | 4 | $2ae = z - dd$ |
| 2 + 4 | 5 | $aa + 2ae + ae = 2z - dd$ |
| 5 ω$^2$ | 6 | $a + e = \sqrt{2z - dd} = s$ |
| 1 + 6 | 7 | $2a = d + \sqrt{2z - dd}$ |
| 7 ÷ 2 | 8 | $a = \frac{d + \sqrt{2z - dd}}{2}$ |
| 6 − 1 | 9 | $2e = \sqrt{2z - dd} - d$ |
| 9 ÷ 2 | 10 | $e = \frac{\sqrt{2z - dd} - d}{2}$ |
| 8 × 10 | 11 | $ae = \frac{z - dd}{2} = p$ |
| 8 ⊚$^2$ | 12 | $aa = \frac{z + d\sqrt{2z - dd}}{2}$ |
| 10 ⊚$^2$ | 13 | $ee = \frac{z - d\sqrt{2z - dd}}{2}$ |
| 12 − 13 | 14 | $aa - ee = d\sqrt{2z - dd} = x$ |
| 8 ÷ 10 | 15 | $\frac{a}{e} = \frac{d + \sqrt{2z - dd}}{\sqrt{2z - dd} - d} = q$ |

*Question*

*Question* 9. Let $d$ and $x$ be given, to find the rest.

| | | |
|---|---|---|
| Viz. | 1 | $a - e = d = 240$ |
| | 2 | $aa - ee = x = 46080$ — Quære $a.e.s.p.q.z.$ |
| $2 \div 1$ | 3 | $a + e = \frac{x}{d} = s$, viz. $a - e)\ aa - ee\ (a + e$ |
| $1 + 3$ | 4 | $2a = \frac{dd + x}{d}$ |
| $4 \div \overline{2}$ | 5 | $a = \frac{dd + x}{2d}$ |
| $3 - 5$ | 6 | $e = \frac{x - dd}{2d}$ |
| $5 \times 6$ | 7 | $ae = \frac{xx - d^4}{4dd} = p$ |
| $5 \div 6$ | 8 | $\frac{a}{e} = \frac{dd + x}{x - dd} = q$ |
| 5 ©$^2$ | 9 | $aa = \frac{d^4 + 2ddx + xx}{4dd}$ |
| 6 ©$^2$ | 10 | $ee = \frac{xx - 2ddx + d^4}{4dd}$ |
| $9 + 10$ | 11 | $aa + ee = \frac{d^4 + xx}{2dd} = z$ |

*Question* 10. Let $p$ and $q$ be given, to find the rest.

| | | |
|---|---|---|
| Viz. | 1 | $ae = p = 5184$ |
| | 2 | $\frac{a}{e} = q = 9$ — Quære $a.e.d.z.x.$ |
| $1 \times 2$ | 3 | $aa = qp$, for $\frac{ae}{1} \times \frac{a}{e} = \frac{aae}{e} = aa$ |
| 3 ω$^2$ | 4 | $a = \sqrt{qp}$ |
| $1 \div 2$ | 5 | $ee = \frac{p}{q}$, for $\frac{a}{1})\ \frac{ae}{1}\ (\frac{aee}{a} = ee$ |
| 5 ω$^2$ | 6 | $e = \sqrt{\frac{p}{q}}$ |
| $4 + 6$ | 7 | $a + e = \sqrt{qp} + \sqrt{\frac{p}{q}} = s$ |

| | | |
|---|---|---|
| $4 - 6$ | 8 | $a - e = \sqrt{qp} - \sqrt{\frac{p}{q}} = d$ |
| $3 + 5$ | 9 | $aa + ee = qp + \frac{p}{q} = z$ |
| $3 - 5$ | 10 | $aa - ee = pq - \frac{p}{q} = x$ |

*Question* 11. Let $p$ and $z$ be given, to find the rest.

| | | |
|---|---|---|
| Viz. { | 1 | $ae = p = 5184$ } Quære $a$ . $e$ . &c. |
| | 2 | $aa + ee = z = 47232$ |
| $1 \times \bar{2}$ | 3 | $2ae = 2p$ |
| $2 + 3$ | 4 | $aa + 2ae + ee = z + 2p$ |
| 4 ɯ$^{2}$ | 5 | $a + e = \sqrt{z + 2p} = s$ |
| $2 - 3$ | 6 | $aa - 2ae + ee = z - 2p$ |
| 6 ɯ$^{2}$ | 7 | $a - e = \sqrt{z - 2p} = d$ |
| $5 + 7$ | 8 | $2a = \sqrt{z + 2p} + \sqrt{z - 2p}$ |
| $8 \div \bar{2}$ | 9 | $a = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{2}$ |
| $5 - 7$ | 10 | $2e = \sqrt{z + 2p} - \sqrt{z - 2p}$ |
| $10 \div \bar{2}$ | 11 | $e = \frac{\sqrt{z + 2p} - \sqrt{z - 2p}}{2}$ |
| $9 \div 11$ | 12 | $\frac{a}{e} = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{\sqrt{z + 2p} - \sqrt{z - 2p}} = q$ |
| 9 ⊚$^{2}$ | 13 | $aa = \frac{z + \sqrt{zz - 4pp}}{2}$ |
| 11 ⊚$^{2}$ | 14 | $ee = \frac{z - \sqrt{zz - 4pp}}{2}$ |
| $aa - ee$ | 15 | $aa - ee = \sqrt{zz - 4pp} = x$ |

*Question* 12. Let $p$ and $x$ be given, to find the rest.

| | | |
|---|---|---|
| Viz. { | 1 | $ae = p = 5184$ } Quære $a$ . $e$ . &c. |
| | 2 | $aa - ee = x = 46080$ |
| 1 ⊚$^{2}$ | 3 | $aaee = pp$ |

| | | |
|---|---|---|
| 2 ⓖ 2 | 4 | $aaaa - 2aaee + eeee = xx$ |
| $3 \times 4$ | 5 | $4aaee = 4pp$ |
| $4 + 5$ | 6 | $aaaa + 2aaee + eeee = xx + 4pp$ |
| 6 ɯ 2 | 7 | $aa + ee = \sqrt{xx + 4pp} = z$ |
| $2 + 7$ | 8 | $2aa = x + \sqrt{xx + 4pp}$ |
| $8 \div \overline{2}$ | 9 | $aa = \frac{x + \sqrt{xx + 4pp}}{2}$ |
| 9 ɯ 2 | 10 | $a = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}}$ |
| $7 - 2$ | 11 | $2ee = \sqrt{xx + 4pp} - x$ |
| $11 \div \overline{2}$ | 12 | $ee = \frac{\sqrt{xx + 4pp} - x}{2}$ |
| 12 ɯ 2 | 13 | $e = \sqrt{:\frac{\sqrt{xx + 4pp} - x}{2}}$ |
| $10 + 13$ | 14 | $a + e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = s$ |
| $10 - 13$ | 15 | $a - e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = d$ |
| $9 + 12$ | 16 | $aa + ee = \sqrt{xx + 4pp} = z$ |

*Question* 13. Having $q$ and $z$ given, to find the rest.

| | | |
|---|---|---|
| *Viz.* | 1 | $\frac{a}{e} = q = 9$ } Quære $a$ . $e$, &c. |
| | 2 | $aa + ee = z =$ 472[illegible]2 |
| $1 \times e$ | 3 | $a = qe$ |
| 3 ⓖ 2 | 4 | $aa = qqee$ |
| $2 - 4$ | 5 | $ee = z - qqee$ |
| $4 + qqee$ | 6 | $qqee + ee = z$ |
| $6 \div \overline{qq+1}$ | 7 | $ee = \frac{z}{qq+1}$, for $\overline{qq + 1} \times ee = qqee + ee$ |

2 — 7

| | | |
|---|---|---|
| 2 — 7 | 8 | $aa = z - \frac{z}{qq+1} = \frac{qqz}{qq+1}$ |
| 8 $\omega^2$ | 9 | $a = \sqrt{\frac{qqz}{qq+1}}$ |
| 7 $\omega^2$ | 10 | $e = \sqrt{\frac{z}{qq+1}}$ |
| 9 + 10 | 11 | $a + e = \sqrt{\frac{qqz}{qq+1}} + \sqrt{\frac{z}{qq+1}} = s$ |
| 9 — 10 | 12 | $a - e = \sqrt{\frac{qqz}{qq+1}} - \sqrt{\frac{z}{qq+1}} = d$ |
| 9 × 10 | 13 | $ae = \sqrt{\frac{qqzz}{q^4 + 2qq + 1}} = p$ |
| 8 — 7 | 14 | $aa - ee = \frac{qqz - z}{qq+1} = x$ |

*Question* 14. When $q$ and $x$ are given, to find the rest.

| | | |
|---|---|---|
| *Viz.* | 1 | $\frac{a}{e} = q = 9$ } Quære $a \cdot e \cdot$ &c. |
| | 2 | $aa - ee = x = 46080$ } |
| 1 × $e$ | 3 | $a = qe$ |
| 3 $\circledS^2$ | 4 | $aa = qqee$ |
| 2 + $ee$ | 5 | $aa = x + ee$ |
| 4 and 5 | 6 | $qqee = x + ee$ |
| 6 — $ee$ | 7 | $qqee - ee = x$ |
| $7 \div \overline{qq-1}$ | 8 | $ee = \frac{x}{qq-1}$ |
| 2 + 8 | 9 | $aa = x + \frac{x}{qq-1} = \frac{qqx}{qq-1}$ |
| 9 $\omega^2$ | 10 | $a = \sqrt{\frac{qqx}{qq-1}}$ |
| 8 $\omega^2$ | 11 | $e = \sqrt{\frac{x}{qq-1}}$ |
| 10 + 11 | 12 | $a + e = \sqrt{\frac{qqx}{qq-1}} + \sqrt{\frac{x}{qq-1}} = s$ |

| | | |
|---|---|---|
| 10 — 11 | 13 | $a - e = \sqrt{\frac{qqx}{qq-1}} - \sqrt{\frac{x}{qq-1}} = d$ |
| 10 × 11 | 14 | $ae = \sqrt{\frac{qqxx}{qqqq - 2qq + 1}} = p$ |
| 8 + 9 | 15 | $aa + ee = \frac{qqx - x}{qq - 1} = z$ |

*Question* 15. When $z$ and $x$ are given, to find the rest.

| | | |
|---|---|---|
| *Viz.* | 1 | $aa + ee = z = 47232$ } Quære $a$ . $e$ . &c. |
| | 2 | $aa - ee = x = 46080$ |
| 1 + 2 | 3 | $2aa = z + x$ |
| $3 \div \overline{2}$ | 4 | $aa = \frac{z+x}{2}$ |
| 1 — 2 | 5 | $2ee = z - x$ |
| $5 \div \overline{2}$ | 6 | $ee = \frac{z-x}{2}$ |
| 4 ɯ² | 7 | $a = \sqrt{\frac{z+x}{2}}$ |
| 6 ɯ² | 8 | $e = \sqrt{\frac{z-x}{2}}$ |
| 7 + 8 | 9 | $a + e = \sqrt{\frac{z+x}{2}} + \sqrt{\frac{z-x}{2}} = s$ |
| 7 — 8 | 10 | $a - e = \sqrt{\frac{z+x}{2}} - \sqrt{\frac{z-x}{2}} = d$ |
| 7 × 8 | 11 | $ae = \sqrt{\frac{zz - xx}{4}} = p$ |
| 7 ÷ 8 | 12 | $\frac{a}{e} = \frac{\sqrt{z+x}}{\sqrt{z-x}} = q$ |

These fifteen Questions are proposed in Dr *Pell's Algebra*; but he pursues only the first Question throughout, and breaks off in the other fourteen, after the Values of what I call $a$ and $e$ are found. But I have proceeded in every one of them, to find the Values of all the unknown Qualities, because they afford

afford ſuch Variety, as being well obſerved by a Learner, will be found very uſeful in the Solution of moſt Queſtions.

*Note*, I have choſe to uſe the ſame Numbers for the reſpective Value of each Quantity throughout all the Queſtions, becauſe they will be more ſatisfactory in proving the Work than various Numbers would have been. Not but that any Numbers may be taken at Pleaſure, provided that the Number repreſented by $a$, be greater than that by $e$, &c. I have omitted the Numerical Calculations purely for the Learner to practiſe on.

*Queſtion* 16. There are two Numbers, the Sum of their Squares is 2368; and the greater of them is in Proportion to the leſs, as 6 to 1. What are theſe Numbers?

Let $a =$ the greater Number, $e =$ the leſſer, and $z = 2368$.

| | | |
|---|---|---|
| Then | 1 | $aa + ee = z$ } by the Queſtion. |
| And | 2 | $a : e :: 6 : 1$ } |
| 2 ·.· | 3 | $1a = 6e$ |
| 3 ⓢ 2 | 4 | $aa = 36ee$ |
| 1 — 4 | 5 | $ee = z - 36ee$ |
| $5 + 36ee$ | 6 | $37ee = z$ |
| $6 \div \overline{37}$ | 7 | $ee = \frac{z}{37} = 64$ |
| 7 ω 2 | 8 | $e = \sqrt{\frac{z}{37}} = 8$ |
| $8 \times \overline{6}$ | 9 | $6e = 6\sqrt{\frac{z}{37}} = 48$ |
| 3 and 9 | 10 | $a = 48$ |

*Proof*

| | |
|---|---|
| If $a =$ | 48 |
| and $e =$ | 8 |
| $aa =$ | 2304 |
| $ee =$ | 64 |
| $aa + ee =$ | 2368 |
| and 48 : 8 :: 6 : 1 | |

*Queſtion* 17. There are three Numbers in continued Proportion, the Sum of the Extreams is 156, and the Mean is 72; What are the two Extreams?

That is, Suppoſe $a \cdot m \cdot e$ in $\div\!\!\div$, and $m = 72$.

| | | |
|---|---|---|
| Then { | 1 | $a + e = 156 = s$ } by the Queſtion. |
| | 2 | $a : m :: m : e$ } Quære $a \cdot e$. &c. |
| 2 ·.· | 3 | $ae = mm$ |
| 1 ⓢ 2 | 4 | $aa + 2ae + ee = ss$ |
| 3 × 4 | 5 | $4ae + 4mm$ |

4 — 5

| | | | |
|---|---|---|---|
| 4 − 5 | 6 | $aa - 2ae + ee = ss - 4mm$ | |
| 6 $ω^2$ | 7 | $a - e = \sqrt{ss - 4mm}$ | |
| 1 + 7 | 8 | $2a = s + \sqrt{ss - 4mm}$ | |
| 8 ÷ 2 | 9 | $a = \frac{s + \sqrt{ss - 4mm}}{2} = 108$ | Or $a = 48$ |
| 1 − 9 | 10 | $e = \frac{s - \sqrt{ss - 4mm}}{2} = 48$ | $e = 108$ |

*Question* 18. There are three Numbers in ∺, their Sum is 74, and the Sum of their Squares is 1924; What are those Numbers?

That is, $a$, $e$, $y$ are in ∺

| | | | |
|---|---|---|---|
| Then | 1 | $a + e + y = s = 74$ | Quære $a$, $e$, $y$. |
| | 2 | $aa + ee + yy = z = 1924$ | |
| | 3 | $a : e :: e : y$ | |
| 5 ∵ | 4 | $ay = ee$ | |
| 1 − $e$ | 5 | $a + y = s - e$ | |
| 2 − $ee$ | 6 | $aa + yy = z - ee$ | |
| 4 × 2 | 7 | $2ay = 2ee$ | |
| 6 + 7 | 8 | $aa + 2ay + yy = z + ee$ | |
| 5 ℰ² | 9 | $aa + 2ay + yy = ss - 2se + ee$ | |
| 8 and 9 | 10 | $z + ee = ss - 2se + ee$ | |
| 10 ± | 11 | $2se = ss - z$ | |
| 11 ÷ 2$s$ | 12 | $e = \frac{ss - z}{2s} = 24$ | |
| 5, | 13 | $a + y = s - e = 50$ | |
| 13 ℰ² | 14 | $aa + 2ay + yy = 2500$ | |
| 4 × 4 | 15 | $4ay = 4ee = 2304$ | |
| 14 − 15 | 16 | $aa - 2ay + yy = 196$ | |
| 16 $ω^2$ | 17 | $a - y = \sqrt{196} = 14$ | |
| 13 + 17 | 18 | $2a = 50 + 14 = 64$ | |
| 18 ÷ 2 | 19 | $a = 32$ | Or $a = 18$ |
| 13 − 19 | 20 | $y = 50 - 32 = 18$ | $y = 32$ |

*Note*, In all Questions about continual Proportionals, (either Arithmetical or Geometrical) where three Terms are sought, the Mean is the easiest found first (as above) and if all the Terms be Affirmative, then it is equal whether the first or last Term be the greatest.

*Question*

*Question* 19. There are three Numbers in ∺ their Sum is 76; and if the Sum of the Extremes be multiplied into the Mean, that Product will be 1248; What are those Numbers?

| | | |
|---|---|---|
| *Viz.* | 1 | $a : e :: e : y$ ⎫ |
| | 2 | $a + e + y = s = 76$ ⎬ by the Question. |
| | 3 | $ae + ye = p = 1248$ ⎭ |
| 1 ∵ | 4 | $ay = ee$ |
| $1 \times e$ | 5 | $ae + ee + ye = se$ |
| $5 - 3$ | 6 | $ee = se - p$ |
| $6 - se$ | 7 | $ee - se = -p$ |
| 7 C □ | 8 | $ee - se + \frac{1}{4}ss = \frac{1}{4}ss - p$ |
| $8\ \omega^2$ | 9 | $e - \frac{1}{2}s = \sqrt{\frac{1}{4}ss - p}$ |
| $9 + \frac{1}{2}s$ | 10 | $e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = \begin{cases} 52. \\ 24 \end{cases}$ per Theorem 3. Chap. 8. |
| $2 - 10$ | 11 | $a + y = 52$ |
| $4 \times 4$ | 12 | $4ay = ee = 2304$ |
| $11 \odot^2$ | 13 | $aa + 2ay + yy = 2704$ |
| $13 - 12$ | 14 | $aa - 2ay + yy = 400$ |
| $14\ \omega^2$ | 15 | $a - y = \sqrt{400} = 20$ |
| $11 + 15$ | 16 | $2a = 52 + 20 = 72$ |
| $16 \div 2$ | 17 | $a = 36$ ⎫ ⎧ Or $a = 16$ |
| $11 - 17$ | 18 | $y = 52 - 36 = 16$ ⎭ ⎩ and $y = 36$ |

*N. B.* If you take $e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = 52$ (at the 10th Step) then it will be $76 - 52 = 24 = a + y$, which is impossible, *viz.* that the Mean should be greater than the Sum of the two Extreams. Therefore it must be $e = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - p} = 24$. (See page 201.)

*Question* 20. There are three Numbers in Arithmetical Progression, the first being added to twice the second, and three times the third, their Sum will be 62; and the Sum of all their Squares is 275; What are those Numbers?

| | | |
|---|---|---|
| Suppose | 1 | $a, e, y$ in Arithmetical Progression. |
| And | 2 | $a + 2e + 3y = 62$ ⎫ by the Question. |
| | 3 | $aa + ee + yy = 275$ ⎭ |
| Then | 4 | $a + y = 2e$, per Sect. 1. Chap. 6. |
| $2 - 4$ | 5 | $2e + 2y = 62 - 2e$ |
| $5 \div 2$ | 6 | $e + y = 31 - e$ |
| $6 - e$ | 7 | $y = 31 - 2e$ |
| $4 - 7$ | 8 | $a = 4e - 31$ |

| | | |
|---|---|---|
| $8 \circledcirc^2$ | 9 | $aa = 16ee - 248e + 961$ |
| $7 \circledcirc^2$ | 10 | $yy = 961 - 124e + 4ee$ |
| $9 + 10$ | 11 | $aa + yy = 20ee - 372e + 1922$ |
| $3 - 11$ | 12 | $ee = 372e - 20ee - 1647$ |
| $12 + 20ee$ | 13 | $21ee = 372e - 1647$ |
| $13 - 372e$ | 14 | $21ee - 372e = -1647$ |
| $14 \div 21$ | 15 | $ee - \frac{124}{7}e = -\frac{549}{7}$ |
| $15\ C\ \square$ | 16 | $ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{549}{7} = \frac{1}{49}$ |
| $16\ \omega^2$ | 17 | $e - \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$, |
| $17 + \frac{62}{7}$ | 18 | $e = \frac{62}{7} \pm \frac{1}{7} = 9$, or $8\frac{5}{7}$ the Mean |
| $18 \times 4$ | 19 | $4e = 36$, or $\frac{244}{7}$ |
| 8 and 19 | 20 | $a = 36 - 31 = 5$, or $34\frac{6}{7} - 31 = 3\frac{6}{7}$ |
| $18 \times 2$ | 21 | $2e = 18$, or $17\frac{3}{7}$ |
| 7 and 21 | 22 | $y = 31 - 18 = 13$, or $31 - 17\frac{3}{7} = 13\frac{4}{7}$ |

*Question* 21. There are three Numbers in Arithmetical Progreſſion; the Square of the firſt Term being added to the Product of the other two is 576; the Square of the Mean being added to the Product of the two Extreams, make 612; and the Square of the laſt Term being added to the Product of the firſt into the ſecond, is 792: What are thoſe Numbers?

| | | |
|---|---|---|
| Suppoſe | 1 | $a$, $e$, $y$ in Arith. Progreſſ. as before. |
| Then | 2 | $aa + ye = 576$ by the Queſtion. |
| | 3 | $ee + ya = 612$ by the Queſtion. |
| | 4 | $yy + ae = 792$ by the Queſtion. |
| $1 \because$ | 5 | $a + y = 2e$, per Sect. 1. Chap. 6. |
| $5 \times e$ | 6 | $ae + ye = 2ee$ |
| $2 + 4$ | 7 | $aa + ye + yy + ae = 1368$ |
| $7 - 6$ | 8 | $aa + yy = 1368 - 2ee$ |
| $3 - ee$ | 9 | $ya = 612 - ee$ |
| $9 \times 2$ | 10 | $2ya = 1224 - 2ee$ |
| $8 + 10$ | 11 | $aa + 2ya + yy = 2592 - 4ee$ |
| $5 \circledcirc^2$ | 12 | $aa + 2ya + yy = 4ee$ |
| 11 and 12 | 13 | $4ee = 2592 - 4ee$ |
| $13 + 4ee$ | 14 | $8ee = 2592$ |
| $14 \div 8$ | 15 | $ee = 324$ |
| $15\ \omega^2$ | 16 | $e = \sqrt{324} = 18$, the Mean |
| 8, | 17 | $aa + yy = 1368 - 2ee = 720$ |
| 10, | 18 | $2ya = 1224 - 2ee = 576$ |
| $17 - 18$, | 19 | $aa - 2ya + yy = 720 - 576 = 144$ |

| | | |
|---|---|---|
| 1 ω² | 20 | $a - y = \sqrt{144} = 12$ |
| 5 + 20 | 21 | $2a = 2e + 12 = 48$ |
| 21 ÷ 2 | 22 | $a = 24$ |
| 5 — 22 | 23 | $y = 2e - 24 = 12$ |

Or $\begin{cases} a = 12 \\ y = 24 \end{cases}$

*Question* 22. It is required to find two such Numbers, that the Sum of their Squares may be $8226\frac{1}{2}$; and their Product being added to the Square of the lesser, may be $6921\frac{1}{2}$.

| | | |
|---|---|---|
| *Viz.* | 1 | $aa + ee = 8226\frac{1}{2}$ } Quære $a$ and $e$ |
| | 2 | $ae + ee = 6921\frac{1}{2}$ |
| 1 — 2 | 3 | $aa - ae = 1305$ |
| 3 ± | 4 | $ae = aa - 1305$ |
| 4 ÷ $a$ | 5 | $e = \frac{aa - 1305}{a}$ |
| 5 ⊛² | 6 | $ee = \frac{a^4 - 2610aa + 1703025}{aa}$ |
| 1 — $aa$ | 7 | $ee = 8226,5 - aa$ |
| 6 and 7 | 8 | $\frac{a^4 - 2610aa + 1703025}{aa} = 8226,5 - aa$ |
| 8 × $aa$ | 9 | $a^4 - 2610aa + 1703025 = 8226,5aa - a^4$ |
| 9 + $a^4$ | 10 | $2a^4 - 2610aa + 1703025 = 8226,5aa$ |
| 10 ± | 11 | $2a^4 - 10836,5aa = -1703025$ |
| 11 ÷ 2 | 12 | $a^4 - 5418,25aa = -85151,25$ |
| 12 C □ | 13 | $a^4 - 5418,25aa + 7339358,26562 = 6487845,765$ |
| 13 ω² | 14 | $aa - 2709,125 = \sqrt{6487845,765625} - 2547,125$ |
| 14+27&c. | 15 | $aa = 2709,125 + 2547,125$ |
| Suppose | 16 | $aa = 2709,125 + 2547,125 = 5256,25$ |
| Then | 17 | $a = \sqrt{5256,25} = 72,5$ |
| And 5, | 18 | $e = \frac{aa + 1305}{a} = \frac{5256,25 - 1305}{72,5} = 54,5$ |
| Or let | 19 | $aa = 2709,125 - 2547,125 = 162$ |
| 10 ω² | 20 | $a = \sqrt{162} = 12,72$ &c. |
| Then | 21 | $e = \frac{162 - 1305}{12,72}$, which is impossible. |
| Therefore | | $a = 72,5$ } as at the 17th and 18th Steps. |
| And | | $e = 54,5$ |

This Question may be performed with less Trouble, by substituting Letters from the known Numbers.

*Viz.* $\begin{cases} aa + ee = z \\ ae + ee = p \end{cases}$ Then let $z - p = d = aa - ae$, &c.

*Question* 23. It is required to find three ſuch Numbers, that the Sum of the firſt and ſecond, being multiplied with the third, may be 37824; and the Sum of the ſecond and third, multiplied with the firſt, may be 59944; alſo, that the Sum of the firſt and third, being multiplied with the ſecond, may be 52456.

Let $a$, $e$, $y$ repreſent the three Numbers.

| | | |
|---|---|---|
| Then | 1 | $ay + ey = 37824 = b$ |
| | 2 | $ea + ya = 59944 = c$ — Quære $a$, $e$, $y$. |
| | 3 | $ae + ye = 52456 = d$ |
| 1 + 2 + 3 | 4 | $2ae + 2ay + 2ye = b + c + d$ |
| Let | 5 | $z = b + c + d$ |
| 4 ÷ 2 | 6 | $ae + ay + ye = \frac{1}{2}z = \frac{b+c+d}{2}$ |
| 6 − 3 | 7 | $ay = \frac{1}{2}z - d = \frac{z-2d}{2}$ |
| 7 ÷ $a$ | 8 | $y = \frac{z-2d}{2a}$ |
| 6 − 2 | 9 | $ye = \frac{1}{2}z - c = \frac{z-2c}{2}$ |
| 6 − 1 | 10 | $ae = \frac{1}{2}z - b = \frac{z-2b}{2}$ |
| 10 ÷ $a$ | 11 | $e = \frac{z-2b}{2a}$ |
| 8 × 11 | 12 | $ye = \frac{z-2d}{2a} \times \frac{z-2b}{2a} = \frac{zz-2dz-2bz+4bd}{4aa}$ |
| 9 and 12 | 13 | $\frac{z-2c}{2} = \frac{zz-2dz-2bz+4bd}{4aa}$ |
| 13 × $4aa$ | 14 | $2zaa - 4caa = zz - 2dz - 2bz + 4bd$ |
| 14 ÷ | 15 | $aa = \frac{zz-2dz-2bz+4bd}{2z-4c} = 55696$ |
| 15 $\omega^2$ | 16 | $a = \sqrt{55696} = 236$ |
| 11 | 17 | $e = \frac{z-2b}{2a} = 158$ |
| 8 | 18 | $y = \frac{z-2d}{2a} = 96$ |

*Question* 24. It is required to find two ſuch Numbers, that their Sum being ſubſtracted from the Sum of their Squares, may leave 14, and if their Product be added to their Sum, it may make 14.

Let $a$ and $e$ be put for the Numbers, and let $y = a + e$

| | | |
|---|---|---|
| Then | 1 | $aa + ee - y = 14$ — by the Queſtion. |
| | 2 | $ae + y = 14$ |

1 + 7

| | | | |
|---|---|---|---|
| $1 + 7$ | 3 | $aa + ee = 14 + y$ | |
| $2 - y$ | 4 | $ae = 14 - y$ | |
| $4 \times 2$ | 5 | $2ae = 28 - 2y$ | |
| $3 + 5$ | 6 | $aa + 2ae + ee = 42 - y$ | |
| $6\ \omega^2$ | 7 | $a + e = \sqrt{42 - y}$ | |
| But | 8 | $a + e = y$, by Subſtitution above. | |
| 7 and 8 | 9 | $y = \sqrt{42 - y}$ | |
| 9 ♋$^2$ | 10 | $yy = 42 - y$ | |
| $10 + y$ | 11 | $yy + y = 42$ | |
| 11 C □ | 12 | $yy + y + \frac{1}{4} = 42 + \frac{1}{4} = 42{,}25$ | |
| $12\ \omega^2$ | 13 | $y + \frac{1}{2} = \sqrt{42{,}25} = 6{,}5$ | |
| $13 - \frac{1}{2}$ | 14 | $y = 6{,}5 - \frac{1}{2} = 6$ | |
| Conſequent | 15 | $a + e = 6$, by Reſtitution from above. | |
| 3 and 14 | 16 | $aa + ee = 14 + 6 = 20$ | |
| 5 and 15 | 17 | $2ae = 28 - 12 = 16$ | |
| $16 - 17$ | 18 | $aa - 2ae + ee = 4$ | |
| $18\ \omega^2$ | 19 | $a - e = \sqrt{4} = 2$ | |
| $15 + 19$ | 20 | $2a = 8$ | Proof: If $a = 4$, and $e = 2$ |
| $23 \div 2$ | 21 | $a = 4$ | Then $aa + ee - a - e = 14$ |
| $1 - 21$ | 22 | $e = 6 - 4 = 2$ | And $ae + a + e = 14$ According to the Queſtion. |

*Queſtion* 25. Three Men diſcourſing of their Money; ſaith the firſt, if 100 *l.* were added to my Money, it would be as much as both your Money put together; ſaid the ſecond Man, if 100 *l.* were added to my Money, I ſhould have twice as much as both you have; ſaith the third Man, if 100 *l.* were added to my Money, I ſhould have then three times as much Money as both you have; How much Money had each Man?

Let $a$ repreſent the firſt Man's Money, $e$ the ſecond, and $y$ the third.

| | | | |
|---|---|---|---|
| Then | 1 | $a + 100 = e + y$ | by the Queſtion. |
| | 2 | $e + 100 = 2a + 2y$ | |
| | 3 | $y + 100 = 3a + 3e$ | |
| $1 - a$ | 4 | $e + y - a = 100 = s$ | Quære $a$, $e$, $y$. |
| $2 - e$ | 5 | $2a + 2y - e = 100 = s$ | |
| $3 - y$ | 6 | $3a + 3e - y = 100 = s$ | |
| 4 and 6 | 7 | $e + y - a = 3a + 3e - y$ | |
| $7 \pm$ | 8 | $2y = 4a + 2e$ | |
| $5 - 8$ | 9 | $2a - e = s - 4a - 2e$ | |
| $9 + 4a - 2e$ | 10 | $6a + e = s = 100$ | |
| $4 + 6$ | 11 | $2a + 4e = 2s = 200$ | |

| | | |
|---|---|---|
| $10 \times \overline{4}$ | 12 | $24a + 4e = 4s = 400$ |
| $12 - 11$ | 13 | $22a = 2s = 200$ |
| $13 \div \overline{22}$ | 14 | $a = \frac{s}{11} = \frac{100}{11} = 9\frac{1}{11}\,l.$ |
| $10 - 6a$ | 15 | $e = s - 6a = 100 - \frac{600}{11} = \frac{500}{11} = 45\frac{5}{11}\,l.$ |
| $8 \div \overline{2}$ | 16 | $y = 2a + e = \frac{200}{11} + \frac{500}{11} = \frac{700}{11} = 63\frac{7}{11}\,l.$ |

Anfwer. The { first / fecond / third } Man had { $9\,l.$ $1\,s.$ $9\frac{9}{11}\,d.$ / $45\,l.$ $9\,s.$ $1\frac{2}{11}\,d.$ / $64\,l.$ $12\,s.$ $8\frac{1}{11}\,d.$ }

*Queftion* 26. Three Men have each fuch a Sum of Money, that if the firft and fecond Mens Money be added to Half of what the third Man hath; that Sum will be 92 *l.* And if the fecond and third Mens Money be added to one third Part of the firft Man's Money, that Sum will be 92 *l.* Laftly, if one fourth Part of the fecond Man's Money be added to the firft and third Man's Money, that Sum will alfo be 92 *l.* How much was each Man's Money?

Put *a* for the 1ft Man's Money, *e* for the 2d, and *y* for the 3d.

| | | |
|---|---|---|
| Then | 1 | $a + e + \frac{1}{2}y = s$ |
| | 2 | $\frac{1}{3}a + e + y = s$ } by the Queftion; and $s = 92$ |
| | 3 | $\frac{1}{4}e + a + y = s$ |
| 1 and 2 | 4 | $a + e + \frac{1}{2}y = \frac{1}{3}a + e + y$ |
| $4 - e$ | 5 | $a + \frac{1}{2}y = \frac{1}{3}a + y$ |
| $5 \times \overline{2} \times \overline{3}$ | 6 | $6a + 3y = 2a + 6y$ |
| $6 \pm$ | 7 | $4a = 3y$ |
| $2 \times \overline{3}$ | 8 | $a + 3e + 3y = 3s$ |
| $8 - 7$ | 9 | $a + 3e = 3s - 4a$ |
| $9 - a$ | 10 | $3e = 3s - 5a$ |
| $10 \div \overline{3}$ | 11 | $e = \frac{3s - 5a}{3}$ |
| $3 \times \overline{4}$ | 12 | $e + 4a + 4y = 4s = 368$ |
| $12 - 2$ | 13 | $3\frac{2}{3}a + 3y = 3s = 276$ |
| 13 and 7 | 14 | $3\frac{2}{3}a + 4a = 3s = 276$ |
| $14 \times 3$ | 15 | $11a + 12a = 9s = 828$ |
| $15 \div 23$ | 16 | $a = \frac{9s}{23} = \frac{828}{23} = 36\,l.$ the 1*ft* Man's Money. |
| 11 | 17 | $e = \frac{3s - 5a}{3} = \frac{96}{3} = 32\,l.$ the 2*d* Man's Money. |
| $7 \div \overline{3}$ | 18 | $y = \frac{4a}{3} = \frac{144}{3} = 48\,l.$ the 3*d* Man's Money. |

*Queftion*

*Question* 27. Four Men walking abroad, found a Purse of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers togther they found, that if the first took 25 Shillings from the second, it would make his Number equal with what the second had then left; if the second took 30 Shillings from the third, his Money would then be triple to what the third had left, and if the third took 40 Shillings from the fourth, his Money would then be double to what the fourth had left; lastly, the fourth taking 50 Shillings from the first, he would then have three times as much as the first had left, and 5 Shillings more: It is required to tell how many Shillings each Man had.

Put $a$ for the first Sum, $e$ the second, $y$ the third, and $u$ the fourth.

| | | | |
|---|---|---|---|
| Then | 1 | $a + 25 = e - 25$ | by the Question. |
| | 2 | $e + 30 = 3y - 90$ | |
| | 3 | $y + 40 = 2u - 80$ | |
| | 4 | $u + 50 = 3a - 145$ | |
| $1 + 25$ | 5 | $a + 50 = e$ | |
| $2 - 30$ | 6 | $3y - 120 = e$ | |
| 5 and 6 | 7 | $a + 50 = 3y - 120$ | |
| $7 + 120$ | 8 | $a + 170 = 3y$ | |
| $8 \div 3$ | 9 | $y = \frac{a + 170}{3}$ | |
| $3 - 40$ | 10 | $y = 2u - 120$ | |
| 9 and 10 | 11 | $2u - 120 = \frac{a + 170}{3}$ | |
| $1 + 120$ | 12 | $2u = \frac{a + 170}{3} + 120 = \frac{a + 530}{3}$ | |
| $12 \div 2$ | 13 | $u = \frac{a + 530}{6}$ | |
| $4 - 50$ | 14 | $u = 3a - 195$ | |
| 13 and 14 | 15 | $3a - 195 = \frac{a + 530}{6}$ | |
| $15 \times 6$ | 16 | $18a - 1170 = a + 530$ | |
| $16 +$ | 17 | $17a = 1700$ | |
| $17 \div 17$ | 18 | $a = 100$ the 1*st* | Man's Number of Shillings. |
| by the 5 | 19 | $e = 150$ 2*d* | |
| by the 9 | 20 | $y = 90$ 3*d* | |
| by the 14 | 21 | $u = 105$ 4*th* | |

*Question*

*Question* 28. Four Men have each a Sum of Money, which being put all together makes 250 Pounds; and if to the first Man's Money be added 8 Pounds, it will be just as much as the second Man's Money decreased by 8 Pounds, and as much as 8 times the third Man's Money, and but as much as one eighth Part of the fourth Man's Money; how much had each Man?

Let $a$, $e$, $y$, $u$, represent the four Men's Money.

| | | | |
|---|---|---|---|
| Then | 1 | $a+e+y+u=s$ | by the Question. Let $s=250$ and $b=8$, or any other Number at Pleasure. |
| | 2 | $a+b=e-b$ | |
| | 3 | $yb=\frac{u}{b}=a+b$ | |
| $2+b$ | 4 | $a+2b=e$ | |
| $3\div b$ | 5 | $y=\frac{a+b}{b}$, because $yb=a+b$ | |
| $3\times b$ | 6 | $u=ba+bb$, for $\frac{u}{b}=a+b$ | |
| $4+5+6$ | 7 | $e+y+u=a+2b+\frac{a+b}{b}+ba+bb$ | |
| $1-a$ | 8 | $e+y+u=s-a$ | |
| 7 and 8 | 9 | $a+2b+\frac{a+b}{b}+ba+bb=s-a$ | |
| $9\times b$ | 10 | $ba+2bb+a+b+bba+bbb=bs-ba$ | |
| $10\pm$ | 11 | $2ba+bba+a=bs-bbb-2bb-b$ | |
| $11\div$ | 12 | $a=\frac{bs-bbb-2bb-b}{bb+2b+1}=16{,}691358$ &c. | |
| by the 4, | 13 | $e=a+2b=32{,}691358$ &c. | |
| by the 5, | 14 | $y=\frac{a+b}{b}=3{,}086419$ &c. | |
| by the 6, | 15 | $u=ba+bb=197{,}530864$ &c. | |

| | | *l.* | *s.* | *d.* |
|---|---|---|---|---|
| That is, | $a=$ | 16 | 13 | 9,92592 |
| | $e=$ | 32 | 13 | 9,92592 |
| | $y=$ | 3 | 1 | 8,74056 |
| | $u=$ | 197 | 10 | 7,40736 |
| Consequently | $a+e+y+u=$ | 249 | 19 | 11,99976 |

which should be just 250 *l.* the Sum proposed in the Question. Now what it wants of that Sum, proceeds from the Imperfection of the Decimal Parts being not continued on to more Places, which would have brought it nearer the Truth, tho' not perhaps exactly so. *Sect.* 5. *Chap.* 5. *Part* 1.

*Question*

*Question* 29. Several Merchants enter into Partnership, every one put into the Stock 65 times as many Pounds as there were Partners; with that Stock they traded and gained as many Pounds per 100*l.* as there were Partners. Now if 10*l.* 10*s.* be added to, and substracted from, their Gain, the Product of that Sum and Difference will be 6491*l.* 6*s.* 3*d.*

*Quære*, How many Merchants there were, *&c.*

| | | |
|---|---|---|
| Let | 1 | $a =$ the Number of Merchants. |
| $1 \times \overline{65}$ | 2 | $65a =$ every one's Sum put into Stock. |
| $2 \times a$ | 3 | $65aa =$ the whole Stock. |
| And | 4 | $100 : a :: 65aa : \frac{65aaa}{100}$, by the Question. |
| *Viz.* | 5 | $\frac{65aaa}{100} =$ the whole Gain. |
| $5 + \overline{10{,}5}$ | 6 | $\frac{65aaa}{100} + 10{,}5$ |
| $5 - \overline{10{,}5}$ | 7 | $\frac{65aaa}{100} - 10{,}5$ |
| $6 \times 7$ | 8 | $\frac{4225aaaaaa}{10000} - 110{,}25 = 6491{,}3125$, by the Quest. |
| $8 \times \overline{10000}$ | 9 | $4225a^6 - 1102500 = 64913125$ |
| $9 +$ | 10 | $4225a^6 = 66015625$ |
| $10 \div \overline{4225}$ | 11 | $a^6 = \frac{66015625}{4225} = 15625$ |
| $11\ \omega^6$ | 12 | $a = \sqrt{15625} = 5$ the Number of Merchants. |
| $12 \times \overline{65}$ | 13 | $65a = 325$ the Number of Pounds each put in. |

*Question* 30. Three Merchants join Stocks together; the first Man's Stock was less than the second Man's by 13*l.* the second and third Man's Stock was 175*l.* in trading they gain 48*l.* more than their whole Stock was; the first Man's proportional Part of the Gain was 78. What was each Man's Stock and Part of the Gain?

Let $a$, $e$, $y$ represent each Man's Stock.

| | | |
|---|---|---|
| Then | 1 | $a + e + y = s$ the whole Stock. |
| | 2 | $s + 48 =$ the whole Gain. |
| And | 3 | $a + 13 = e$ } by the Question. |
| | 4 | $e + y = 175$ |
| $4 + a$ | 5 | $a + e + y = 175 + a$ |
| 1 and 5 | 6 | $s = 175 + a$ |

6 and 2

| | | |
|---|---|---|
| 6 and 2 | 7 | $s + 48 = 223 + a$ |
| But | 8 | $175 + a : 223 + a :: a : 78$ per Question. |
| 8 ∵ | 9 | $aa + 223a = 78a + 13650$ |
| $9 - 78a$ | 10 | $aa + 145a = 13650$ |
| 10 G □ | 11 | $aa + 145a + 5256,25 = 18906,25$ |
| 11 $\omega^2$ | 12 | $a + 72,5 = \sqrt{18906,25} = 137,5$ |
| $12 - 72,5$ | 13 | $a = 137,5 - 72,5 = 65$ |
| 3, | 14 | $e = a + 13 = 78$ |
| $4 - 14$ | 15 | $y = 97$ |
| Then | 16 | $65 : 78 :: 78 : 93$ *l.* 12 *s.* = *e*'s Gain. |
| Again | 17 | $65 : 78 :: 97 : 116$ *l.* 8 *s.* = *y*'s Gain. |
| Proof | 18 | 116 *l.* 8 *s.* + 93 *l.* 12 *s.* + 78 *l.* = 288 *l.* the Gain, |
| | 19 | $65 + 78 + 97 = 240$. the whole Stock. |
| $18 - 19$ | 20 | $288 - 240 = 48$ the Gain more than the Stock. |

*Question* 31. A Father at his Death left his three Sons his Money in this manner; to the eldest he gave half of it, wanting 44 Pounds; to the second he gave one third of it, and 14 Pounds more; to the youngest he gave the Remainder, which was less than the Share of the second Son, by 82 Pounds: What was each Son's Share?

Let $a$, $e$, $y$ be the three Shares, and $z =$ the whole Sum.

| | | |
|---|---|---|
| Then | 1 | $a + e + y = z$ (by the Question.) |
| | 2 | $a = \frac{1}{2}z - 44$ (by the Question.) |
| | 3 | $e = \frac{1}{3}z + 14$ (by the Question.) |
| | 4 | $y = \frac{1}{3}z + 14 - 82$ (by the Question.) |
| $2 + 3 + 4$ | 5 | $a + e + y = \frac{2z}{3} + \frac{z}{2} - 98$ |
| 1 and 5 | 6 | $z = \frac{2z}{3} + \frac{z}{2} - 98$ |
| $6 \times \overline{3}$ | 7 | $3z = 2z + \frac{3z}{2} - 294$ |
| $7 \times \overline{2}$ | 8 | $6z = 4z + 3z - 588$ |
| 8 + | 9 | $z = 588$, the whole Sum that was left. |
| 2 and 9 | 10 | $a = \frac{588}{2} - 44 = 250$, the eldest Son's Share. |
| 3 and 9 | 11 | $e = \frac{588}{3} + 14 = 210$, the second Son's Share. |
| 4 and 9 | 12 | $y = \frac{588}{3} + 14 - 82 = 128$, the youngest &c. |

*Question* 32. A Man playing at Hazard or Dice, won the first Throw just so much Money as he had in his Pocket; the second

ſecond Throw he won the Square Root of what he then had, and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was 112 *l.* 16 *s.* What Money had he when he began to play?

| | | |
|---|---|---|
| Suppoſe | 1 | $a =$ his firſt Sum. Then |
| $1 \times \overline{2}$ | 2 | $2a =$ his Sum after the firſt Throw. |
| And | 3 | $5 + \sqrt{2a} =$ the Winnings at the 2d Throw. |
| $2 + 3$ | 4 | $2a + 5 + \sqrt{2a} =$ the Sum after the 2d Throw. |
| $4 \circledS^2$ | 5 | $4aa + 22a + 25 + 4a\sqrt{2a} : + 10\sqrt{2a} =$ the Winnings at the 3d Throw; and therefore |
| $4 + 5$ | 6 | $4aa + 24a + 30 + 4a\sqrt{2a} + 11\sqrt{2a} = 2256$ Shil. |

But to avoid theſe Surd Quantities, let us, inſtead of ſuppoſing $a =$ the firſt Sum, make a ſecond Trial, *viz.*

| | | |
|---|---|---|
| Let | 1 | $2aa =$ the firſt Sum. |
| $1 \times \overline{2}$ | 2 | $4aa =$ the Sum after the firſt Throw. |
| Then | 3 | $2a + 5 =$ the Sum won at the 2d Throw. |
| $2 + 3$ | 4 | $4aa + 2a + 5 =$ his Sum after the 2d Throw. |
| $4 \circledS^2$ | 5 | $16a^4 + 16a^3 + 44aa + 20a + 25 =$ the Winnings at the 3d Throw; and therefore |
| $4 + 5$ | 6 | $16a^4 + 16a^3 + 48aa + 22a + 30 = 2256$ Shil. |

Yet again, to avoid theſe high Equations, let us make a third Suppoſition; thus,

| | | |
|---|---|---|
| Let | 1 | $\frac{aa}{2} =$ the firſt Sum. |
| $1 \times \overline{2}$ | 2 | $aa =$ the Sum after the firſt Throw. |
| Then | 3 | $a + 5 =$ the Winnings at the 2d Throw. |
| $2 + 3$ | 4 | $aa + a + 5 =$ the Sum after the 2d Throw. |
| Subſti. | 5 | $e = aa + a + 5$. |
| $5 \circledS^2$ | 6 | $ee =$ the Winnings at the 3d Throw. Then |
| $5 + 6$ | 7 | $ee + e = 2256$ Shillings by the Queſtion. |
| $7\ C\ \square$ | 8 | $ee + e + 0,25 = 2256,25$ |
| $8\ \omega^2$ | 9 | $e + 0,5 = \sqrt{2256,25} = 47,5$ |
| $9 - 0,5$ | 10 | $e = 47$ |
| 5 and 10 | 11 | $aa + a + 5 = 47$ |
| $11 - \overline{5}$ | 12 | $aa + a = 42$ |
| $12,\ C\ \square$ | 13 | $aa + a + 0,25 = 42,25$ |
| $1\ \omega\ \overline{2}$ | 14 | $a + 0,5 = \sqrt{42,25} = 6,5$ |
| $14 - 0,5$ | 15 | $a = 6$ |
| $15 \circledS^2$ | 16 | $aa = 36$ |
| $16 \div \overline{2}$ | 17 | $\frac{aa}{2} = \frac{36}{2} = 18$ { The Shillings he had in his Pocket when he began to play. |



*Note,*

*Note*, In resolving of the last Question, I have made three different Suppositions for the Thing sought, purely as an Instance, to shew the young Learner how well he ought to consider the Nature of the Question, when he first states it, and make choice of representing the Thing sought, so as to avoid running it into Surds, if possible, *viz.* as in the first Supposition of $a =$ the first Sum, *&c.* Not but that such Equations may be solved, as shall be shewed in the next Chapter. However, it is most like an Artist to perform Things of this Nature the nearest and easiest way they can be done.

Question 33. *Suppose there were two equal Circles, whose Peripheries* (viz. *Circumferences*) *are divided into* 44310 *equal Parts; and that those Circles were so placed upon one Axis, as to move the contrary way to each other; and suppose one of them to move but one of these equal Parts the first Day, two Parts the second Day, three Parts the third Day, and so on in Arithmetical Progression*, viz. 1, 2, 3, 4, 5, &c. *and the other to move every Day the Cube of those Parts*, 1, 8, 27, 64, 125, &c. *of the same Parts: How many Parts and how many Days must each Circle move, before the same two Points meet that were together when they began to move?*

In order to give a ready Solution to this Question (or any other in this Kind) it will be convenient to premise this *Lemma.*

## *L E M M A.*

The Sum of any Series of Cubes whose Roots are in Arithmetick Progression (the first Term, and common Difference being Unity or 1) is equal to the Square of the Sum of all those Roots.

As in these

| Terms in Arith. &c. | their Cubes. |
|---|---|
| 1 | 1 |
| 2 | 8 |
| 3 | 27 |
| 4 | 64 |
| 5 | 125 |
| 6 | 216 &c. |
| 21 × 21 = | 441 Sum of their Cubes. |

| | | |
|---|---|---|
| Let | 1 | $a =$ the Sum of all the Parts the 1st Circle moves. |
| Then | 2 | $aa =$ the Sum of all the Parts the 2d moves. |
| Consequen. | 3 | $aa + a = 44310$ by the Quest. (*per Lem.* |
| 2 C □ | 4 | $aa + a + 0{,}25 = 44310{,}25$ |

4 ш 2

| | | |
|---|---|---|
| 4 ωω² | 5 | $a + 0,5 = \sqrt{44310,25} = 210,5$ |
| 5 — 0,5 | 6 | $a = 210$ { the Number of Parts the first Circle must move. |
| 6 ⊚² | 7 | $aa = 44100$ { the Number of Parts the second Circle moves. |

Next to find the Number of Days they moved; there is given the first Term $= 1$, the common Difference $= 1$, and the Sum of all the Terms $= 210$, thence to find the last Term, which in this Case is the same with the Number of all the Terms.

Let $a = 1$ the first Term, $e = 1$ the common Difference, and $s = 210$ the Sum of all the Terms, to find $y =$ the last Term; as per Sect. 1. Chap. 6. Then $yy + ey = 2s + aa - ae$ by the 16 Step, Page 186; that is, $yy + y = 210 \times 2 = 420$ &c. Hence $y = 20$ the Number of Days required.

---

I shall now proceed to give an Example or two of the Method used in arguing about unlimited Questions; *viz.* such Questions which admit of various Answers, such as those in *Alligation Alternate* promised in Page 117.

In order to shorten that Work, it will be convenient for the Learner to know the two Signs of Comparison, $>$ and $<$. The Sign $>$ is of **Greater than**; as $b > a$ signifies that $b$ is greater than $a$. The Sign $<$ is of **Lesser than**; as $b < d$ signifies that $b$ is lesser than $d$, *&c.*

## *EXAMPLE* 1.

*Question* 34. *A Tobacconist hath three Sorts of Tobacco*, viz. *one of* 2s. 8d. *the Pound*, *another of* 20 d. *the Pound*, *and a third Sort of* 16 d. *the Pound*; *of these he would make a Mixture to contain* 56 *Pound*, *that may be sold for* 22 d. *the Pound: How much of each Sort may he take?*

Let $a =$ the Quantity of that worth 32 Pence the Pound, $e =$ that of 20 Pence the Pound, and $y =$ that of 16 Pence the Pound;

Then $a + e + y = 56$
And $32a + 20e + 16y = 1232$ } { *viz.* each Quantity multiplied into it's own Price, equals their Sum multiplied into the mean Price.



This

This Queſtion being thus ſtated, it appears by Rule 1, Page 176, that it is capable of innumerable Anſwers; becauſe for any one of theſe three Letters, $a$, $e$, $y$, there may be taken any Number at Pleaſure, provided it be leſs than 56. But although that may be truly done, yet there are ſeveral Ways of arguing about theſe Sorts of Queſtions, which will limit or bound them to all their proper or poſſible Anſwers in whole Numbers. Thus,

| | | |
|---|---|---|
| Let | 1 | $a + e + y = 56$ } as above. |
| And | 2 | $32a + 20e + 16y = 1232$ } as above. |
| $1 - a$ | 3 | $e + y = 56 - a$ |
| $2 - 32a$ | 4 | $20e + 16y = 1232 - 32a$ |
| $3 \times 16$ | 5 | $16e + 16y = 896 - 16a$ |
| $4 - 5$ | 6 | $4e = 336 - 16a$ |
| $6 \div 4$ | 7 | $e = 84 - 4a$; hence $a < 21$ |
| $3 - 7$ | 8 | $5 = 3a - 28$; hence $a > \frac{28}{3}$ or $9\frac{1}{3}$ |

From the two laſt Steps it appears, that the Quantity ſignified by $a$, ought to be leſs than 21, and greater than $9\frac{1}{3}$; that is, any Number betwixt $9\frac{1}{3}$ and 21, may be taken for the Value of $a$: Conſequently there may be eleven Anſwers to this Queſtion in whole Numbers.

Suppoſe $a = 10$, then $e = 84 - 40 = 44$, per 7th Step; and $y = 30 - 28 = 2$, per 8th Step. Again, if $a = 11$, then $e = 84 - 44 = 40$, per 7th Step; and $y = 33 - 28 = 5$, per 8th Step: and ſo on for the reſt, which will be as in the following Table.

| $a$ | $e$ | $y$ | | $a$ | $e$ | $y$ | | $a$ | $e$ | $y$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 44 | 2 | | 14 | 28 | 14 | | 18 | 12 | 26 |
| 11 | 40 | 5 | | 15 | 24 | 17 | | 19 | 8 | 29 |
| 12 | 36 | 8 | | 16 | 20 | 20 | | 20 | 4 | 32 |
| 13 | 32 | 11 | | 17 | 16 | 23 | | | | |

Thus it will be eaſy to find out and collect all the limited Anſwers to any Queſtion (of this Kind) wherein there are only three Quantities propoſed to be mixed: But when there are more than three, then the Work requires a little more Trouble; becauſe the ſingle Limits of all the Quantities above two muſt be found; that is, if there are four Quantities concerned in the Queſtion, the Limits of two of them muſt be found; if five Quantities are concerned, then the Limits of three of them muſt be found, *&c.* As in the following Queſtion.

*Queſtion*

*Question* 35. *Suppose it were required to mix four Sorts of Wines together*; viz. *one Sort worth* 7 s. 4 d. *the Gallon, another Sort worth* 4 s. 7 d. *the Gallon, a third Sort worth* 3 s. 8 d. *the Gallon, and a fourth Sort worth* 2 s. 9 d. *the Gallon: How much of each Sort may be taken to make a Mixture of* 63 *Gallons, so as that the whole Quantity may be sold for* 5 s. 6 d. *the Gallon, without Loss*, &c.

First, let all these several Rates, and the mean Rate, be reduced to one Denomination, *viz.* into Pence.

Viz. $\left\{\begin{matrix} 7s.\,4d. = 88d. & 4s.\,7d. = 55d. \\ 3s.\,8d. = 44d. & 2s.\,9d. = 33d. \end{matrix}\right\}$ and $5s.\,6d. = 66.$

Put $a =$ the Quantity of that worth $88d.$ the Gallon; $e =$ that of $55d.$ the Gallon, $y =$ that of $44d.$ the Gallon, and $u =$ that of $33d.$ the Gallon.

| | | |
|---|---|---|
| Then | 1 | $a + e + y + u = 63$ by the Question. |
| And | 2 | $88a + 55e + 44y + 33u = 4158 = 63 \times 66$ |
| $1 - a$ | 3 | $e + y + u = 63 - a$ |
| $2 - 88a$ | 4 | $55e + 44y + 33u = 4158 - 88a$ |
| $3 \times 33$ | 5 | $33e + 33y + 33u = 2079 - 33a$ |
| $4 - 5$ | 6 | $22e + 11y = 2079 - 55a$ |
| $6 \div 11$ | 7 | $2e + y = 189 - 5a$; hence $a < \frac{189}{5}$ or $37\frac{4}{5}$ |
| $3 \times 55$ | 8 | $55e + 55y + 55u = 3465 - 55a$ |
| $8 - 4$ | 9 | $11y + 22u = 33a - 693$ |
| $9 \div 11$ | 10 | $y + 2u = 3a - 63$; hence $a > \frac{63}{3}$ or 21 |

From the 7th and 10th Steps it appears, that the Quantity of that Sort of Wine denoted by $a$, must be less than $37\frac{4}{5}$ Gallons, and greater than 21 Gallons: that is, it may be $a =$ any Number of Gallons betwixt 21 and $37\frac{4}{5}$. Whence it follows, that there may be collected 16 Answers to this Question from the Limits of $a$ only.

Next to find the Limits of $e$, $y$, and $u$.

| | | |
|---|---|---|
| Suppose | 11 | $a = 22$, then will $5a = 110$, and $3a = 66$ |
| But | 12 | $2e + y = 189 - 5a = 79$, per 7th Step. |
| $12 - 2e$ | 13 | $y = 79 - 2e$; hence $e < \frac{79}{2}$ or $39\frac{1}{2}$ |
| Again | 14 | $e + y + u = 63 - a = 41$, per 3d Step. |
| $14 - e$ | 15 | $y + u = 41 - e$ |
| $15 - 13$ | 16 | $u = e - 38$; hence $e > 38$ |

From the 13th and 16th Steps it appears, that if $a = 22$, then $e = 39$, $y = 79 - 2e = 1$, and $u = e - 38 = 1$.

Again,

Again,

| | | |
|---|---|---|
| Suppose | 17 | $a = 23$, then $5a = 115$, and $3a = 69$ |
| But | 18 | $2e + y = 189 - 5a = 74$, per 7th Step. |
| $18 - 2e$ | 19 | $y = 74 - 2e$; hence $e < \frac{74}{2} = 37$ |
| Again | 20 | $e + y + u = 63 - a = 40$, per 3d Step. |
| $20 - e$ | 21 | $y + u = 40 - e$ |
| $21 - 19$ | 22 | $u = e - 34$, hence $e > 34$. |

From the 19th and 22d Steps it appears, that if $a = 13$, then $e$ may be either 35 or 36.

Once more for a further Illustration.

| | | |
|---|---|---|
| Let | 23 | $a = 24$, then $5a = 120$, and $3a = 72$ |
| But | 24 | $2e + y = 189 - 5a = 69$, per 7th Step. |
| $24 - 2e$ | 25 | $y = 69 - 2e$; hence $e < \frac{69}{2}$ or $34\frac{1}{2}$ |
| Again | 26 | $e + y + u = 63 - a = 39$, per 3d Step. |
| $26 - e$ | 27 | $y + u = 39 - e$ |
| $27 - 25$ | 28 | $u = e - 30$, hence $e > 30$. |

From hence it appears, that if $a = 24$, then $e$ may be either 31, 32, 33, or 34, *viz.* it may be any Number betwixt 30 and $34\frac{1}{2}$ by the 25th and 28th Steps; from whence the Values of $y$ and $u$ may be easily found.

That is, if
$$\begin{cases} e = 31 . & \text{then } y = 7 . & \text{And } u = 1 \\ e = 32 . & y = 5 . & u = 2 \\ e = 33 . & y = 3 . & u = 3 \\ e = 34 . & y = 1 . & u = 4 \end{cases}$$

Proceeding on in this manner with all the other single Values of $a$, there may be found above 120 Answers to this Question in whole Numbers: and if you please to put $a =$ Fractions, there may be found an innumerable Set of Answers; whereas the Rule of *Alligation* in *Vulgar Arithmetick* affords but only one Answer in Fractions, to wit, that of $a = 31\frac{1}{2}$, $e = 10\frac{1}{2}$, $y = 10\frac{1}{2}$, $u = 10\frac{1}{2}$; as may be easily tried per Rule *Page* 115, &c.

These two Examples being well understood (especially if the last be thoroughly pursued) may suffice to shew the Method of limiting the Answers to all Sorts of Questions of this Kind. I shall therefore conclude this Chapter of Questions with giving a Solution to the Enigma (or Riddle) proposed (but not answered) by Mr *John Kersey*, in the Close of the *Appendix* to his *Arithmetick*, which

which affords several pretty Questions, the Solution whereof will discover a certain Sentence consisting of three Words, which must be found by the Help of Figures placed (or supposed to be placed) over the twenty-four Letters of the Alphabet.

| | | |
|---|---|---|
| Thus | 1 . 2 . 3 . 4 . 5 . 6 . 7 . &c. | called Indices. |
| | *a* . *b* . *c* . *d* . *e* . *f* . *g* . &c. | to the last Letter. |

So that if the Index of that Letter be once found, the Letter to which it belongs is consequently known.

## The Enigma.

1. If the Difference between the Indices of the second Letter of the second Word, and the third Letter of the first Word, be multiplied into the Difference of their Squares, the Product will be 576; and if their Sum be multiplied into the Sum of their Squares, that Product will be 2336, the Index of the said third Letter being the greatest.

| | | |
|---|---|---|
| Let | 1 | $a =$ the greater Index, or that of the 3d Letter. |
| And | 2 | $e =$ the lesser, or that of the 2d Letter. |
| Then | 3 | $\overline{a-e} \times \overline{aa-ee} = 576$ } by the Question. |
| | 4 | $\overline{a+e} \times \overline{aa+ee} = 2336$ } by the Question. |
| $3 \times$ | 5 | $aaa - aae - aee + eee = 576$ |
| $4 \times$ | 6 | $aaa + aae + aee + eee = 2336$ |
| $6 - 5$ | 7 | $2aae + 2ee = 1760$ |
| $6 + 7$ | 8 | $aaa + 3aae + 3aee + eee = 4096$ |
| $8\ \omega\omega^3$ | 9 | $a + e = \sqrt[3]{4096} = 16$ |
| $4 \div \overline{a+e}$ | 10 | $aa + ee = \frac{2336}{a+e} = \frac{2336}{16} = 146$ |
| $9\ \circledast^2$ | 11 | $aa + 2ae + ee = 256$ |
| $11 - 10$ | 12 | $2ae = 110$ |
| $10 - 12$ | 13 | $aa - 2ae + ee = 36$ |
| $13\ \omega\omega^2$ | 14 | $a - e = \sqrt{36} = 6$ |
| $9 + 14$ | 15 | $2a = 22$ — *From hence it appears, that the* 3d |
| $15 \div 2$ | 16 | $a = 11$ — *Letter of the* 1st *Word is* l, *and the* |
| $9 - 16$ | 17 | $e = 5$ — 2d *Letter of the* 2d *Word is* e. |

Note, *In order to set down the Letters (as they become found) in their proper Places, it may be convenient to supply the vacant Places with Stars.*

| | | | |
|---|---|---|---|
| Thus | First Word. | Second Word. | Third Word. |
| | * * *l* * * | * * *e* * * * | * * * * * |

2. The

2. The Indices laſt found, are the two Extreams of four Numbers in Arithmetical Progreſſion, the leſſer Mean being the Index of the firſt Letter of the third Word; and the greater Mean is the Index of the fourth and laſt Letter of the firſt Word. *Viz.* 5 . 7 . 9 . 11 are the four Terms in Arithmetical Progreſſion. Whence it appears, that *G* (whoſe Index is 7) is the firſt Letter of the third Word; and that *i* (whoſe Index is 9) is the fourth or laſt Letter of the firſt Word; which being placed down, will ſtand thus,

** * li   * e * * *.   G * * *.*

3. The ſecond Letter of the third Word is the ſame with the third Letter of the firſt Word; and the fifth Letter of the third Word is the ſame with the laſt Letter of the firſt Word: Whence the Letters will ſtand thus,

** * li.   * e * * *.   Gl * * * i *.*

4. The Sum of the Squares of the Indices of the firſt and ſecond Letters of the firſt Word is 520, and the Product of the ſame Indices is ſeven Ninths of the Square of the greater Index, which is the Index of the ſaid firſt Letter.

Let $a =$ the greater, and $e =$ the leſſer Index.

| | | |
|---|---|---|
| Then | 1 | $aa + ee = 520$ } according to the *Data* |
| And | 2 | $ae = \frac{7}{9} aa$ } |
| $2 \div a$ | 3 | $e = \frac{7}{9} a$ |
| $3 \circledS^{2}$ | 4 | $ee = \frac{49}{81} aa$ |
| $1 - 4$ | 5 | $aa = 520 - \frac{49}{81} aa$ |
| $5 \times 81$ | 6 | $81aa = 42120 - 49aa$ |
| $6 + 49aa$ | 7 | $130aa = 42120$ |
| $7 \div 130$ | 8 | $aa = \frac{42120}{130} = 324$ |
| $8\ \omega^{2}$ | 9 | $a = \sqrt{324} = 18$, whoſe Letter is *s*. |
| 3 and 9 | 10 | $e = \frac{7}{9} a = 14$, whoſe Letter is *o*. |

Hence the Letters will ſtand thus,

*Soli.   * e * * *.   Gl. * * i *.*

5. The Difference between the two laſt Indices, is the Index of the firſt Letter of the ſecond Word, *viz.* $18 - 14 = 4$ being the Index of the Letter *D*. Then the Letters will ſtand thus,

*Soli.   De * * *.   Gl. * * i *.*

6. The

6. The third and laſt Letter of the ſecond Word, alſo the third Letter of the third Word, are the ſame with the ſecond Letter of the firſt Word; hence the Letters will ſtand thus,

*Soli Deo Glo* * *i* *.

7. The Sum of the Indices of the fourth Letter of the third Word, and the ſixth or laſt Letter of the ſame Word, being added to their Product is 35; and the Difference of their Squares is 288; the Index of the laſt Letter being the leaſt.

Put $a =$ the greater, and $e =$ the leſſer Index, as before.

| | | |
|---|---|---|
| Then | 1 | $ae + a + e = 35$ } by the *Data*. |
| And | 2 | $aa - ee = 288$ } |
| $1 - a$ | 3 | $ae + e = 35 - a$ |
| $3 \div \overline{a+1}$ | 4 | $e = \frac{35 - a}{a + 1}$, for $e \times \overline{a + 1} = ae + e$ |
| 4 ☉² | 5 | $ee = \frac{1225 - 70a + aa}{aa + 2a + 1}$ |
| $2 + 5$ | 6 | $aa = 288 + \frac{1225 - 70a + aa}{aa + 2a + 1}$ |
| $6 \times aa$ &c. | 7 | $\left\{ \begin{array}{l} a^4 + 2a^3 + aa = 288aa + 576a + 288 \\ + 1225 - 70a + aa \end{array} \right.$ |
| $7 +$ | 8 | $a^4 + 2a^3 - 288aa - 506a = 1513$ |

This laſt Equation being reſolved according to the Method which ſhall be ſhewed in the next Chapter, it will be $a = 17$ it's Letter; and from the 4th Step $e = \frac{35 - a}{a + 1} = 1$, the Index of the Letter *a*. Then theſe two Letters being placed according to the *Data* above, are all that are required by the Enigma to compleat theſe Words

*Soli Deo Gloria.*

## CHAP. X.

### *The Solution of* Adfected Equations *in Numbers.*

BEFORE we proceed to the Solution of Adfected Equations, it may not be amiss to shew the Investigation (or Invention) of those Theorems or Rules for extracting the Roots of Simple Powers, made use of in Chapter 11. Part 1. I shall here make choice of the same Letters to represent the Numbers both given and sought, as in my Compendium of *Algebra.*

*Viz.* Let $G$, always denote the given Resolvend.
$r =$ any Number taken as near the true Root as may be, whether it be greater or less.
$e =$ the unknown Part of the Root sought by which $r$ is to be either increased or decreased.

Then if $r$ be any Number less than the true Root, it will be $r + e =$ the Root sought. But if $r$ be taken greater than the true Root, it will then be $r - e =$ the Root sought. And put $D$ for the Dividend that is produced from $G$, after it is lessened and divided by $r$, *&c.* (into the Co-efficients of Adfected Equations) according as the Nature of the Root requires. These Things being premised, we may proceed to raising the Theorems.

## SECT. I.

I. FOR the Square Root, *viz.* $aa = G$. Quære $a$.

| | | |
|---|---|---|
| Let | 1 | $r + e = a$ |
| 1 $\text{Ⓢ}^2$ | 2 | $rr + 2re + ee = aa = G$ |
| $2 - rr$ | 3 | $2re + ee = G - rr$. Call it $D$, viz. $D = G - rr$. |
| Then | 4 | $\dfrac{D}{2r + e} = e$ This shews the 1st Method of extracting the Square Root, Sect. 5. Chap. 11. Part 1. |
| $3 \div 2$ | 5 | $re + \frac{1}{2}ee = \dfrac{G - rr}{2} = D.$ |

Which gives this Theorem $\dfrac{D}{r + \frac{1}{2}e} = e.$

The Arithmetical Operations of both these Theorems, you have in the Examples of Section 2. Page 126, to which I refer the

the Learner, ſuppoſing him by this Time to underſtand them without any more Words than what is there expreſſed.

II. To extract the Cube Root; *viz.* $aaa = G$. Quære $a$.

| | | |
|---|---|---|
| Let | 1 | $r + e = a$, ſuppoſing $r$ leſs than the true Root. |
| 1 $\circledS^{2}$ | 2 | $rrr + 3rre + 3ree + eee = aaa = G$ |
| $2 - rrr$ | 3 | $3rre + 3ree + eee = G - rrr$ |
| $3 \div 3r$ | 4 | $re + ee + \frac{eee}{3r} = \frac{G - rr}{3r} = D$ |

Let $\frac{eee}{3r}$ be rejected or caſt off, as being of ſmall Value; then it will be, $re + ee = D$, which gives this following

$$\textit{Theorem} \quad \frac{D}{r + e} = e$$

By this Theorem or Rule, the 1ſt and 2d Examples in Caſe 1. Page 132, are performed; the which being compared with this Theorem may be eaſily underſtood.

Again, Suppoſe $aaa = G$, as before, and let $r$ be taken greater than the true Root.

| | | | |
|---|---|---|---|
| Then | 1 | $r - e = a$ | $eee$ being rejected as before. |
| 1 $\circledS^{3}$ | 2 | $rrr - 3rre + 3ree = a^3 = G$ | |
| $2 \pm$ | 3 | $3rre - 3ree = rrr - G$ | |
| $3 \div 3r$ | 4 | $re - ee = \frac{rrr - G}{3r} = D$ | |

Which gives this *Theorem* $\frac{D}{r - e} = e$.

By this Theorem the third Example in Caſe 2. Page 133, is performed.

III. To extract the *Biquadrate* Root; *viz.* $a^4 = G$. Quære $a$.

| | | | |
|---|---|---|---|
| Let | 1 | $r - e = a$ ſuppoſing $r$ leſs than juſt. | |
| 1 $\circledS^{4}$ | 2 | $r^4 + 4rrre + 6rree = a^4 = G$ | rejecting all the Powers of $e$ above $ee$. |
| $2 - r^4$ | 3 | $4rrre + 6rree = G - r^4$ | |
| $3 \div 2rr$ | 4 | $2re + 3ee = \frac{G - r^4}{2rr} = e.$ | |

Which gives this *Theorem* $\frac{D}{2r + 3e} = e.$



By

By this Theorem the Biquadrate Root of any Number may be extracted. But, as I have already said, Page 134, those Extractions may be very well performed by two Extractions of the Square Root. *Vide Example, Page* 135.

IV. To extract the *Surfolid* Root, *viz.* $a^5 = G$. Quære $a$.

If $r$ be taken less than just, then $r + e = a$, as before, and $\frac{G - r^5}{5r^3} = D$, which gives this *Theorem* $\frac{D}{r - 2e} = e$. By this Theorem the Surfolid Root, Example 1, Page 136, is extracted. But if $r$ be taken greater than just; then $r - e = a$, and $\frac{r^5 - G}{5r^3} = D$, which gives this *Theorem* $\frac{D}{r - 2e} = e$. By this last Theorem the Example in Page 137 is performed.

I presume it needless to pursue the raising of those Theorems, for extracting the Roots of Simple Powers, any further; because the Method of doing it is general, how high soever they are; and therefore it may be easily understood by what is already done.

## SECT. 2.

NOtwithstanding I have already shewed the Solution of Quadratick Equations, two several Ways, *viz.* by casting off the lowest Term; and by compleating the Square, *vide* Section 2. Page 195, &c. Yet it may not be amiss to shew, how those Equations may be resolved into Numbers by this Universal Method of continued Series; wherein, if the first $r$ be taken equal to the first true Root, or single Side of the Resolvend; and every single Value of $e$ (as it becomes found) be still added to it, for a new $r$, then those Roots may be extracted without repeating a second Operation, as before in the single Powers.

*Case* 1. Let $aa + 2ba = G$. It is required to find the Value of $a$.

| | | |
|---|---|---|
| Put | 1 | $r + e = a$ |
| 1 ⓢ² | 2 | $rr + 2re + ee = aa$ |
| 1 × 2$b$ | 3 | $2br + 2be = 2ba$ |
| 2 + 3 | 4 | $rr + 2br + 2re + 2be + ee = aa + 2ba = G$ |
| 4 − $rr$ &c. | 5 | $2re + 2be + ee = G - rr - 2br$ |
| 5 ÷ 2 | 6 | $re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D$ |

Which gives this *Theorem* $\frac{D}{r + b + \frac{1}{2}e} = e$.

Suppose

Suppofe $b = 364$, and $G = 38692865$: If $r = 6000$, then $rr = 36000000$, and $2br = 4368000$. But $36000000 + 4368000 = 40368000 > 38692865 = G$. Therefore the firft $r < 6000$. Let $r = 5000$, then

| | | |
|---|---|---|
| 1ft $r = 5000$ | $19346432,5 = \frac{1}{2}G$ | |
| $b = 364$ | $-1432000, = \frac{1}{2}rr + br$ | |
| 1ft $r + b = 5364$ | $5026432,5 = D$ | $(800 = e$ |
| $+ \frac{1}{2}e = 400$ | 46112 | |
| 1 Divifor 5764) | 41523 | $(60 = e$ |
| 2d $r + b = 6164$ | 37164 | |
| $+ \frac{1}{2}e = 30$ | 4359 | $(7 = e$ |
| 2 Divifor 6194) | 43592,5 | $867 = e$ |
| 3d $r + b = 6224$ | (0) | |
| $+ \frac{1}{2}e = 3,5$ | | |
| 3 Divifor 6227,5 | | |

$$\left.\begin{array}{r} \text{First } r = 5000 \\ + e = 867 \end{array}\right\} = 5867 = a \text{ as was required.}$$

*Cafe* 2. If $aa - 2ba = G$, then proceeding as above, there will arife this *Theorem* $\frac{D}{r - b + \frac{1}{2}e} = e$, &c. And in *Cafe* 3, viz. $2ba - aa = G$, you will have this *Theorem* $\frac{D}{b - r - \frac{1}{2}e}$ &c. as above.

I think it needlefs to trouble the Reader with the Work of thefe two Theorems in Numbers; becaufe if the laft Example of *Cafe* 1, be underftood, the other will be eafy. Not but that the Method of compleating the Square is very ready and eafy, as you may obferve by the Work in feveral Queftions of this Chapter.

## SECT. 3.

IN the Solution of all Adfected Equations, that are above (or higher than) Quadraticks, it will be the beft way to take $r =$ the next neareft Root of the Equation: And then it will be $r + e = a$, if $r$ be lefs than juft; or $r - e = a$ if $r$ be greater than juft (as at the Beginning of this Chapter). And all the Powers of the unknown Part of the Root, (*viz.* $e$) above it's Square ($ee$) are to be rejected or caft off, as before in raifing the Theorems for the Simple

Simple Powers. And therefore it is, that to ſupply the want of thoſe Powers (above $ee$ in the Theorem) the Operation muſt be repeated: as in the Example of extracting the Cube Root, Page 133, *viz.* when the Figures in the Root conſiſt of more than three Places. (*vide* Page 140, and 141.)

Suppoſe $aaa + ba = G$. Quære $a$.

| | | |
|---|---|---|
| Let | 1 | $r + e = a$ viz. let $r$ be ſuppoſed leſs than juſt. |
| 1 ⓒ$^3$ | 2 | $rrr + 3rre + 3ree = aaa$ |
| $1 \times b$ | 3 | $br + be = ba$ |
| $2 + 3$ | 4 | $rrr + br + 3rre + be + 3ree = a^3 + ba = G$ |
| $4 \div 3r$ | 5 | $\frac{1}{3}rr + \frac{1}{3}b + re + \frac{be}{3r} + ee = \frac{G}{3r}$ |
| 5 — &c. | 6 | $re + \frac{be}{3r} + ee = \frac{G}{3r} - \frac{1}{3}rr - \frac{1}{3}b = D$ |

Which gives this *Theorem* $\dfrac{D}{r + \frac{b}{3r} + e} = e.$

But if $r$ be taken greater than juſt, then it will be $re + \frac{be}{3r} - ee = \frac{1}{2}rr + \frac{1}{3}b - \frac{G}{3r} = D$, which produces this *Theorem* $\dfrac{D}{r + \frac{b}{3r} - e} = e.$

By either of theſe two Theorems the Value of $a$ may be eaſily found. Or rather otherwiſe, as in the following Example.

Let $aaa + 24a = 587914$. Here $b = 24$. Suppoſe the firſt $r = 90$, then $r^3 = 729000 > 587914$ without the $24 \times 90$ being added to it: Therefore $r < 90$ Again, Suppoſe $r = 80$ then $r^3 = 512000$, and $24r = 1920$. But $512000 + 1920 = 513920 < 58791$, hence $> 70$, but nearer to it than 90. Therefore

| | | |
|---|---|---|
| it muſt be | 1 | $r + e = a$ leſs than juſt. |
| 1 ⓒ$^3$ | 2 | $rrr + 3rre + 3ree = aaa$ |
| $1 \times 24$ | 3 | $24r + 24e = 24a$ |
| 2 in Numb. | 4 | $512000 + 19200e + 240ee = aaa$ |
| 3 in Numb. | 5 | $1920 + 24e = 24a$ |
| $4 + 5$ | 6 | $513920 + 19224e + 240ee = 587914$ |
| $5 - 513920$ | 7 | $19224e + 240ee = 73994$ |
| $7 \div 240$ | 8 | $80{,}1e + ee = 308{,}31 = D$ |
| $8 \div$ | 9 | $e = \frac{D}{80{,}1 + e}$ |

Operation

| | | | |
|---|---|---|---|
| Operation | 80,1 | | |
| $+e=$ | 3, | | $\left(\begin{array}{l}80, \quad = r\\ 3,68 \text{ \&c.} = e\end{array}\right.$ |
| 1 Divisor | 83,1) | 308,31 | |
| $+e=$ | 3,6 | 249,3 | 83,68 &c. $= r + e$ |
| 2 Divisor | 86,7) | 59,01 | |
| $+e$ | ,67 | 52,02 | |
| | 87,37) | 6,99 &c. | |

Or rather new $r = 83,7$ for a second Operation, which being involved and tried (as above) will be found greater than just: therefore

| | | |
|---|---|---|
| it must be | 1 | $r - e = a$ |
| 1 $\text{S}^3$ | 2 | $rrr - 3rre + 3ree = aaa$ |
| 1 × 24 | 3 | $24r - 24e = 24a$ |
| 2 in Numb. | 4 | $586376,253 - 21017,07e + 251,1ee = aaa$ |
| 3 in Numb. | 5 | $2008,8 - 24e = 24a$ |
| 4 + 5 | 6 | $588385,053 - 21041,07e + 251,1ee = 587914$ |
| 6 + | 7 | $21041,07e - 251,1ee = 471,053$ |
| 7 ÷ 251,1 | 8 | $83,7955e - ee = 1,87595778 = D$ |
| 8 ÷ | 9 | $e = \dfrac{D}{83,7955 - e}$ |

| | | | |
|---|---|---|---|
| 2d Operation | 83,7955 | | |
| $-e=$ | ,02 | | $\left(\begin{array}{l}83,70000000 = r\\ 00,02239331 = e\end{array}\right.$ |
| 1st Divisor | 83,7755) | 1,87595778 | |
| $-e=$ | ,022 | 1,675510 | $83,67760669 = a = r - e$ |
| 2d Divisor | 83,7535) | ,2004477 | |
| $-e=$ | ,0023 | ,1675070 | |
| 3d Divisor | 83,7512) | ,03294078 | |
| $-e=$ | 3 &c. | ,02512536 | |
| | 83,751 | ,00781542 | |
| | ... | 00753760 | |
| | | 27782 | |
| | | 25125 | |
| | | 2657 | |
| | | 2512 | |
| | | 145 | |
| | | 83 | |

Here the new Divisors are rejected, as insignificant.

All the remaining Examples of extracting Roots (except Page 260) are left in the Author's own Method; which by this Time, it is presumed, the Learner will easily know how to correct of himself, if he takes due Notice of what has been delivered Page 131, 132, &c.

But

But if more Exactneſs be required, you may make the new $r = 83{,}6776067$, and proceed with it to a third Operation; which will afford twenty-ſeven Places of Figures for the Value of $a$; that is, every Operation will produce triple the Places of Figures to thoſe of the Precedent $r$. And this tripling the Places of Figures in the Root, at every Operation, holds good, and is to be obſerved in the Solution of all Adfected Equations (how high ſoever they are) according to this Method of reſolving them. *See Page* 141.

*Example* 2. Suppoſe $aaa - ba = G$. Quære $a$. If $r + e = a$, then $re - \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem $\frac{D}{r - \frac{\frac{1}{3}b}{r} + e} = e$. But if $r - e = a$, then $re + \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem $\frac{D}{r + \frac{\frac{1}{3}b}{r} + e}$.

Or you may proceed otherwiſe, as in the laſt Example. Let $aaa - 6438a = 104785688$, here $b = 6438$. Suppoſe the firſt $r = 500$, $rrr = 125000000$, and $br = 3219000$, then $125000000 - 3219000 = 121781000$. But $121781000 > 104785688$, therefore $r < 500$. Again, ſuppoſe $r = 400$, $rrr = 64000000$, and $br = 2575200$, then will $64000000 - 2575200 = 61424800$. But $61424800 < 104785688$, hence $r > 400$; conſequently $r$ is betwixt 400 and 500. But 500 is the next neareſt; therefore, let $r = 500$ being greater than juſt.

| | | |
|---|---|---|
| Then | 1 | $r - e = a$ |
| $1\ @^{2}$ | 2 | $rrr - 3rre + 3ree = aaa$ |
| $1 \times b$ | 3 | $br - be = ba$ |
| 2 in Numb. | 4 | $125000000 - 750000e + 1500ee = aaa$ |
| 3 in Numb. | 5 | $3219000 - 6438e = 6438a$ |
| $4 - 5$ | 6 | $121781000 - 743562e + 1500ee = 104785688$ |
| $6 +$ | 7 | $743562 - 1500ee = 16995312$ |
| $7 \div 1500$ | 8 | $495e - ee = 11330 = D$ |
| $8 \div$ | 9 | $e = \frac{D}{495 - e}$ |

Operation

| | | | |
|---|---|---|---|
| Operation | 495 | | |
| $-e=$ | 20 | | (500,0 $= r$ |
| 1 Divisor | 475) | 11330 | 23,8 $= e$ |
| $-e=$ | 3 | 950 | 476,2 $= r - e = a$ |
| | 472) | 1830 | |
| | | 1416 | |
| | | 414,0 | |
| | | 377,6 | |

Let new $r = 476$ for a 2d Operation, then $r^3 = 107850176$ and $br = 3064488$: but $107850176 - 3064488 = 104785688$ the same with the Resolvend. Consequently $a = 476$ just.

*Example* 3. Let $ba - aaa = G$. Quære $a$. If $r + e = a$, then $\frac{\frac{1}{3}be}{r} - re - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$, which gives this *Theorem* $\frac{D}{\frac{\frac{1}{3}b}{r} - re} = e$. But if $r - e = a$, then $re - \frac{\frac{1}{3}be}{r} - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$, which gives this *Theorem* $\frac{D}{r - \frac{\frac{1}{3}b}{r} - e} = e$.

Or otherwise as before in the two last Examples. Thus, let $123456a - aaa = 12272861$. Here $b = 123456$. Suppose the first $r = 200$, then $rrr = 8000000$, and $br = 24691200$; then $24691200 - 8000000 = 16691200$, but $16691200 > 12272861$, therefore $r$ is here less than just, because the highest Power is —, or Negative. Again, Suppose $r = 300$, then $r^3 = 27000000$, and $br = 37036800$, then $37036800 - 27000000 = 10036800 < 12272861$. Consequently $r < 300$, and $r > 200$. Let $r = 300$, being the next nearest, but more than just.

| | | |
|---|---|---|
| Then | 1 | $r - e = a$ |
| 1 ⊚,3 | 2 | $rrr - 3rre + 3ree = aaa$ |
| $1 \times b$ | 3 | $br - be = ba$ |
| 2 in Numb. | 4 | $27000000 - 270000e + 900ee$ |
| 3 in Numb. | 5 | $37036800 - 123456e$ |
| $5 - 4$ | 6 | $10036800 + 146544e - 900ee = 12272861$ |
| $6 - \overline{\phantom{0}}$ | 7 | $146544e - 900ee = 2236061$ |
| $7 \div 900$ | 8 | $162e - ee = 2484 = D$ |
| $1 \div$ &c. | 9 | $e = \frac{D}{162 - e}$ |

$$
\begin{array}{lr}
\text{Operation} & 162 \\
-e= & 10 \\ \hline
\text{1st Divisor} & 152) \\
-e= & 6 \\ \hline
\text{2d Divisor} & 646)
\end{array}
\qquad
\begin{array}{r}
2484 \\
152 \\ \hline
964 \\
876 \\ \hline
88,0 \\
86,6 \\ \hline
\end{array}
\left(
\begin{array}{l}
300,0 = r \\
16,6 = e \\ \hline
283,4 = r - e = a.
\end{array}
\right.
$$

Or new $r = 283$, which being involved, &c. will appear to be the true Root, that is, $a = 283$ just.

*Note*, These are usually called the three Forms of Cubick Equations; and in the Solution of the third or last Form, *viz.* $ba - aaa = G$, you may meet with some seeming Difficulties; especially in making Choice of the first $r$, because this Equation is an ambiguous Equation, and hath two Affirmative Roots, *viz.* a greater and lesser Root. But having once found either of them, the other may be easily obtained by Division only; as in the Quadratick Equations. *Vide* Chap. 8. As for Instance, in the last Example, $a = 283$ and $123456a - aaa = 12272861$. Make these two Equations $= 0$, to wit, let $a - 283 = 0$, and $-aaa + 123456a - 12272861 = 0$.

$$
\begin{array}{rl}
\text{Then, } a - 283) & -aaa + 123456a - 12272861 \quad (-aa \\
& -aaa + 283aa \\ \hline
& -283aa + 123456a \quad (-283a \\
& -283aa + 80089a \\ \hline
& +43367a - 12272861 \quad (+43367 \\
& +43367a - 12272861 \\ \hline
& (0) \qquad (0)
\end{array}
$$

Hence it appears that $-aa - 283a + 43367 = 0$. Consequently $aa + 283a = 43367$ this Equation being solved, $a = 110$, 2722 &c. which is the lesser Root of the aforesaid Equation $ba - aaa - G$, &c. After this Manner all the possible and impossible Roots of any Equation may be easily discovered, any one of it's Roots being once found. I shall therefore omit inserting more Examples of that kind.

Suppose $aaa + baa + ca = G$. Quære $a$. Let $b = 74$, $c = 8729$, and $G = 560783$. By Trial (as before) it will be found that the next nearest $r = 40$ being something less than just.

Therefore

| | | |
|---|---|---|
| Therefore | 1 | $r + e = a$ |
| $1 \times c$ | 2 | $cr + ce = ca$ |
| $1 \circledS^{2} : \times b$ | 3 | $brr + 2bre + bee = baa$ |
| $1 \circledS^{3}$ | 4 | $rrr + 3rre + 3ree = aaa$ |
| 2 in Numb | 5 | $349160 + 8729e$ |
| 3 in Numb. | 6 | $118400 + 5920e + 74ee$ |
| 4 in Numb. | 7 | $64000 + 4800e + 120ee$ |
| $5 + 6 + 7$ | 8 | $531560 + 19449e + 194ee = 560783$ |
| $8 - 531560$ | 9 | $19449e + 194ee = 29223$ |
| $9 \div 194$ | 10 | $100{,}2e + ee = 153{,}06 = D$ |
| $10 \div$ | 11 | $e = \dfrac{D}{100{,}2 + e}$ |

Operation 100,2
$+ e =$ 1
1ſt Diviſor 101,2) 153,06 (40,0 $= r$
$+ e =$ ,5   1,5 $= e$
2d Diviſor 101,7)   101,2   41,5 $= r + e = a$

51,86
50,85
1,01

Or new $r = 41{,}5$ for a ſecond Operation, which being duly involved, &c. will be found more than juſt.

| | | |
|---|---|---|
| Therefore | 1 | $r - e = a$ |
| Then | 2 | $cr - ce = ca$ |
| | 3 | $brr - 2bre + bee = baa$ |
| | 4 | $rrr - 3rre + 3ree = aaa$ |

Theſe being turned into Numbers, &c. as above, they will be $20037{,}75e - 198{,}5ee = 390{,}375$, which being divided by 198,5 the Co-efficient of $ee$, will become $100{,}946e - ee = 1{,}966624$, &c. $= D$.

Operation 100,946
$- e =$ 01
1ſt Diviſor 100,936) 1,966624 (41,5000000 $= r$
$- e =$ ,009   ,0194847 $= e$
2d Diviſor 100,927)   1,00936   41,4805153 $= r - e = a$

957264
908343
* 489210
403708
855020
807416
476040
403708
72332 &c.

* Here I proceed by plain Diviſion without forming new Diviſors.

Let the laſt Equation in the Enigma, Chap. 9. be here propoſed for a Solution. *Viz.* $aaaa + baaa - caa - da = G$; $b = 2$, $c = 288$, $d = 506$, and $G = 1513$, Quære $a$. By Trials it will be found, that the next neareſt $r = 20$, being ſomething more than juſt.

| | | |
|---|---|---|
| Therefore | 1 | $r - e = a$ |
| $1 \times d$ | 2 | $dr - de = da$ |
| 1 ⓔ$^2 \times c$ | 3 | $crr - 2cre + cee = caa$ |
| 1 ⓔ$^3 \times b$ | 4 | $brrr - 3brre + 3bree = baaa$ |
| 1 ⓔ$^4$ | 5 | $r^4 - 4rrre + 6rree = aaaa$ |

Theſe being turned into Numbers, and thoſe duly collected, according as the Signs of the Equation direct, they will become $50680 - 22374e + 2232ee = 1513$, which being all divided by 2232 the Co-efficient of $ee$, will be $10e - ee = 22 = D$.

$$\text{Then } \frac{D}{10 - e} = e.$$

Operation 10
$-e =$ 3
Diviſor 7) 22 (
21
1

$20 = r$
$3 = e$
$17 = r - e = a$ juſt.

See the End of Chap. 9.

By what hath been already done about the Solution of theſe few Equations (being carefully obſerved) I preſume the Learner will eaſily conceive how to proceed in the Solution of all Kinds of Equations, be they never ſo high, or affected; therefore I ſhall not here propoſe many various Examples, but only take them as they fall in Courſe, when I come to the next Part, wherein you will (perhaps) find ſuch Equations with their Solutions as are not common.

## CHAP. XI.

### *Of Simple* Intereſt, Annuities, *or Penſions*, &c.

INTEREST, or the Uſe paid for the Loan of Money, is either Simple, or Compound.

### Sect. 3. *Of Simple* Intereſt.

SIMPLE Intereſt, is that which is paid for the Loan of any Principal or Sum of Money, lent out for ſome Time, at any Rate *per Cent.* agreed on between the Borrower and the Lender; which, according to the late Laws of *England*, ought to be ſix Pounds for the Uſe of 100 *l.* for one Year, and twelve Pounds for the Uſe of 100 *l.* for two Years; and ſo on for a greater, or leſſer Sum, proportionable to the Time propoſed.

There are ſeveral Ways of computing (or anſwering Queſtions about) Simple Intereſt; as by the ſingle and double Rule of Three (See Page 96, &c.) others make uſe of Tables compoſed at ſeveral Rates *per Cent.* as Sir *Samuel Moreland*, in his Doctrine of Intereſt, both ſimple and compound, all performed by Tables; wherein he hath detected ſeveral material Errors committed by Sir *Iſaac Newton*, Mr *Kerſey* upon *Wingate*, and Mr *Clavil*, &c. in the Buſineſs of computing Intereſt, *&c.* by their Tables, too tedious to be here repeated. But I ſhall in this Tract take other Methods, and ſhew that all Computations relating to Simple Intereſt are grounded upon Arithmetick Progreſſion; and from thence raiſe ſuch general Theorems, as will ſuit with all Caſes. In order to that

Let
- $P =$ any Principal or Sum put to Intereſt.
- $R =$ the Ratio of the Rate, *per Cent. per Annum.*
- $t =$ the Time of the Principal's Continuance at Intereſt.
- $A =$ the Amount of the Principal, and it's Intereſt.

*Note*, The Ratio of the Rate, is only the Simple Intereſt of 1 *l.* for one Year, at any given Rate; and is thus found.

*Viz.* 100 : 6 :: 1 : 0,06 = the Ratio at 6 *per Cent. per Annum.*
Or 100 : 7 :: 1 : 0,07 = the Ratio at 7 *per Cent.* &c.
Again 100 : 7,5 :: 1 : 0,075 = the Ratio at 7 and $\frac{1}{2}$ *per Cent.*

And if the given Time be whole Years; then $t =$ the Number of whole Years: but if the Time given, be either pure Parts of a Year, or Parts of a Year mixed with Years; thoſe Parts muſt be turned into Decimals; and then $t =$ thoſe Decimals, *&c.*

Now

Now the common Parts of a Year may be easily turned or converted into Decimal Parts, if it be considered

That one { Day is the $\frac{1}{365}$ Part of a Year = 0,00274 *ferè*
Month is the $\frac{1}{12}$ Part of a Year = 0,0833333 &c.
Quarter is the $\frac{1}{4}$ Part of a Year = 0,25 }

These Things being premised, we may proceed to raising the *Theorems.*

Let $R =$ the Interest of 1 *l.* for one Year, as before.
Then $2\,R =$ the Interest of 1 *l.* for two Years.
And $3\,R =$ the Interest of 1 *l.* for three Years.
$4\,R =$ the Interest of 1 *l.* for four Years. And so on for any Number of Years proposed.

Hence it is plain, that the Simple Interest of one Pound is a Series of Terms in Arithmetic Progression increasing; whose first Term and common Difference is $R$, and the Number of all the Terms is $t$. Therefore the last Term will always be $tR =$ the Interest of 1 *l.* for any given Term signified by $t$.

Then { *As one Pound : is to the Interest of* 1 l. :: *so is any Principal or given Sum : to it's Interest.* }

That is, $1\,l. : tR :: P : tRP =$ the Interest of $P$. Then the Principal being added to it's Interest, their Sum will be $= A$ the Amount required; which gives this general *Theorem.*

*Theorem* 1. $tRP + P = A.$

From whence the three following *Theorems* are easily deduced.

*Theorem* 2. $\frac{A}{tR + 1} = P.$ *Theorem* 3. $\frac{A - P}{tp} = R.$

*Theorem* 4. $\frac{A - P}{RP} = t.$

These four *Theorems* resolve all Questions about Simple Interest.

*Question* 1. *What will* 256 l. 10 s. *amount to in* 3 *Years, one Quarter,* 2 *Months, and* 18 *Days, at* 6 *per Cent. per Annum.*

Here is given $P = 256,5$; $R = 0,06$; and $t = 3,46599$
For 3 Years = 3 Quære *A. per Theorem* 1.
one Quarter = 0,25
2 Months = 0,16667 = 0,08333 × 2
18 Days = 0,04932 — 0,00274 × 18
Hence $t =$ 3,46599 : × 0,06 = 0,2079594 $= tR$

Then 0,2079594 × 256,5 = 53,341586 $= tRP$
And 53,341586 + 256,5 = 309,841586 $= tRP + P = A.$
That is, 309,841586 = 309 *l.* 16 *s.* 10 *d.* being the Answer required.

*Question*

*Queſtion* 2. *What Principal or Sum being put to Intereſt, will raiſe a Stock of* 309 l. 16 s. 10 d. in three Years, one Quarter, two Months, and 18 Days; at 6 *per Cent. per Annum?*

Or the ſame Queſtion otherwiſe ſtated thus.

*What is* 309 l. 16s. 10d. *due* 3 *Years, one Quarter,* 2 *Months and* 18 *Days hence, worth in ready Money; abating or diſcounting* 6 *per Cent.* &c.

Here is given $A = 309{,}841586$; $R = 0{,}06$; $t = 3{,}46599$ (found as before) thence to find $P$. *Per Theorem* 2. Firſt $3{,}46599 \times 0{,}06 = 0{,}2079594 = tR$. Then

$tR + 1 = 1{,}2079594)\ 309{,}841586 = A\ (256{,}5 = P$;

that is, $256{,}5 = 256\,l.\ 10\,s.$ the Anſwer required.

*Queſtion* 3. *At what Rate or Intereſt, per Cent.* &c. *will* 256 l. 10 s. *amount to* 309 l. 16 s. 10 d. *in three Years, one Quarter, two Months, and* 18 *Days?*

Here is given, $P = 256{,}5$; $A = 309{,}841586$; and $t = 3{,}46599$ to find $R$. *Per Theorem* 3. Firſt $309{,}841586 - 256{,}5 = 53{,}341586 = A - P$. Next $3{,}46599 \times 256{,}5 = 889{,}026435 = tR$. And $tR = 889{,}026435)\ 53{,}341586\ (00{,}06 =$ the Ratio. Then $1\,l. : 0{,}06 :: 120 : 6 =$ the Rate required.

*Queſtion* 4. *In what Time will* 256 l. 10 s. *raiſe a Stock of* (*or amount to*) 309 l. 16 s. 10 d. *at* 6 *per Cent.* &c.

Here is given, $P = 256{,}5$; $A = 309{,}841586$, and $R = 0{,}06$ to find $t$. *Per Theorem* 4. Firſt $309{,}841586 - 256{,}5 = 53{,}341586 = A - P$. And $256{,}5 \times 0{,}06 = 15{,}39 = PR$. Then $15{,}39)\ 53{,}341586\ (3{,}46599 = t$; that is $t = 3$ Years and ,46599 Decimal Parts of a Year; which may be brought into common Parts of a Year, thus

| | |
|---|---|
| 0,46599 | And 0,08333) 0,21599 (2 Months. |
| 0,25 = one Quarter. | ,16666 |
| 0,21599 | 0,02074) ,04933 . (18 Days. |

Hence $t = 3$ Years, one Quarter, 2 Months, and 18 Days; the Anſwer required.

It muſt needs be eaſy to conceive, that what is here done at 6 *per Cent.* may be done at any other Rate of Intereſt, by forming the Ratio (*viz.* $R$) accordingly.

SCHOLIUM.

## SCHOLIUM.

Although it be according to the Laws and Custom of *England*, to compute Interest at the Proportion of 6 *per Cent.* (as above) yet he that takes up Money at Interest for any Time less than even or compleat Years, pays more Interest than seems reasonably due, according to the Rules of Art. As for Instance; if 100 *l.* be forborne at Interest one whole Year, it amounts to 106 *l.* But (I say) if it be paid at the half Year's End, it should not amount to 103; as appears from this following Proportion.

Let $a =$ the Amount due at the half Year's End; then it will be $100 : a :: a : 106$ the Amount at the Year's End. *Ergo* $aa = 10600$, and $a = \sqrt{10600} = 102,9563 = 102\,l.\ 19\,s.\ 1\frac{1}{2}\,d.$ which is less than $103\,l.$ by $10\frac{1}{2}\,d.$ And if it be paid in less than half a Year's Time, the Error must needs be the greater.

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## Sect. 2. *Of* Annuities, *or* Pensions *in Arrears, computed at Simple Interest.*

ANNUITIES, or Pensions, *&c.* are said to be in Arrears, when they are payable or due, either Yearly, or Half-yearly, *&c.* and are unpaid for any Number of Payments. Therefore the Business is, to compute what all those Payments will amount unto, allowing any Rate of Simple Interest for their Forbearance, from the Time each particular Payment became due: Now in order to that,

Put $\begin{cases} u = \text{the Annuity, Pension, or Yearly Rent, \&c.} \\ t = \text{the Time of it's Continuance, or being unpaid.} \\ R = \text{the Ratio, or Interest of } 1\,l. \text{ for } 1 \text{ Year, as before.} \\ A = \text{the Amount of the Annuity and it's Interest.} \end{cases}$

Then if $u =$ the first Year's Rent, due without Interest.

$\left.\begin{array}{r} Ru = \text{the Interest} \\ 2u = \text{the Rent} \end{array}\right\}$ due at the End of the second Year.

$\left.\begin{array}{r} 2Ru = \text{the Interest} \\ 3u = \text{the Rent} \end{array}\right\}$ due at the End of the third Year.

$\left.\begin{array}{r} 3Ru = \text{the Interest} \\ 4u = \text{the Rent} \end{array}\right\}$ due at the End of the fourth Year.

$\left.\begin{array}{r} 4Ru = \text{the Interest} \\ 5u = \text{the Rent} \end{array}\right\}$ due at the End of the fifth Year.

And so on for any Number of Years. Hence it is evident, that $Ru + 2Ru + 3Ru + 4Ru + 5u = A$ the Sum of all the Rents and their Interest, being forborne 5 Years.

From

From whence it follows, that $Ru+2Ru+3Ru+4Ru=A-tu$. Here $t=5$. Divide by $u$, then $R+2R+3R+4R=\frac{A-tu}{u}$.

Next to find the Sum of this Progression (See Page 185) thus, Let $R+2R+3R+4R$ &c. $=z$, then $1+2+3+4$ &c. $=\frac{z}{R}$.

Here the Sum of the first and last Terms are $4+1=5=t$, and the Numbers of all the Terms is $4=t-1$. Therefore $\frac{t-1}{2}\times t=$ the Sum of all the Terms; that is, $\frac{tt-t}{2}=\frac{z}{R}$: hence $\frac{ttR-tR}{2}=z$. Consequently $\frac{ttR-tR}{2}=\frac{-tu}{u}$.

Now from this Equation it will be easy to deduce the following *Theorems.*

*Theorem* 1. $\frac{ttRu-tRu+2tu}{2}=A$, or $\frac{ttu-tu}{2}=R:+tu=A$.

*Theorem* 2. $\frac{2A}{ttR-tR+2t}=u$. *Theorem* 3. $\frac{2A-2tu}{ttu-tu}=R$:

Let $\frac{2}{R}-1=x$, then $t=\sqrt{\frac{2A}{Ru}+\frac{xx}{4}}:-\frac{1}{2}x$ *Theorem* 4.

*Question* 1. *If* 250l. *yearly Rent (or Pension,* &c.) *be forborn or unpaid seven Years; what will it amount to in that Time, at* 6 per Cent. *for each Payment, as it becomes due?*

Here is given $u=250$, $t=7$, and $R=0{,}06$; to find $A$. *Per Th.* 1. First $250\times 7=1750=tu$, $1750\times 7=12250=ttu$. Again $12250-1750=10500=ttu-tu$, and $\frac{10500}{2}\times 0{,}06=315$. Lastly $315+1750=2065=A$; *Viz.* 2065 *l.* is the *Answ.* required.

But if the Annuity, Rent, or Pension, is to be paid by Quarterly or half yearly Payments, *&c.* Then $\frac{0{,}06}{2}=0{,}03=R$ for half yearly Payments: and $\frac{0{,}06}{4}=0{,}015=R$ for quarterly; or $0{,}045=R$ for three quarterly Payments. Example of half yearly Payments.

*Suppose* 250l. per Annum, *to be paid by half yearly Payments, were in Arrears, or unpaid for seven Years; what would it amount to, allowing* 6 per Cent. per Annum *for each Payment, as it becomes due?*

In this Example there is given $u = 125 = \frac{250}{2}$; $t = 14$ the Number of Payments; and $R = 0{,}03 = \frac{0{,}06}{2}$; thence to find $A$.

First $125 \times 14 = 1750 = tu$; $1750 \times 14 = 24500 = ttu$: again $24500 - 1750 = 22750 = ttu - tu$; then $\frac{22750}{2} = 11375$, and $11375 \times 0{,}03 = 341{,}25$. Laftly $341{,}25 + 1750 = 2091{,}25$; that is, $A = 2091\,l.\ 5\,s.$ the Anfwer required.

*N. B.* Hence it may be obferved, that half yearly Payments are more advantageous than yearly. For 2091 *l.* 5 *s.* $>$ 2065 *l.* by 26 *l.* 5 *s.* confequently, quarterly Payments are more advantageous than half yearly Payments.

*Queftion* 2. *What yearly Rent, Penfion,* &c. *being forborn or unpaid feven Years, will raife a Stock of* 2065 l. *allowing* 6 per Cent. per Annum *for each Payment, as it becomes due?*

Here is given $A = 2065$, $t = 7$, and $R = 0{,}06$; to find $u$. *Per Theorem* 2. Firft $7 \times 0{,}06 = 0{,}42 = tR$, and $0{,}42 \times 7 = 2{,}94 = ttR$. Then $ttR - tR = 2{,}52$. Laftly $ttR - tR + 2t = 16{,}52)\ 4130 = 2A\ (250 = u$; that is, 250 *l.* *per Annum*, &c. will raife 2065 *l.* the Stock required.

*Queftion* 3. *In what Time will* 250 l. *yearly Rent raife a Stock of* 2065 l. *allowing* 6 per Cent. *&c. for the Forbearance of the Payments as they become due?*

Here is given $u = 250$, $A = 2065$, and $R = 0{,}06$; to find $t$. *Per Theorem* 4. Firft $\frac{2}{R} = \frac{,06}{2} = 33{,}3333$; and $33{,}3333 - 1 = 32{,}3333 = x = \frac{2}{R} - 1$. Then $16{,}16666$ &c. $= \frac{1}{2}x$; $261{,}3605$ &c. $= \frac{1}{4}xx$. Again $\frac{4130}{15} = 275{,}333 = 2A \div Ru$, and $275{,}3333 + 261{,}3605 = 536{,}6938 = \frac{2A}{Ru} + \frac{1}{4}xx$. Then $\sqrt{536{,}6938} = 23{,}1666$. Laftly, $23{,}1666 - 16{,}1666 = 7 = t$ the Time required.

*Queftion* 4. *If* 250 l. *yearly Rent, being forborn feven Years, will amount to* 2065 l. *allowing Simple Intereft for every Payment as it becomes due; what muft the Rate of the Intereft be* per Cent. *&c.?*

Here is given $u = 250$, $A = 2065$, and $t = 7$; to find $R$: *Per Theorem* 3.

$$\text{Thus}\left\{\begin{array}{l} ttu = 12250 \\ tu = \underline{\ 1750} \end{array}\right\}\left\{\begin{array}{l} 4130 = 2A \\ \underline{3500} = 2tu \end{array}\right.$$

$$ttu - tu = 10500)\quad 630 = 2A - 2tu\ (0{,}06 = R.$$

Then $1 : 0{,}06 :: 100 : 6$ the Rate required.

Sect.

## Sect. 3. *The* Preſent *Worth of* Annuities *or Penſions,* &c. *computed at Simple Intereſt.*

THE Buſineſs of purchaſing Annuities, or taking of Leaſes, *&c.* for any aſſigned Time, depends upon the true equating of the Principal or Money laid out on the Purchaſe, with the Annuity or Yearly Rent, by allowing (or diſcompting) the ſame Rate of Intereſt to both Parties. Which may be eaſily performed by duly applying the reſpective *Theorems* of the two laſt Sections together; as will fully appear by the following Queſtion.

*Queſtion* 1. *What is* 75 l. *yearly Rent, to continue nine Years, worth in ready Money, at* 6 per Cent. per Annum *Simple Intereſt?*

1. *Per Theorem* 1. of the laſt Section, find what the propoſed yearly Rent would amount to, if it were forborn 9 Years, at 6 *per Cent.*

Thus $u = 75$, $t = 9$, and $R = 0{,}06$: Quære $A$.

$$ttu = 6075$$
$$tu = 675$$
$$ttu - tu = 5400$$

Then 2) 5400 (2700 and $R = 0{,}06$ — Multiply

$$162, \; + tu = 675, \; \Big\} = 837 = A.$$

2. Then by *Theorem* 2. *Section* 1. find what Principal, being put to Intereſt for the ſame Time, and at the ſame Rate, will amount to 837 $l. = A$. Thus $tR = 0{,}54 = 9 \times 0{,}06$; $tR + 1 = 1{,}54$) 837 (543,5064 $= P$: that is, $P = 543\,l.\ 10\,s.\ 1\frac{1}{2}\,d.$ which is the Worth of 75 *l.* a Year, as was required.

From the Work of theſe two Operations (duly conſidered) it muſt needs be eaſy to conceive, how the two *Theorems* by which they were performed, may be combined in one.

For 1. $\dfrac{ttRu - tRu + 2tu}{2} = A$; and 2. $PtR + P = A$,

Conſequently $PtR + P = \dfrac{ttRu - tRu + 2tu}{2}$. And from this Equation may be deduced the following *Theorems.*

*Theorem* 1. $\dfrac{ttRu - tRu + 2tu}{2tR + 2} = P$, or $\dfrac{ttR - tR + 2t}{2tR + 2} \times u = P$.

By this *Theorem* all Queſtions of the ſame Kind with the laſt (*viz.* that above) may be eaſily and readily anſwered at one Operation.

*Theorem* 2. $\frac{2PtR + 2P}{ttR - tR + 2t} = u$, or $\frac{tR + 1}{ttR - tR + 2t} : \times 2P = u$.

*Theorem* 3. $\frac{2P - 2tu}{ttu - tu - 2Pt} = R$.

Let $\frac{2}{R} - \frac{2P}{u} - 1 = x$, then will $tt \pm xt = \frac{2P}{Ru}$.

Which gives this *Theorem* 4. $\sqrt{\frac{2P}{Ru} + \frac{xx}{u}} : \pm \frac{x}{2} = t$.

By the fecond and fourth *Theorems*, two very ufeful Queftions may be eafily anfwered.

1. *As for Inftance: If it be required to find what Annuity, or yearly Rent,* &c. *may be purchafed, for any propofed Sum, to continue any affigned Time, allowing any Rate of Intereft?*

This Queftion may be anfwered by *Theorem* 2.

2. *Again: If it be required to find how long any yearly Rent, Penfion, or Annuity,* &c. *may be purchafed (or enjoyed) for any propofed Sum, at any given Rate of Intereft?*

All Queftions of this Kind are eafily anfwered by *Theorem* 4.

In thefe Queftions it is fuppofed, that the Purchafe, or yearly Rent, is to commence or be immediately entered upon. But if it be required to find the Value or Purchafe of an Annuity or yearly Rent, *&c.* in Reverfion; that is, when it is not to be entered upon until after fome Time, or Number of Years are paft; then you muft firft find what the Sum propofed to be laid out in the Purchafe, would amount to, if it were put to Intereft, during the Time the Annuity, *&c.* is not to be put in prefent Poffeffion; and make that Amount the Sum for the Purchafe, proceeding with it as in either of the two laft Queftions, *&c.*

Note, *From the firft Queftion of this Section it will be eafy to conceive how to perform the Equation of Payments, between Debtor or Creditor, at any Rate of Intereft, without doing any Damage to either Party.*

That is, when feveral Sums of Money are to be paid, at feveral different Times, to find the Time when all the Payments may be truly difcharged at once: as if one Sum were to be paid at the End of two Months, another at fix Months, and perhaps a third Sum at eight Months end, *&c.* And if it were required to find the Time when all thofe Sums may be truly difcharged at one Payment without Lofs, *&c.*

CHAP.

# CHAP. XII.

## Of Compound *Intereſt,* *and* Annuities, *&c.*

COMPOUND Intereſt is that which ariſes from any Principal and it's Intereſt put together, as the Intereſt ſo becomes due; ſo that at every Payment, or at the Time when the Payments became due, there is created a new Principal; and for that Reaſon it is called Intereſt upon Intereſt, or Compound Intereſt.

As for Inſtance; Suppoſe 100 *l.* were lent out for two Years, at 6 *per Cent. per Annum*, Compound Intereſt: then at the End of the firſt Year, it will only amount to 106 *l.* as in Simple Intereſt. But for the ſecond Year this 106 *l.* becomes Principal, which will amount to 112 *l.* 7 *s.* 2 $\frac{1}{2}$ *d.* at the ſecond Year's End, whereas by Simple Intereſt it would have amounted to but 112 *l.*

And altho' it be not lawful to let out Money at Compound Intereſt; yet in purchaſing of Annuities or Penſions, *&c.* and taking Leaſes in Reverſion, it is very uſual to allow Compound Intereſt to the Purchaſer for his ready Money; and therefore it is very requiſite to underſtand it.

### Sect. 1. *Of* Compound *Intereſt.*

Let $\begin{cases} P = \text{the Principal put to Intereſt.} \\ t = \text{the Time of it's Continuance.} \\ A = \text{the Amount of the Principal and Intereſt.} \end{cases}$ as before.

$R = \begin{cases} \text{the Amount of } 1\,l. \text{ and it's Intereſt for 1 Year, at} \\ \text{any given Rate, which may be thus found.} \end{cases}$

*Viz.* 100 : 106 :: 1 : 1,06 = the Amount of 1 *l.* at 6 *per Cent.*
Or 100 : 105 :: 1 : 1,05 = the Amount of 1 *l.* at 5 *per Cent.*
and ſo on for any other aſſigned Rate of Intereſt.

Then if $R$ = the Amount of 1 *l.* for one Year, at any Rate.
$R^2$ = the Amount of 1 *l.* for two Years.
$R^3$ = the Amount of 1 *l.* for three Years.
$R^4$ = the Amount of 1 *l.* for four Years.
$R^5$ = the Amount of 1 *l.* for five Years. Here $t = 5$

For $1 : R :: R : RR :: RR : RRR :: RRR : R^4 :: R^4 : R^5$ : &c. in $\div\!\!\div$.

That is $\begin{cases} \text{As one Pound : is to the Amount of one Pound at one} \\ \text{Year's End :: ſo is that Amount : to the Amount of} \\ \text{one Pound at two Years End, \&c.} \end{cases}$

Whence

Whence it is plain, that Compound Intereſt is grounded upon a Series of Terms, increaſing in Geometrical Proportion continued; wherein $t$ (*viz.* the Number of Years) does always aſſign the Index of the laſt and higheſt Term: *Viz.* the Power of $R$, which is $R^t$.

Again, As $1 : R^t :: P : PR^t = A$ the Amount of $P$ for the Time that $R^t =$ the Amount of 1 *l.*

That is { *As one Pound : is to the Amount of one Pound for any given Time :: ſo is any propoſed Principal (or Sum) to it's Amount for the ſame Time.*

From the Premiſes (I preſume) the Reaſon of the following *Theorems*, may be very eaſily underſtood.

*Theorem* 1. $PR^t = A$, as above.

From hence the two following *Theorems* are eaſily deduced.

*Theorem* 2. $\frac{A}{R^t} = P$. *Theorem* 3. $\frac{A}{P} = R^t$.

By theſe three *Theorems*, all Queſtions about Compound Intereſt may be truly reſolved by the Pen only, *viz.* without Tables; tho' not ſo readily as by the Help of Tables, calculated on purpoſe; as will appear farther on.

*Queſtion* 1. *What will* 256 l. 10 s. *amount to in ſeven Years, at* 6 per Cent. per Annum, *Compound Intereſt?*

Here is given $P = 256,5$; $t = 7$; and $R = 1,06$ which being involved until it's Index $= t$ (*viz.* 7.) will become $R^7 = 1,50363$. Then $1,50363 \times 256,5 = 385,6811 = A = 385$ *l.* 13 *s.* $7\frac{1}{2}$ *d.* which is the Anſwer required.

*Queſtion* 2. *What Principal or Sum of Money muſt be put (or let) out to raiſe a Stock of* 385 l. 13 s. $7\frac{1}{2}$ d. *in ſeven Years, at* 6 per Cent. per Annum, *Compound Intereſt?*

Here is given $A = 385,6811$; $R = 1,06$; and $t = 7$; to find $P$, by *Theorem* 2. Thus $R^t = 1,50363$) $385,6811 = A$ ($256,5 = P$. That is, $P = 256$ *l.* 10 *s.* which is the Principal or Sum, as was required.

*Queſtion*

*Question* 3. *In what Time will* 256 l. 10 s. *raise a Stock of (or amount to)* 385 l. 13s. $7\frac{1}{2}$d. *allowing* 6 per Cent. per Annum, *Compound Interest?*

Here is given $P = 256{,}5$; $A = 385{,}6811$; $R = 1{,}06$; to find $t$ by the third *Theorem* $R^t = \frac{A}{P} = \frac{385{,}6811}{256{,}5} = 1{,}50363$, which being continually divided by $R = 1{,}06$ until nothing remain, the Number of those Divisions will be $7 = t$. Thus 1,06) 1,50363 (1,41852. And 1,06) 1,41852 (1,338225. Again 1,06) 1,338225 (1,262477. And so on until it become 1,06) 1,06 (1. which will be at the seventh Division. Therefore it will be $t = 7$ the Number of Years required by the Question.

*Question* 4. *If* 256 l. 10 s. *will amount to (or raise a Stock of)* 385 l. 13s. $7\frac{1}{2}$ d. *in seven Years Time; what must the Rate of Interest be,* per Cent. per Annum?

Here is given $P = 256{,}5$; $A = 385{,}6811$, and $t = 7$, Quære $R$. By *Theorem* 3. $\frac{A}{P} = R^t = 1{,}50363$; as before in the last Question. And if $R^t = R^7 = 1{,}50363$, then $R = \sqrt[7]{1{,}50363}$, which may be thus extracted.

| | | |
|---|---|---|
| Put | 1 | $r + e = R$, then |
| 1 ⊛ 7 | 2 | $r^7 + 7r^6e + 21r^5ee = R^7 = 1{,}50363 = G$ |
| $2 - r^7$ | 3 | $7r^6e + 21r^5ee\,G - r^7$ |
| $3 \div 7r^5$ | 4 | $re + 3ee = \frac{G - r^7}{7r^5} = D$ |
| $4 \div$ | 5 | $e = \frac{D}{r + 3e}$; let $r = 1$, then $D = 0{,}0719$ |

Operation $r = 1{,}00$
$+3e = 0{,}18$
Divisor 1,18) 0,0719 (1,00 = $r$
0,06 = $e$
708   1,06 = $r + e = R$
11 to be rejected.

Then 1 : 0,06 :: 100 : 6 the Rate *per Cent.* required.

The first three Questions may be much more easily performed by the following Table, which is only the Amounts of one Pound for thirty-nine Years.

That

That is, of $R . RR . RRR . R^4 . R^5$ . and fo on to $R^{39}$.

| Years = *t*. | The Amounts of 1 *l*. at 6 *perCent*. &c. Compound Intereft. | Years = *t*. | The Amounts of 1 *l*. at 6 *perCent*. &c. Compound Intereft. | Years = *t*. | The Amounts of 1 *l*. at 6 *perCent*. &c. Compound Intereft. |
|---|---|---|---|---|---|
| 1 | $1.06 = R$ | 14 | 2.2609039557 | 27 | 4.8223459407 |
| 2 | $1.1236 = RR$ | 15 | 2.3965581931 | 28 | 5.1116866971 |
| 3 | $1.191016 = R^3$ | 16 | 2.5403516847 | 29 | 5.4183878990 |
| 4 | 1.26247696 | 17 | 2.6927727857 | 30 | 5.7434911729 |
| 5 | 1.3382255776 | 18 | 2.8543391529 | 31 | 6.0881006432 |
| 6 | 1.4185191122 | 19 | 3.0255995021 | 32 | 6.4533866818 |
| 7 | 1.5036302590 | 20 | 3.2071354722 | 33 | 6.8405898828 |
| 8 | 1.5938480745 | 21 | 3.3995636005 | 34 | 7.2510252757 |
| 9 | 1.6894789590 | 22 | 3.6035374166 | 35 | 7.6860867923 |
| 10 | 1.7908476965 | 23 | 3.8197496616 | 36 | 8.1472519998 |
| 11 | 1.8982985583 | 24 | 4.0489346413 | 37 | 8.6360871198 |
| 12 | 2.0121964718 | 25 | 4.2918707197 | 38 | 9.1542523470 |
| 13 | 2.1329282601 | 26 | 4.5493829629 | 39 | 9.7035074878 |

The Title of this Table fhews it's Conftruction, and it's Ufe will eafily appear by an Example or two.

*EXAMPLE* 1.

*What will* 375 l. 10 s. *amount to in nine Years, at* 6 per Cent. per Annum, &c.?

The tabular Number againft 9 Years is 1,689479 which being multiplied with the Principal 375,5 will produce 634,3993 &c. *viz.* 634 *l.* 8 *s.* *ferè*, being the Amount or Anfwer required.

*EXAMPLE* 2.

*What Principal (or Sum) muft be put to Intereft to raife a Stock of* 634 *l.* 8 *s.* *in nine Years Time, at* 6 per Cent. per Annum, *&c.*

If the propofed Stock (*viz.* 634,4) be divided by the tabular Number that is againft the given Number of Years (*viz.* 9.) the Quotient will be the Principal (or Sum) required. *Viz.* againft 9 is 1,689479. Then 1,689479) 634,4 (375,5 = 375 *l.* 10 *s.* the Principal (or Sum) required.

*EXAMPLE* 3.

*In what Time will* 375 l. 10 s. *raife a Stock of (or amount to)* 634 l. 8 s. *at* 6 per Cent. *&c?*

Divide

Divide the propoſed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5) and the Quotient will ſhew the tabular Number that ſtands over againſt the Time ſought. Thus

375,5) 634,4 (1,689479 &c.

This Number being ſought in the Table, will be found to ſtand againſt 9 Years, which is the Time required.

But if the Quotient cannot be truly found in the Table of Amounts for Years, as above; then take out of that Table the neareſt Number that is leſs, and make it a Diviſor, by which you muſt divide the firſt Quotient; and then ſeek the ſecond Quotient in the Table of Amounts for Days (which is inſerted a little further on) and it will aſſign the Number of Days; as in this Example.

*In what Time will* 563 l. *amount to* 860 l. *at* 6 per Cent. per Annum, *Compound Intereſt?*

*Anſwer.* In 7 Years and 99 Days.

Thus 563) 860 (1,52753 which ſhews the Time to be more (or above) ſeven Years; for over againſt 7 Years is 1,50363 which being made the new Diviſor: *Viz.*

1,50363) 1,52753 (1,01589 &c.

This Number is the neareſt Amount to 99 Days.

Note, *If the Stock, Principal, and Time be given; the Rate of Intereſt will be beſt found by extracting the Root,* &c. *as before in the fourth Queſtion.*

The next Thing that I ſhall here propoſe, is to make this Table (which is only calculated for the Rate of 6 *per Cent.*) univerſally uſeful for all the Rates of Compound Intereſt, *which I may preſume to ſay, is a new Improvement of my own,* being well ſatisfied it never was publiſhed before; and not only ſo, but I have heard ſeveral very good Artiſts affirm it was impoſſible to be done.

The Method of performing it is briefly thus, Let $x =$ the Difference between $1,06 = R$, the Amount of 1 *l.* for one Year (in the Table), and any other propoſed Amount of 1 *l.* for one Year; which admits of two Caſes.

*Caſe* 1. If the propoſed Rate be greater than the $1,06 = R$, then will $R + x =$ the true Amount of 1 *l.* for one Year at that Rate.

*Caſe* 2. But if the propoſed Rate be leſs than $1,06 = R$, then it will be $R - x =$ the Amount of 1 *l.* &c.

Make $\begin{cases} t - 1 = b,\ t - 2 = c,\ t - 3 = d,\ t - 4 = f, \text{ \&c.} \\ \frac{1}{2}\, t b = g,\ \frac{1}{4}\, c g = m,\ \frac{1}{4}\, d m = n,\ \frac{1}{4} f n = s, \text{ \&c.} \end{cases}$



Then

Then will $R^t + t R^b x + g R^c x^2 + m R^d x^4$ &c. = the Amount of 1 *l.* at the given Rate, for any Time denoted by *t*, in Case 1. And $R^t - t R^b x + g R^c x^2 - m R^d x^3$ &c. = the Amount of 1 *l.* in Case 2.

Which is no more but this: Let $R + x$ or $R - x$ (which soever it is) be involved (as directed in *Sect.* 5. *Chap.* 2.) to the same Power or Height as the Index *t* the given Time in the Question denotes: rejecting all the Powers of *x* above *xxx* or *xxxx* at most, as useless. Then multiply that Power of $R + x$ or $R - x$ into the given Principal, and their Product will be the Amount required.

An Example or two in each Case will render all easy.

## *EXAMPLE* 1.

*Suppose it were required to find what* 256 l. *would amount to in fifteen Years, at* 8 l. per Cent. per Annum *Compound Interest?* Here $t = 15$.

First 100 : 108 :: 1 : 1,08 the Amount of 1 *l.* at 8 *per Cent.* Next $1,08 - 1,06 = 0,02 = x$. And $R + x = 1,08$ as in Case 1. Then $R^{15} + 15 R^{14} x + 105 R^{13} xx + 455 R^{12} xxx$ &c. = the Amount of 1 *l.* for 15 Years, at 8 *per Cent.*

Here $x = 0,02$ . $xx = 0,0004$ . and $xxx = ,000008$

By the Table $R^{15} = 2,396558$

And $\begin{cases} 15 R^{14} x = 2,260904 \times 15 \times ,02 = 0,678271 \\ 105 R^{13} xx = 2,132928 \times 105 \times ,0004 = 0,089583 \\ 455 R^{12} xxx = 2,012196 \times 455 \times ,000008 = 0,007324 \end{cases}$

Sum = 3,171736

Then $3,171736 \times 256 = 811,964416 = A$.

That is, 811 *l.* 9 *s.* 3½ *d.* *ferè.* Which is the Answer required.

## *EXAMPLE* 2.

*What will* 365 *l. amount to in seven Years at four and a half* per Cent. *&c.*

First 100 : 104,5 :: 1 : 1,045 the Amount of 1 *l.* at 4½ *l.* *per Cent.*

Next $1,06 - 1,045 = 0,015 = x$. Consequently $R - x = 1,045$ as in Case 2.

Then $R^7 - 7 R^6 x + 21 R^5 xx - 35 R^4 xxx$ &c. = the Amount of 1 *l.* for 7 Years, at 4½ *per Cent.*

Here

Here $x = ,015$ ; $xx = ,000225$ ; and $xxx = 000003375$

$$\begin{aligned} &\text{By the Table } R^7 = + 1,503630 \\ \text{And} &\begin{cases} - 7 R^6 x = - 0,148944 \\ + 21 R^5 xx = + 0,006323 \\ - 35 R^4 xxx = - 0,000141 \end{cases} \end{aligned}$$

$$R^7 - 7 R^6 x + 21 R^5 xx - 35 R^4 xxx = 1,360868$$

Then $1,360868 \times 365 = 496,71682 = A$.

That is, 496 *l.* 14 *s.* $3\frac{1}{4}$ *d.* is the Anſwer required.

If the Reaſon of theſe two Operations be but well underſtood, it will be very eaſy to conceive how to find $P$, the Principal, by having $A$, $t$, and $x$ given (becauſe $R$ and it's Powers are always given by the Table).

For $\overline{R^t \mp t R^b x + g R^c xx \mp m R^d xxx} \times P = A$ (as above).

Therefore $\dfrac{A}{R^t \mp t R^b x + g R^c xx \mp m R^d xxx} = P$.

Or if $A$, $P$, and $t$, be given, $x$ may be found.

For $R^t \mp t R^b x + g R^c xx \mp m R^d xxx = \dfrac{A}{P}$. This Equation being ſolved (as in Chap. 10.) the Value of $x$ will be found; and then either $R + x$, or $R - x$ will ſhew the Rate of Intereſt, &c.

But I ſhall leave the numerical Operations to the Learner's Practice, ſuppoſing enough done to ſhew how all Queſtions of this Kind that are limited by whole Years may be computed.

And if the Time given or ſought be not terminated by whole Years, but by Weeks, Months, Quarters, or Half-Years, &c. for reſolving ſuch Queſtions, the beſt Way will be to reduce thoſe Parts of a Year into Days; that done, find an Anſwer according to the Demand of the Queſtion (and agreeing to 1 *l.* as before) for that Number of Days; and in order to that, it will be requiſite to find the Amount of 1 *l.* for one Day (*as in my Compendium of Algebra*, Page 110) which I ſhall here inſert.

Put $a$ = the Amount ſought, then it will be

$$1 : a :: a : aa :: aa : aaa :: aaa : aaaa \div \text{ to } a^{365}.$$

That is { *As one Pound is to it's Amount for one Day :: ſo is that Amount : to the Amount of two Days :: and ſo is that of two Days : to that of three Days. And ſo on in* ÷ *to 365 Days.*

 Then

Then the laſt of the Terms will be $a^{365} = 1{,}06$

| | | |
|---|---|---|
| Put | 1 | $r + e = a$. And let $r = 1$ |
| 1 ◉ 365 | 2 | $r^{365} + 365\, r^{364} e + 66430\, r^{363} ee = a^{365} = 1{,}06$ |
| 2 in Numb. | 3 | $1 + 365\, e + 66430\, ee = 1{,}06$ |
| $3 - 1$ | 4 | $365\, e + 66430\, ee = 0{,}06$ |
| $4 \div 66430$ | 5 | $,00549\, e + ee = 0{,}0000009032 = D$ |
| $5 \div$ | 6 | $e = \dfrac{D}{,00549 + e}$ |

Operation ,00549
$+ e =$ ,0001

1ſt Diviſor ,00559) 0,0000009032 (1,0000000 = $r$
$+ e =$ ,00015   559   0,0001598 = $e$
2d Diviſor ,00574   )3442   1,0001598 = $r + e = a$
$+ e =$ ,000059   2870   true to the 7th Figure
3d Diviſor ,005799   )57200   and only too much by
&c.   &c.   2 in the 8th, at one Operation.

Now $r = 1{,}00016$ for a ſecond Operation. Then

| | | |
|---|---|---|
| 2 in Numb. | 7 | $1{,}0601340140 7 + 386{,}887\, e + 70402{,}172\, ee = 1{,}06$. Hence it appears that $r - e = a$. |
| Therefore | 8 | $1{,}06013401407 - 386{,}887\, e + 70402{,}172\, ee = 1{,}06$ |
| $8 \pm$ | 9 | $386{,}887\, e - 70402{,}172\, ee = 0{,}00013401407$ |
| $9 \div$ | 10 | $,0054953 - ee = ,000000019035503$ |
| $10 \div$ | 11 | $e = \dfrac{,000000019035503}{,0054953 - e}$ |

Operation ,0054953
$- e =$ 3

1ſt Diviſor ,0054950) 0,000000019035503 (1,00016 = $r$
34   164850   0,000000346417 = $e$
,00549466   )2550503   1,000159653583 = $r - e$
$- e =$ 46   2197864
2d Diviſor ,005494614   )35263900
$- e =$ 64   32967684
3d Diviſor ,00549460   )2296216
2197840
98376
54946
&c.

Which being further purſued to a third Operation will give $a = 1{,}00015965358745 3$ &c. This

This Value of *a* is the Amount of 1 *l.* for one Day, from which, if 1 *l.* be substracted, the Remainder = ,000159653587 &c. will be the Interest of 1 *l.* for one Day. Consequently, if any proposed Principal be multiplied into either of these, the respective Product will be the Amount or Interest of that Principal for one Day, at 6 *per Cent.* &c.

And that the Amount (or Interest) of any Principal or Sum may be easily computed for any Number of Days less than a Year; I have here inserted the following Table, which with a great deal of Care (and I believe Exactness) is calculated from the last found (1,000159653587453) Amount of 1 *l.* for one Day. To which also is annexed a Table of the Amounts of 1 *l.* for Months.

| Days | Amounts of 1 *l.* &c. | Days | Amounts of 1 *l.* &c. | Days | Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 1 | 1.0001596536 | 26 | 1.0041592879 | 51 | 1.0081749166 |
| 2 | 1.0003193326 | 27 | 1.0043196055 | 52 | 1.0083358753 |
| 3 | 1.0004790372 | 28 | 1.0044799487 | 53 | 1.0084968597 |
| 4 | 1.0006387673 | 29 | 1.0046403175 | 54 | 1.0086578699 |
| 5 | 1.0007985229 | 30 | 1.0048007120 | 55 | 1.0088189057 |
| 6 | 1.0009583039 | 31 | 1.0049611320 | 56 | 1.0089799673 |
| 7 | 1.0011181105 | 32 | 1.0051215776 | 57 | 1.0091410545 |
| 8 | 1.0012779426 | 33 | 1.0052820488 | 58 | 1.0093021675 |
| 9 | 1.0014378002 | 34 | 1.0054425457 | 59 | 1.0094633062 |
| 10 | 1.0015976834 | 35 | 1.0056030682 | 60 | 1.0096244707 |
| 11 | 1.0017575920 | 36 | 1.0057636164 | 61 | 1.0097856608 |
| 12 | 1.0019175262 | 37 | 1.0059241901 | 62 | 1.0099468767 |
| 13 | 1.0020774859 | 38 | 1.0060847895 | 63 | 1.0101081184 |
| 14 | 1.0022374712 | 39 | 1.0062454146 | 64 | 1.0102693858 |
| 15 | 1.0023974820 | 40 | 1.0064060653 | 65 | 1.0104306789 |
| 16 | 1.0025575184 | 41 | 1.0065667416 | 66 | 1.0105919978 |
| 17 | 1.0027175803 | 42 | 1.0067274436 | 67 | 1.0107533424 |
| 18 | 1.0028776677 | 43 | 1.0078881712 | 68 | 1.0109147128 |
| 19 | 1.0030377808 | 44 | 1.0070489245 | 69 | 1.0110761090 |
| 20 | 1.0031979193 | 45 | 1.0072097035 | 70 | 1.0112375309 |
| 21 | 1.0033580850 | 46 | 1.0073705082 | 71 | 1.0113989786 |
| 22 | 1.0035182732 | 47 | 1.0075313385 | 72 | 1.0115604521 |
| 23 | 1.0036784885 | 48 | 1.0076921945 | 73 | 1.0117219513 |
| 24 | 1.0038387294 | 49 | 1.0078530762 | 74 | 1.0118834764 |
| 25 | 1.0039989958 | 50 | 1.0080139835 | 75 | 1.0120450272 |

Days

| Days | Amounts of 1*l.* &c. | Days | Amounts of 1*l.* &c. | Days | Amounts of 1*l.* &c. |
|---|---|---|---|---|---|
| 76 | 1.0122066038 | 116 | 1.0186908655 | 156 | 1.0252166658 |
| 77 | 1.0123682062 | 117 | 1.0188535031 | 157 | 1.0253803453 |
| 78 | 1.0125398344 | 118 | 1.0190161667 | 158 | 1.0255440509 |
| 79 | 1.0126914885 | 119 | 1.0191788563 | 159 | 1.0257077827 |
| 80 | 1.0128531683 | 120 | 1.0193415719 | 160 | 1.0258715406 |
| 81 | 1.0130148739 | 121 | 1.0195043134 | 161 | 1.0260353247 |
| 82 | 1.0131766054 | 122 | 1.0196670809 | 162 | 1.0261991349 |
| 83 | 1.0133383627 | 123 | 1.0198298745 | 163 | 1.0263629713 |
| 84 | 1.0135001458 | 124 | 1.0199926934 | 164 | 1.0265268338 |
| 85 | 1.0136619547 | 125 | 1.0201555389 | 165 | 1.0266907225 |
| 86 | 1.0138927895 | 126 | 1.0203184110 | 166 | 1.0268546374 |
| 87 | 1.0139856501 | 127 | 1.0204813084 | 167 | 1.0270185784 |
| 88 | 1.0141475365 | 128 | 1.0206442319 | 168 | 1.0271825456 |
| 89 | 1.0143094488 | 129 | 1.0208071814 | 169 | 1.0273465389 |
| 90 | 1.0144713869 | 130 | 1.0209701569 | 170 | 1.0275105585 |
| 91 | 1.0146333511 | 131 | 1.0211331585 | 171 | 1.0276746046 |
| 92 | 1.0147953408 | 132 | 1.0212961861 | 172 | 1.0278386764 |
| 93 | 1.0149573565 | 133 | 1.0214592397 | 173 | 1.0280027746 |
| 94 | 1.0151193981 | 134 | 1.0216223193 | 174 | 1.0281668989 |
| 95 | 1.0152814655 | 135 | 1.0217854250 | 175 | 1.0283310494 |
| 96 | 1.0154435589 | 136 | 1.0219485567 | 176 | 1.0284952262 |
| 97 | 1.0156056781 | 137 | 1.0221117144 | 177 | 1.0286594291 |
| 98 | 1.0157678232 | 138 | 1.0222748982 | 178 | 1.0288236583 |
| 99 | 1.0159299941 | 139 | 1.0224381081 | 179 | 1.0289879137 |
| 100 | 1.0160921910 | 140 | 1.0226013440 | 180 | 1.0291521953 |
| 101 | 1.0162544138 | 141 | 1.0227646060 | 181 | 1.0293160231 |
| 102 | 1.0164166624 | 142 | 1.0229278940 | 182 | 1.0294908372 |
| 103 | 1.0165789370 | 143 | 1.0230902081 | 183 | 1.0296451975 |
| 104 | 1.0167412375 | 144 | 1.0232545483 | 184 | 1.0298095841 |
| 105 | 1.0169035638 | 145 | 1.0234179146 | 185 | 1.0299739969 |
| 106 | 1.0170659161 | 146 | 1.0235813069 | 186 | 1.0301384359 |
| 107 | 1.0172282944 | 147 | 1.0237447253 | 187 | 1.0303029012 |
| 108 | 1.0173906985 | 148 | 1.0239081699 | 188 | 1.0304673928 |
| 109 | 1.0175531086 | 149 | 1.0240716405 | 189 | 1.0306319206 |
| 110 | 1.0177155846 | 150 | 1.0242351372 | 190 | 1.0307964557 |
| 111 | 1.0178780665 | 151 | 1.0243986600 | 191 | 1.0309610251 |
| 112 | 1.0180405744 | 152 | 1.0245622089 | 192 | 1.0311256216 |
| 113 | 1.0182031083 | 153 | 1.0247257830 | 193 | 1.0312902445 |
| 114 | 1.0183656680 | 154 | 1.0248893851 | 194 | 1.0314548937 |
| 115 | 1.0185282578 | 155 | 1.0250530124 | 195 | 1.0316195692 |

| Days | Amounts of 1 *l.* &c. | Days | Amounts of 1 *l.* &c. | Days | Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 196 | 1.0317842709 | 236 | 1.0383939484 | 276 | 1.0450459680 |
| 197 | 1.0319489990 | 237 | 1.0385597318 | 277 | 1.0452128133 |
| 198 | 1.0321137534 | 238 | 1.0387255415 | 278 | 1.0453796853 |
| 199 | 1.0322785341 | 239 | 1.0388913778 | 279 | 1.0455446584 |
| 200 | 1.0324433410 | 240 | 1.0390572405 | 280 | 1.0457135092 |
| 201 | 1.0326081742 | 241 | 1.0392231298 | 281 | 1.0458804611 |
| 202 | 1.0327730339 | 242 | 1.0393890454 | 282 | 1.0460474397 |
| 203 | 1.0329379198 | 243 | 1.0395549876 | 283 | 1.0462144449 |
| 204 | 1.0331028321 | 244 | 1.0397209563 | 284 | 1.0463814768 |
| 205 | 1.0332677706 | 245 | 1.0398869515 | 285 | 1.0465484353 |
| 206 | 1.0334327355 | 246 | 1.0400529732 | 286 | 1.0467156206 |
| 207 | 1.0335977268 | 247 | 1.0402190214 | 287 | 1.0468827325 |
| 208 | 1.0337627444 | 248 | 1.0403850961 | 288 | 1.0470498711 |
| 209 | 1.0339277883 | 249 | 1.0405511973 | 289 | 1.0472170363 |
| 210 | 1.0340928586 | 250 | 1.0407173250 | 290 | 1.0473842283 |
| 211 | 1.0342579552 | 251 | 1.0408834793 | 291 | 1.0475514469 |
| 212 | 1.0344230782 | 252 | 1.0410496601 | 292 | 1.0477186923 |
| 213 | 1.0345882275 | 253 | 1.0412158674 | 293 | 1.0478859643 |
| 214 | 1.0347534033 | 254 | 1.0413821012 | 294 | 1.0480532631 |
| 215 | 1.0349186054 | 255 | 1.0415483616 | 295 | 1.0482205885 |
| 216 | 1.0350838338 | 256 | 1.0417146485 | 296 | 1.0483879407 |
| 217 | 1.0352490887 | 257 | 1.0418809620 | 297 | 1.0485553196 |
| 218 | 1.0354143699 | 258 | 1.0420473021 | 298 | 1.0487227252 |
| 219 | 1.0355796775 | 259 | 1.0422136687 | 299 | 1.0488901576 |
| 220 | 1.0357450115 | 260 | 1.0423800618 | 300 | 1.0490576166 |
| 221 | 1.0359103719 | 261 | 1.0425464815 | 301 | 1.0492251025 |
| 222 | 1.0360757587 | 262 | 1.0427129278 | 302 | 1.0493926150 |
| 223 | 1.0362411719 | 263 | 1.0428794007 | 303 | 1.0495601543 |
| 224 | 1.0364066116 | 264 | 1.0430459001 | 304 | 1.0497277204 |
| 225 | 1.0365710776 | 265 | 1.0432124261 | 305 | 1.0498953132 |
| 226 | 1.0367375701 | 266 | 1.0433789787 | 306 | 1.0500629327 |
| 227 | 1.0369030889 | 267 | 1.0435455579 | 307 | 1.0502305790 |
| 228 | 1.0370686342 | 268 | 1.0437121637 | 308 | 1.0503082521 |
| 229 | 1.0372342059 | 269 | 1.0438787961 | 309 | 1.0505659519 |
| 230 | 1.0373998041 | 270 | 1.0440454551 | 310 | 1.0507336786 |
| 231 | 1.0375654287 | 271 | 1.0442121407 | 311 | 1.0509014320 |
| 232 | 1.0377310798 | 272 | 1.0443788529 | 312 | 1.0510692121 |
| 233 | 1.0378967573 | 273 | 1.0445455918 | 313 | 1.0512370191 |
| 234 | 1.0380624612 | 274 | 1.0447123572 | 314 | 1.0514048529 |
| 235 | 1.0382241916 | 275 | 1.0448791493 | 315 | 1.0515727134 |

| Days | Amounts of 1 *l.* &c. | Days | Amounts of 1 *l.* &c. | Days | Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 316 | 1.0517406008 | 339 | 1.0556094165 | 362 | 1.0594924636 |
| 317 | 1.0519085150 | 340 | 1.0557779484 | 363 | 1.0596616154 |
| 318 | 1.0520764559 | 341 | 1.0559465071 | 364 | 1.0598307942 |
| 319 | 1.0522444237 | 342 | 1.0561150927 | 365 | 1.06 |
| 320 | 1.0524124183 | 343 | 1.0562837053 | Months | The Amounts of 1 *l.* at 6 *per Cent.* For Months. |
| 321 | 1.0525804397 | 344 | 1.0564523448 | 1 | 1.0048675505 |
| 322 | 1.0527484880 | 345 | 1.0566210112 | 2 | 1.0097587942 |
| 323 | 1.0529165631 | 346 | 1.0567897045 | 3 | 1 0146738462 |
| 324 | 1.0530846650 | 347 | 1.0569584248 | 4 | 1.0196128224 |
| 325 | 1.0532527937 | 348 | 1.0571271720 | 5 | 1.0245758394 |
| 326 | 1.0534209493 | 349 | 1.0572959594 | 6 | 1.0295630141 |
| 327 | 1.0535891317 | 350 | 1.0574647472 | 7 | 1.0345744641 |
| 328 | 1.0537573410 | 351 | 1.0576335753 | 8 | 1.0396103076 |
| 329 | 1 0539255771 | 352 | 1.0578024303 | 9 | 1.0446706634 |
| 330 | 1.0540938401 | 353 | 1.0579713122 | 10 | 1.0497556507 |
| 331 | 1.0542621300 | 354 | 1.0581402211 | 11 | 1.0548653894 |
| 332 | 1.0544304467 | 355 | 1.0583091570 | 12 | 1.06 |
| 333 | 1.0545987903 | 356 | 1.0584781199 | | |
| 334 | 1.0547671608 | 357 | 1.0586471097 | | |
| 335 | 1.0549355582 | 358 | 1.0588161265 | | |
| 336 | 1.0551039824 | 359 | 1.0589851703 | | |
| 337 | 1.0552724336 | 360 | 1.0591542411 | | |
| 338 | 1.0554409116 | 361 | 1.0593233389 | | |

The uſe of this Table is in all reſpects like that of whole Years, in finding the Amount of any given Sum for any propoſed Number of Days leſs than a Year.

*EXAMPLE* 1.

*Suppoſe it were required to find the Amount of* 375 *l. for* 210 *Days, at* 6 per Cent.

The Amount of 1 *l.* for 210 Days is 1,0340928 *&c.* per Table. Then 1,0340928 × 375 = 387,7848 *&c.* = 387 *l.* 15 *s.* 8 ¼ *d.* which is the Amount required. And the reſt of the Variations may be performed juſt as in the Examples of whole Years.

But if the Time given conſiſts of Years, and Parts of a Year; as Quarters, Months, *&c.* Then reduce the odd Time or Parts of the Year into Days; and the Anſwer may then be found at two Operations; as in the following *Example.*

*EXAMPLE.*

Example 2. *Suppose it were required to find what* 265 *l. would amount to in five Years and* 135 *Days at* 6 per Cent. *&c.*

First, the Amount of 1 *l.* for { 5 Years is 1,338225, *&c.* / 135 Days is 1,021785, *&c.*

Then 1,338225 × 1,021785 × 265 *l.* = 362,355232, *&c.* being the *Amount* or *Answer* required.

Or, if the *Amount* and *Time* are given, to find the *Principal*: Then *multiply* the *Amount* of 1 *l.* for the *Years*, and the *Amount* of 1 *l.* for the odd Days together: And by their *Product* divide the given *Amount*, the *Quotient* will be the *Principal* required.

Example 3. *What Principal will raise a Stock of* 362 *l.* 7 *s.* 1¼ *d. Or* 362,355232 *l. in* 5 *Years and* 135 *Days, at* 6 per Cent. *&c.*

The Amount of 1 for { 5 Years is 1,338225, *&c.* / 135 Days is 1,021785, *&c.*

Then 1,338225 × 1,021785 = 1,367378, &c. the *Divisor.* Next 1,367378) 362,355232 = *A* (265 *l.* the *Principal* required.

Again, if the *Principal* and its *Amount* are given, to find the *Time*, at 6 *per Cent.* &c. you must divide the *Amount* by its *Principal*, and then proceed as in the *Third Example*, Page 256, for the *Answer required.*

But if the *Amount* and its *Principal*, with the *Time* of its being at *Interest*, are given, to find the *Rate* of *Interest*: Then proceed as in the Fourth *Question*, Page 255, *&c.*

Now in order to make this *Table* of *Amounts* for Days useful for all *Rates* of *Interest (as before in that for Years)* you must first find the Simple *Interest* of 1 *l.* for one Day, both at the given *Rate*, and also at 6 *per Cent.* And call their Difference *x*.

Thus, suppose the given *Ratio* were 8 *per Cent. per Annum*, First 100 : 8 : : 1 : 0,08 And 100 : 6 : : 1 : 0,06 the Two Simple *Interests* for one Year.

Then 365) 0,08 (0,00021917, *&c.* the *Simple Interest* of 1 *l.* for one Day, at 8 *per Cent.*

And 365) 0,06 (0,00016438, *&c.* the *Simple Interest* of 1 *l.* for one Day, at 6 *per Cent.*

Their *Difference* 0,00005479 = *x*, which may do indifferently well for ordinary small *Questions*: But where Exactness is required, it will be convenient to make Use of this *Proportion*:

*Viz.* { As the Simple Intereſt of 1 *l.* for one Day at 6 *per Cent.* : Is to the Tabular Intereſt of 1 *l.* for one Day : : So is the Simple Intereſt of 1 *l.* for one Day, at any given Rate : To a Fourth Number.

That is, 0,00016438 : 0,00015965 : : 0,00021917 : 0,00021286
Then 0,00021286—0,00015965 = 0,00005321 = $x$.

This $x$ being *involved* with the reſpective *Amounts* for Days, in the ſame Manner as was done with thoſe for *Years* (*vide Page* 258) the Reſult will be the *Anſwer* to the *Queſtion.*

## Sect. 2. Annuities or Penſions *in Arrear, computed at Compound Intereſt.*

When *Annuities*, &c. are ſaid to be in *Arrear*, ſee *Page* 248. And I ſhall here make Uſe of the ſame Letters to repreſent the ſame Things as before in that *Page*, ſave only that $R$ is here equal to the *Amount* of 1 *l.* as in *Section* 1. of this *Chapter.*

Suppoſe $u$ = the Firſt *Year's Rent* of any Annuity without *Intereſt.*

Then will $Ru + u =$ { the *Amount* of the Firſt *Year's Rent*, and its *Intereſts* ; More the 2d *Year's Rent.*

And $RRu + Ru + u =$ { the *Amount* of the 1ſt and 2d *Years Rents*, with their *Intereſts* ; More the 3d *Year's Rent*, &c.

Here $RRu + Ru + u = A$, the *Amount* of any *Yearly Rent* or *Annuity*, being forborne *Three Years.* And from hence may be deduced theſe *Proportions :*

*Viz.* $u : Ru :: Ru : RRu :: RRu : RRRu$, and ſo on in ∺ for any *Number* of *Terms* or *Years* denoted by $t$, wherein the laſt Term will always be $uR^{t-1}$.

Conſequently, $A - uR^{t-1}$ = the Sum of all the Antecedents

And $A - u$ = the Sum of all the Conſequents in the Series.

And therefore it would be $u : uR :: A - uR^{t-1} : A - u$, Vide *Page* 188.

*Ergo* $Au - uu = RuA - uuR^t$, which, being divided all by $u$, will become $A - u = RA - uR^t$.

From this laſt *Æquation* it will be eaſy to raiſe the following *Theorems :*

*Theorem* 1. $\left\{ \dfrac{uR^t - u}{R - 1} = A. \right.$ *Theorem* 2. $\left\{ \dfrac{RA - A}{R - 1} = u. \right.$

*Theorem*

*Theorem* 3. $\left\{ \frac{RA + u - A}{u} = R^t \right.$. If this *Equation* be continually *divided* by *R*, until nothing *remain*, the *Number* of those *Divisions* will be *t*. See *Page* 255

*Theorem* 4. $\left\{ \frac{A}{u} R - R^t = \frac{A - u}{u} \right.$. If this *Æquation* be resolved into *Numbers*, according to the Method proposed in *Sect.* 3. *Chap.* 10. the *Root* will shew the *Value* of *R*.

QUESTION 1. *If* 30 *l. Yearly Rent, or Annuity*, &c. *be forborne* (i. e. *remain unpaid*) *Nine Years*; *what will it amount to*, *at* 6 per Cent. per Annum, *Compound Interest?*

Here is given $u = 30$, $t = 9$, and $R = 1,06$; to find *A*. per *Theorem* 1.

$R^9 = 1,689479$ By the Table of *Amounts* for *Years*
$30 = u$

$R^9 u = 50,684370$
$-u = 30,$

$R - 1 = 0,06)\ 20,684370\ (344,7395 = 344\,l.\ 14\,s.\ 9\tfrac{1}{2}\,d. = A$, the Amount required.

QUESTION 2. *What Yearly Rent or Annuity*, &c. *being forborne or unpaid Nine Years, will raise a Stock of* 344 *l.* 14 *s.* 9½ *d.* = 344,7395, *at* 6 per Cent. &c.

Here is given $A = 344,7395$, $t = 9$, and $R = 1,06$; to find *u*. per *Theorem* 2.

$AR = 344,7395 \times 1,06 = 365,42387$
$-A = 344,7395$

$R^t - 1 = 1,689479 - 1 = 0,689479)\ 20,68437\ (30 = u.$

QUESTION 3. *In what Time will* 30 *l. Yearly Rent raise a Stock or Amount to* 344 *l.* 14 *s.* 9½ *d. allowing* 6 per Cent. *for the Forbearance of Payments?*

Here is given $u = 30$, $A = 344,7395$, and $R = 1,06$; to find *t*. per *Theorem* 3.

First $AR + u - A = 365,42387 + 30 - 344,7395 = 50,68437$. And $u = 30)\ 50,68437\ (1,689479 = R^t$. Then $R = 1,06)\ 1,689479\ (1,593848$. And $1,06)\ 1,593848\ (1,50363$; and so on until it become $1,06)\ 1,06\ (1$. which will be at the *Ninth Division*; therefore $t = 9$.

 Or

Or $R^t = 1{,}689479$, being ſought in the *Table* of *Amounts* for *Years*, will be found to ſtand over-againſt 9 *Years*, which is the *Time required.*

QUESTION 4. *If* 30 *l.* per Annum, *being unpaid Nine Years, will amount to* 344 *l.* 14 *s.* 9½ *d. allowing Compound Intereſt for every Payment as it becomes due, What muſt the Rate of Intereſt be* per Cent. *&c.*

Here is given $u = 30$, $A = 344{,}7395$, and $t = 9$; to find $R$ by the laſt of the Four *Æquations*, *Viz.* $\left\{ \frac{A}{u} R - R^t = \frac{A-u}{u} \right.$.

Firſt $\frac{A}{u} = \frac{344{,}7395}{30} = 11{,}491317$. And $\frac{A-u}{u} = 10{,}491317$.

Hence there is this *Æquation*; $11{,}491317\, R - R^9 = 10{,}491317$.

| | | |
|---|---|---|
| Let | 1 | $r + e = R$, and ſuppoſe $r = 1$ |
| 1 ⊚ 9 | 2 | $r^9 + 9r^8 e + 36 r^7 ee = R^9$ |
| $1 \times \frac{z}{u}$, in Numb. | 3 | $11{,}491317 + 11{,}491317\, e = 11{,}491317\, R$ |
| 2 in Numb. | 4 | $1{,}000000 + 9{,}000000\, e + 36\, ee = R^9$ |
| 3 — 4 | 5 | $10{,}491317 + 2{,}491317\, e - 36\, ee = 10{,}491317$ |
| Whence | 6 | $36\, ee = 2{,}491317\, e$ |
| 6 ÷ 36 *e* | 7 | $e = 0{,}06$, &c. |

Firſt $r = 1$, $+\, e = 0{,}06$ $\Big\} = 1{,}06 = R$ $\Big\{$ *As may be eaſily try'd by involving it, and ordering it, as the Æquation above directs.*

## Section 3. *To find the* Preſent Worth *of Annuities, Penſions, or Leaſes,* &c. *at Compound Intereſt.*

Let $P$ = the *preſent Worth* of any *Annuity*, or *Leaſe*, &c. and the reſt of the Letters as before.

Then, from what has been ſaid in *Section* 3. *Chap.* 11. about *Purchaſing* of *Annuities*, &c. at Simple *Intereſt*, it will be eaſy to form the like *Theorems* here at *Compound Intereſt*, *viz.* by Combining *Theorem* 1. *Page* 266. and *Theorem* 1. *Page* 254. into one *Theorem*.

For $\left\{ \frac{uR^t - u}{R - 1} = A \right.$ $\Big\{$ *The Amount of any Yearly Rent being unpaid any Number of Years. Per* Theorem 1. *of the laſt Section.* Page 266.

And $P R^t = A$ $\Big\{$ *The Amount of any Principal or Sum being put to Intereſt, for the ſame Number of Years. Per* Theorem 1. *Page* 254.

Hence

Hence it follows, That $P\,R^t = \frac{u\,R^t - u}{R - 1}$,
Viz. $P\,R^{t+1} - P\,R^t = u\,R^t - u$ being the very ſame *Æquation* with that in my *Compendium* of *Algebra*, *Page* 112. which is there raiſed from the Conſideration of purchaſing *Annuities*, or taking of *Leaſes*, &c. to be grounded upon a *Rank* or *Series* of *Geometrical Proportionals* continually decreaſing. Thus $\frac{u}{R}$ is the *Firſt* and *Greateſt Term*; $R$ the common *Ratio* of all the *Terms*; and $P$ is the *Sum* of all the *Series*.

That is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$, &c. in $\div$ until the laſt *Term* $= \frac{u}{R^t}$. Then will $P - \frac{u}{R^t}$ be the *Sum* of all the *Antecedents*, and $P - \frac{u}{R}$ the *Sum* of all the *Conſequents*. Therefore it will be

$\frac{u}{R} : \frac{u}{RR}$. Or (in the ſame Ratio) $u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R}$, which produces $P\,R^{t+1} - u\,R^t = P\,R^t - u$. As above.
From this *Æquation* may be deduced the following *Theorems*:

*Theorem* 1. $\left\{ \frac{u - \frac{u}{R^t}}{R - 1} = P. \right.$ *Theorem* 2. $\left\{ \frac{P\,R^t \times R : - P\,R^t}{R - 1} = u. \right.$

*Theorem* 3. $\left\{ \frac{u}{P + u - P\,R} = R^t \right.$ $\left\{ \right.$ *Which, being continually divided by* R, *will give t.*

*Theorem* 4. $\left\{ \frac{u}{P} = \frac{u}{P}\,R^t + R^t - R^{t+1}. \right.$ The Reſolving of which *Æquation* will diſcover the Value of $R$.

Queſtion 1. *What is* 30 *l. Yearly Rent, to continue Seven Years, worth in ready Money, allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

Here is given $u = 30$. $t = 7$. And $R = 1{,}06$ to find $P$. per *Theorem* 1. *Viz.* $\frac{u}{R^t} = \frac{30}{1{,}50363} = 19{,}9517$.

And $30 - 19{,}9517 = 10{,}483 = u - \frac{u}{R^t}$,

Then

Then $R - 1 = 0{,}06)\ 10{,}0483\ (167{,}4716 = P = 167\ l.\ 9s.\ 5d.$ being the Anſwer required.

Queſtion 2. *What Annuity or Yearly Rent, to continue Seven Years, may be purchaſed for* 167 *l.* 9 *s.* 5 *d. allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

In this *Queſtion* there is given $P = 167{,}4716\ .\ t = 7$
And $R = 1{,}06$ to find $u$. By the Second *Theorem.*
Firſt $P R^t \times R = 251{,}8153 \times 1{,}06 = 266{,}9242$
And $- P R^t = 167{,}4716 \times 1{,}50363) = 251{,}8153$

Then $R^t - 1 = 0{,}50363)\quad 15{,}1089\quad (30 = u$
That is $u = 30\ l.$ the *Anſwer required.*

Queſtion 3. *How long may one have a Leaſe of* 30 *l. Yearly Rent, for* 167 *l.* 9 *s.* 5 *d. allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

Here is given $P = 167{,}4716\ .\ u = 30\ .$ And $R = 1{,}06$ to find $t$. By the Third *Theorem.*
Firſt $P + u = 167{,}4716 + 30 = 197{,}4716$
And $- P R = 177{,}5199$

Then $19{,}9517)30 = u(1{,}50363 = R^t$

If this $1{,}50363 = R^t$ be either continually *divided* by $1{,}06 = R$ until nothing *remain* (As before in *Page* 255.) Or if it be ſought in the *Table* of *Amounts* for *Years*, &c. it will diſcover $t = 7$ which is the true *Anſwer* required.

Queſtion 4. *Suppoſe one ſhould give* 167 *l.* 9 *s.* 5 *d. for the Purchaſe of a Penſion, or Annuity of* 30 *l.* per Annum, *to continue Seven Years: At what Rate of Intereſt,* per Cent. *would that Purchaſe be made, allowing Compound Intereſt to the Purchaſer?*

In this *Queſtion* there is given, $P = 167{,}4716\ .\ u = 30$ and $t = 7$ to find $R$. Per *Theorem* 4 in this *Æquation* $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1} \right.$, which being brought into *Numbers*, and its *Root extracted*, as in the fourth *Queſtion* of the laſt *Section*; the Value of $R$ will be found 1,06, and then it will be $1 : 0{,}06 :: 100 : 6$, the Rate *per Cent.* as was required.

Theſe

Theſe Four *Queſtions* include all the *Varieties* that can be propoſed about purchaſing *Annuities* or *Leaſes*, &c. which are to be either immediately enter'd upon, or in Poſſeſſion at the Time when the *Purchaſe* is made.

But ſuch *Queſtions* as relate to *Annuities*, or a taking of *Leaſes*, &c. in *Reverſion*, muſt be *parted* or *divided* into *two diſtinct Queſtions*, each to be ſeparately conſider'd by itſelf (See *Page* 252.) As in the following *Examples* :

Example 1. *Suppoſe it were required to compute the preſent Worth of* 75 *l. Yearly Rent, which is not to commence or be enter'd upon, until Ten Years hence; and then to continue Seven Years after that Time: at* 6 per Cent. *&c. Compound Intereſt ?*

The Firſt Work in this *Queſtion* is, to find what 75 *l. per Annum*, to continue *Seven Years*, is worth in ready Money ; as if it were to be immediately enter'd upon : And to perform that, there is given $u = 75$. $R = 1{,}06$. and $t = 7$. to find $P$. as in the Firſt *Queſtion* of this *Section*.

Thus, $\frac{u}{R^t} = \frac{75}{1{,}50363} = 49{,}8793$ And $75 - 49{,}8793 = 25{,}1207$ $= u - \frac{u}{R}$.

Then, $R - 1 = 0{,}06)\,25{,}1207 = 418{,}6783 = 418\,l.\ 14\,s.\ 6\frac{3}{4}d.$ the *Anſwer* to the Firſt Part of the *Queſtion*.

Then the next Work will be, to find what *Principal* or *Sum* being put out Ten *Years*, at 6 *per Cent.* &c. will amount to 418 *l.* 14 *s.* 6 $\frac{3}{4}$ *d.* Here is given $A = 418{,}6783$, $R = 1{,}06$, $t = 10$. to find $P$. Per *Theorem* 2. *Page* 254.

Thus $R^{10} = 1{,}790847)\ 418{,}6783 = A\ (233{,}7884 = 233\,l.$ 15 *s.* 9 *d.* the *preſent Worth of* 75 *l. per Annum in Reverſion*, &c. As was required.

Example 2. *What Annuity or Yearly Rent, to be enter'd upon Ten Years hence, and then to continue Seven Years, may be purchaſed for* 233 *l.* 15 *s.* 9 *d. Ready Money, at* 6 per Cent. *&c. Compound Intereſt ?*

In the 1ſt Work of this *Queſtion* there is given, $P = 233{,}7884$ $R = 1{,}06$. And $t = 10$ (*the Time which the Annuity is not to be enter'd upon*) to find $A$. Per *Theorem* 1. Page 254.

Thus, $P\,R^t = 233{,}7884 \times 1{,}790847 = 418{,}6783 = A$ the

*Amount* of 233*l.* 15*s.* 9*d.* put to *Interest Ten Years*, at 6 *per Cent.* &c. Then, for the Second Work of the *Question*, there is given $P = 418,6783$ . $R = 1,06$ . And $t = 7$ *(the Time that the Annuity is to be enjoy'd)* to find $u$. Per *Theorem* 2. of this *Section*.

Thus $PR^t \times R = 418,6783 \times 1,50363 \times 1,06 = 667,3095$
$-PR^t = 418,6783 \times 1,50363 = 629,5372$

$R^t - 1 = 0,50363)\ 37,7723(75 = u$

That is, $u = 75\,l.$ the *Yearly Rent* required by the *Question*.

These *Two Examples* of finding $P$ and $u$ do fully shew the *Method* that must be used in *Resolving* the two *General*, and indeed, the most useful *Questions* about *Annuities* or *Leases* in *Reversion*: And if there be Occasion, either the *Rate*, or the *Time*, viz. $R$ or $t$, may be found by a due Application of their respective *Theorems*.

Note, *That which hath been done in the two last Sections about Annuities or Yearly Rents*, &c. *at* 6 per Cent. *may also be done for any Rate of Interest, by applying the Difference of the Rates* (viz. $x$.) *As directed in the First Section of this Chapter.*

Now because that *Rents* and *Annuities*, &c. are usually paid either by *Quarterly* or *Half Yearly Payments*, and the Method of computing them by the Pen may be thought a little troublesome; I have inserted the following Tables of the *Amounts* of 1 *l.* for each, at 6 *per Cent.*

| Half Years $= t$. | Annuities of 1 *l.* at 6 *per Cent.* Compound Interest. | Half Years $= t$. | Annuities of 1 *l.* at 6 *per Cent.* Compound Interest. | Half Years $= t$. | Annuities of 1 *l.* at 6 *per Cent.* Compound Interest. |
|---|---|---|---|---|---|
| 1 | 1,0295630141 | 11 | 1,3777875592 | 21 | 1,8437905523 |
| 2 | 1,06 | 12 | 1,4185191122 | 22 | 1,8982985583 |
| 3 | 1,0613367949 | 13 | 1,4604548127 | 23 | 1,9544179853 |
| 4 | 1,1236 | 14 | 1,5036302590 | 24 | 2,0121964718 |
| 5 | 1,1568170026 | 15 | 1,5480821017 | 25 | 2,0716830644 |
| 6 | 1,191016 | 16 | 1,5938480745 | 26 | 2,1329282601 |
| 7 | 1,2262260228 | 17 | 1,6409670276 | 27 | 2,1959840483 |
| 8 | 1,26247696 | 18 | 1,6894789589 | 28 | 2,2609039557 |
| 9 | 1,2997995842 | 19 | 1,7394250493 | 29 | 2,3277430912 |
| 10 | 1,3382255776 | 20 | 1,7908476965 | 30 | 2,3965581001 |

Quar-

Quarterly Amounts.

| Quarters of a Year = $t$. | Amounts of 1*l.* at 6 *per Cent.* &c. Compound Intereſt. | Quarters of a Year = $t$. | Amounts of 1*l.* at 6 *per Cent.* &c. Compound Intereſt. | Quarters of a Year = $t$. | Amounts of 1*l.* at 6 *per Cent.* &c. Compound Intereſt. |
|---|---|---|---|---|---|
| 1 | 1,0146738461 | 21 | 1,3578024938 | 41 | 1,8171263199 |
| 2 | 1,0295630141 | 22 | 1,3777875592 | 42 | 1,8437905523 |
| 3 | 1,0446706634 | 23 | 1,3980050019 | 43 | 1,8708460509 |
| 4 | 1,06 | 24 | 1,4185191122 | 44 | 1,8982985583 |
| 5 | 1,0755542769 | 25 | 1,4393342435 | 45 | 1,9261538989 |
| 6 | 1,0913367949 | 26 | 1,4604548127 | 46 | 1,9544179853 |
| 7 | 1,1073509032 | 27 | 1,4818853020 | 47 | 1,9830968140 |
| 8 | 1,1236 | 28 | 1,5036302590 | 48 | 2,0121964718 |
| 9 | 1,1400875335 | 29 | 1,5256942978 | 49 | 2,0417231330 |
| 10 | 1,1568170026 | 30 | 1,5480821017 | 50 | 2,0716830644 |
| 11 | 1,1737919574 | 31 | 1,5707984203 | 51 | 2,1020826228 |
| 12 | 1,191016 | 32 | 1,5938480745 | 52 | 2,1329282601 |
| 13 | 1,2084927856 | 33 | 1,6172359557 | 53 | 2,1642265211 |
| 14 | 1,2262260228 | 34 | 1,6409670276 | 54 | 2,1959840483 |
| 15 | 1,2442194748 | 35 | 1,6650463253 | 55 | 2,2282075801 |
| 16 | 1,26247696 | 36 | 1,6894789589 | 56 | 2,2609039557 |
| 17 | 1,2810023527 | 37 | 1,7142701133 | 57 | 2,2940801123 |
| 18 | 1,2997995842 | 38 | 1,7394250493 | 58 | 2,3277430912 |
| 19 | 1,3188726433 | 39 | 1,7649491048 | 59 | 2,3619000349 |
| 20 | 1,3382255776 | 40 | 1,7908476965 | 60 | 2,3965581931 |

Either of theſe *Tables* may alſo be made uſeful for any propoſed *Rate* of *Intereſt*; by making the $\frac{1}{2}$ or $\frac{1}{4}$ of the *Difference* of the *Rate* $= x$, &c.

As for Inſtance, ſuppoſe any of the aforeſaid *Queſtions* about *Annuities* or *Rents*, &c. were to be computed at 8 *per Cent. per Ann.*

Then $1{,}08 - 1{,}06 = 0{,}02 = x$ for *Yearly Payments*; as before. *Conſequently* 2) 0,02 (0,01 $= x$ for *Half Year's Payments.*

Or 4) 0,02 (0,005 $= x$ for *Quarterly Payments.*

Now theſe Values of $x$, although they are not really true, yet they may ſerve indifferently well for ſmall *Rents*; as I have already ſaid, *Page* 265. But if you would work exactly;

Then $\surd$ 1,08 = 1,0392304845, &c.

— $\surd$ 1,06 = 1,0295680141, *Vide Table, Page* 272.

Difference = 0,0096624704 = $x$ for *Half Yearly Payments.*

 And

And $\sqrt{}:\sqrt{}$ 1,08 = 1,0194263092, &c.
— $\sqrt{}:\sqrt{}$ 1,06 = 1,0146738461. See the *Last Table.*

Their Difference 0,0047524631 = *x*, for *Quarterly Payments.*

Thefe are the true *Values* of *x*, which being *involved* with their refpective *Amounts* (as before for *Years*, &c.) according as the *Queftion* requires, the *Refult* will be the *Anfwer* at 8 *per Cent.* &c. The like may be done for any other *Rate*, either *Greater* or *Lefs* than 6.

Now, although the Method ufed here (and in *Page* 257 and 258, &c.) be really true (by which the *Tables* calculated only for 6 *per Cent.* are made effectual for all Rates of *Compound Intereft*) yet it was rather propos'd to fhew what may poffibly be perform-ed by the Pen, without a great many *Tables* of feveral *Rates*, than intended for common Practice.

For it muft needs be confefs'd, that *Tables*, calculated on Pur-pofe for any defigned *Rate* of *Intereft*, are much more ready and ufeful in common Practice. And therefore fince the Legiflative Power hath thought fit to reduce the *Rate* of *Intereft*, and hath fettled it by an Act of Parliament, at 5 *per Cent.* I've therefore been at the Trouble *(which was not a little)* to calculate the following *Tables* for that *Rate*; but don't think it convenient to take the *Tables* at 6 *per Cent.* out of the Book, becaufe the Ex-amples are all fuited to them; and not only fo, but they may be found ufeful in the taking of *Leafes* for Houfes, *&c.* For in thofe Cafes, the *Purchafer* is allowed more *Intereft* for his purchafe Mo-ney, than the common *Rate* paid upon the Loan of Money.

Here

Here follow New *Tables* of the *Amounts* of one Pound at the *Rate* of 5 *per Cent. per Annum* Compound Intereſt. For *Years*, *Half Years*, *Quarters*, *Months*, and *Days*.

I. The *Table* of the *Yearly Amounts* of 1 *l.* &c.

| Years = t. | The Amounts of 1 *l.* &c. | Years = t. | The Amounts of 1 *l.* &c. | Years = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 1 | 1,05 = *R* | 14 | 1,97993160 | 27 | 3,73345632 |
| 2 | 1,1025 = *R R* | 15 | 2,07892818 | 28 | 3,92012914 |
| 3 | 1,157625 = $R^3$ | 16 | 2,18287459 | 29 | 4,11613599 |
| 4 | 1,21550625 | 17 | 2,29201832 | 30 | 4,32194239 |
| 5 | 1,27628156 | 18 | 2,40661923 | 31 | 4,53803949 |
| 6 | 1,34009564 | 19 | 2,52695019 | 32 | 4,76494147 |
| 7 | 1,40710042 | 20 | 2,65329770 | 33 | 5,00318854 |
| 8 | 1,47745544 | 21 | 2,78596259 | 34 | 5,25334797 |
| 9 | 1,55132822 | 22 | 2,92526072 | 35 | 5,51601536 |
| 10 | 1,62889463 | 23 | 3,07152375 | 36 | 5,79181613 |
| 11 | 1,71033936 | 24 | 3,22509994 | 37 | 6,08140694 |
| 12 | 1,79585633 | 25 | 3,38635494 | 38 | 6,38547729 |
| 13 | 1,88564914 | 26 | 3,55567269 | 39 | 6,70475115 |

II. The *Table* of the *Half Yearly Amounts* of 1 *l.* &c.

| Half Yrs. = t. | The Amounts of 1 *l.* &c. | Half Yrs. = t. | The Amounts of 1 *l.* &c. | Half Yrs. = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 1 | 1,02469507 | 11 | 1,30779943 | 21 | 1,66912030 |
| 2 | 1,05 | 12 | 1,34009564 | 22 | 1,71033936 |
| 3 | 1,07592983 | 13 | 1,37318940 | 23 | 1,75257632 |
| 4 | 1,1025 | 14 | 1,40710042 | 24 | 1,79585633 |
| 5 | 1,12972632 | 15 | 1,44184887 | 25 | 1,84020513 |
| 6 | 1,57625 | 16 | 1,47745544 | 26 | 1,88564914 |
| 7 | 1,18621264 | 17 | 1,51394132 | 27 | 1,93221539 |
| 8 | 1,21550625 | 18 | 1,55132822 | 28 | 1,97993160 |
| 9 | 1,24552327 | 19 | 1,58963838 | 29 | 2,02882616 |
| 10 | 1,27628156 | 20 | 1,62889463 | 30 | 2,07892818 |

### III. The *Table* of the *Quarterly Amounts* of 1 *l.* &c.

| Quarters = t. | The Amounts of 1 *l.* &c. | Quarters = t. | The Amounts of 1 *l.* &c. | Quarters = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 1 | 1,01227223 | 21 | 1,29194439 | 41 | 1,64888480 |
| 2 | 1,02469507 | 22 | 1,30779943 | 42 | 1,66912031 |
| 3 | 1,03727037 | 23 | 1,32384905 | 43 | 1,68960414 |
| 4 | 1,05 | 24 | 1,34009564 | 44 | 1,71033936 |
| 5 | 1,06288585 | 25 | 1,35654161 | 45 | 1,73132904 |
| 6 | 1,07592983 | 26 | 1,37318940 | 46 | 1,75257632 |
| 7 | 1,08913389 | 27 | 1,39004151 | 47 | 1,77408435 |
| 8 | 1,1025 | 28 | 1,40710042 | 48 | 1,79585633 |
| 9 | 1,11603014 | 29 | 1,42436869 | 49 | 1,81789549 |
| 10 | 1,12972632 | 30 | 1,44184887 | 50 | 1,84020513 |
| 11 | 1,14359059 | 31 | 1,45954358 | 51 | 1,86278856 |
| 12 | 1,157625 | 32 | 1,47745544 | 52 | 1,88564914 |
| 13 | 1,17183164 | 33 | 1,49558712 | 53 | 1,90879027 |
| 14 | 1,18621264 | 34 | 1,51394132 | 54 | 1,93221539 |
| 15 | 1,20077012 | 35 | 1,53252076 | 55 | 1,95592799 |
| 16 | 1,21550625 | 36 | 1,55132822 | 56 | 1,97993160 |
| 17 | 1,23042323 | 37 | 1,57036648 | 57 | 2,00422978 |
| 18 | 1,24552327 | 38 | 1,58963838 | 58 | 2,02882616 |
| 19 | 1,26080862 | 39 | 1,60914680 | 59 | 2,05372439 |
| 20 | 1,27628156 | 40 | 1,62889463 | 60 | 2,07892818 |

### IV. The *Table* of the *Monthly Amounts* of 1 *l.* &c.

| Months = t. | The Amounts of 1 *l.* &c. | Months = t. | The Amounts of 1 *l.* &c. | Months = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 1 | 1,00407412 | 5 | 1,02053728 | 9 | 1,03727037 |
| 2 | 1,00816485 | 6 | 1,02469507 | 10 | 1,04149634 |
| 3 | 1,01227223 | 7 | 1,02886981 | 11 | 1,04573953 |
| 4 | 1,01639636 | 8 | 1,03306155 | 12 | 1,05 |

*NOTE:* The *Amount* of one *Pound*, for one *Day*, is 1,0001336807225, *&c.* (found as that in *Page* 260) but in the following *Table*, I take only Nine of thoſe *Figures*, as being ſufficient in Practice, for computing the *Intereſt* of any *Sum* not exceeding One Hundred Millions of *Pounds*.

V. The

## V. The *Table* of the *Daily Amounts* of 1 *l.* &c.

| Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 1 | 1,00013368 | 36 | 1,00482376 | 71 | 1,00953587 |
| 2 | 1,00026738 | 37 | 1,00495810 | 72 | 1,00967082 |
| 3 | 1,00040109 | 38 | 1,00509245 | 73 | 1,00980579 |
| 4 | 1,00053483 | 39 | 1,00522681 | 74 | 1,00994079 |
| 5 | 1,00066858 | 40 | 1,00536119 | 75 | 1,01007579 |
| 6 | 1,00080235 | 41 | 1,00549558 | 76 | 1,01021083 |
| 7 | 1,00093614 | 42 | 1,00563000 | 77 | 1,01034587 |
| 8 | 1,00106994 | 43 | 1,00576443 | 78 | 1,01048093 |
| 9 | 1,00120377 | 44 | 1,00589888 | 79 | 1,01061602 |
| 10 | 1,00133761 | 45 | 1,00603335 | 80 | 1,01075112 |
| 11 | 1,00147147 | 46 | 1,00616784 | 81 | 1,01088623 |
| 12 | 1,00160535 | 47 | 1,00630234 | 82 | 1,01102137 |
| 13 | 1,00173924 | 48 | 1,00643687 | 83 | 1,01115652 |
| 14 | 1,00187315 | 49 | 1,00657141 | 84 | 1,01129169 |
| 15 | 1,00200708 | 50 | 1,00670597 | 85 | 1,01142688 |
| 16 | 1,00214103 | 51 | 1,00684055 | 86 | 1,01156209 |
| 17 | 1,00227500 | 52 | 1,00697514 | 87 | 1,01169732 |
| 18 | 1,00240899 | 53 | 1,00710975 | 88 | 1,01183256 |
| 19 | 1,00254299 | 54 | 1,00724438 | 89 | 1,01196783 |
| 20 | 1,00267701 | 55 | 1,00737903 | 90 | 1,01210311 |
| 21 | 1,00281105 | 56 | 1,00751370 | 91 | 1,01223841 |
| 22 | 1,00294510 | 57 | 1,00764839 | 92 | 1,01237372 |
| 23 | 1,00307918 | 58 | 1,00778309 | 93 | 1,01250906 |
| 24 | 1,00321327 | 59 | 1,00791781 | 94 | 1,01264441 |
| 25 | 1,00334738 | 60 | 1,00805255 | 95 | 1,01277978 |
| 26 | 1,00348151 | 61 | 1,00818731 | 96 | 1,01291517 |
| 27 | 1,00361565 | 62 | 1,00832208 | 97 | 1,01305058 |
| 28 | 1,00374982 | 63 | 1,00845687 | 98 | 1,01318600 |
| 29 | 1,00388400 | 64 | 1,00859168 | 99 | 1,01332145 |
| 30 | 1,00401820 | 65 | 1,00872651 | 100 | 1,01345691 |
| 31 | 1,00415242 | 66 | 1,00886136 | 101 | 1,01359239 |
| 32 | 1,00428665 | 67 | 1,00899623 | 102 | 1,01372788 |
| 33 | 1,00442091 | 68 | 1,00913111 | 103 | 1,01386340 |
| 34 | 1,00455518 | 69 | 1,00926601 | 104 | 1,01399893 |
| 35 | 1,00468947 | 70 | 1,00940093 | 105 | 1,01413448 |

| Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 106 | 1,01427005 | 146 | 1,01970775 | 186 | 1,02517459 |
| 107 | 1,01440564 | 147 | 1,01984406 | 187 | 1,02531164 |
| 108 | 1,01454125 | 148 | 1,01998039 | 188 | 1,02544870 |
| 109 | 1,01467687 | 149 | 1,02011675 | 189 | 1,02558578 |
| 110 | 1,01481252 | 150 | 1,02025312 | 190 | 1,02572288 |
| 111 | 1,01494818 | 151 | 1,02038950 | 191 | 1,02586000 |
| 112 | 1,01508386 | 152 | 1,02052591 | 192 | 1,02599714 |
| 113 | 1,01521955 | 153 | 1,02066234 | 193 | 1,02613430 |
| 114 | 1,01535527 | 154 | 1,02079878 | 194 | 1,02627147 |
| 115 | 1,01549100 | 155 | 1,02093524 | 195 | 1,02640866 |
| 116 | 1,01562675 | 156 | 1,02107172 | 196 | 1,02654588 |
| 117 | 1,01576252 | 157 | 1,02120822 | 197 | 1,02668310 |
| 118 | 1,01589831 | 158 | 1,02134473 | 198 | 1,02682015 |
| 119 | 1,01603412 | 159 | 1,02148127 | 199 | 1,02695762 |
| 120 | 1,01616994 | 160 | 1,02161782 | 200 | 1,02709490 |
| 121 | 1,01630578 | 161 | 1,02175439 | 201 | 1,02723221 |
| 122 | 1,01644164 | 162 | 1,02189098 | 202 | 1,02736953 |
| 123 | 1,01657752 | 163 | 1,02202758 | 203 | 1,02750686 |
| 124 | 1,01671349 | 164 | 1,02216421 | 204 | 1,02764422 |
| 125 | 1,01684933 | 165 | 1,02230085 | 205 | 1,02778160 |
| 126 | 1,01698527 | 166 | 1,02243751 | 206 | 1,02791899 |
| 127 | 1,01712122 | 167 | 1,02257419 | 207 | 1,02805640 |
| 128 | 1,01725719 | 168 | 1,02271089 | 208 | 1,02819384 |
| 129 | 1,01739317 | 169 | 1,02284761 | 209 | 1,02833129 |
| 130 | 1,01752918 | 170 | 1,02298434 | 210 | 1,02846875 |
| 131 | 1,01766521 | 171 | 1,02312109 | 211 | 1,02860624 |
| 132 | 1,01780125 | 172 | 1,02325787 | 212 | 1,02874375 |
| 133 | 1,01793731 | 173 | 1,02339466 | 213 | 1,02888127 |
| 134 | 1,01807338 | 174 | 1,02353147 | 214 | 1,02901881 |
| 135 | 1,01820948 | 175 | 1,02366829 | 215 | 1,02915637 |
| 136 | 1,01834559 | 176 | 1,02380514 | 216 | 1,02929395 |
| 137 | 1,01848173 | 177 | 1,02394200 | 217 | 1,02943154 |
| 138 | 1,01861788 | 178 | 1,02407888 | 218 | 1,02956916 |
| 139 | 1,01875405 | 179 | 1,02421578 | 219 | 1,02970679 |
| 140 | 1,01889024 | 180 | 1,02435270 | 220 | 1,02984445 |
| 141 | 1,01902644 | 181 | 1,02448964 | 221 | 1,02998212 |
| 142 | 1,01916267 | 182 | 1,02462659 | 222 | 1,03011980 |
| 143 | 1,01929891 | 183 | 1,02476356 | 223 | 1,03025751 |
| 144 | 1,01943517 | 184 | 1,02490055 | 224 | 1,03039524 |
| 145 | 1,01957145 | 185 | 1,02503756 | 225 | 1,03053298 |

| Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 226 | 1,03067074 | 266 | 1,03619636 | 306 | 1,04175160 |
| 227 | 1,03080852 | 267 | 1,03633488 | 307 | 1,04189086 |
| 228 | 1,03094632 | 268 | 1,03647342 | 308 | 1,04203015 |
| 229 | 1,03108414 | 269 | 1,03661197 | 309 | 1,04216944 |
| 230 | 1,03122197 | 270 | 1,03675055 | 310 | 1,04230876 |
| 231 | 1,03135983 | 271 | 1,03688914 | 311 | 1,04244810 |
| 232 | 1,03149770 | 272 | 1,03702775 | 312 | 1,04258245 |
| 233 | 1,03163559 | 273 | 1,03716638 | 313 | 1,04272683 |
| 234 | 1,03177350 | 274 | 1,03730503 | 314 | 1,04286622 |
| 235 | 1,03191143 | 275 | 1,03744370 | 315 | 1,04300563 |
| 236 | 1,03204938 | 276 | 1,03758239 | 316 | 1,04314506 |
| 237 | 1,03218734 | 277 | 1,03772109 | 317 | 1,04328451 |
| 238 | 1,03232533 | 278 | 1,03785982 | 318 | 1,04342397 |
| 239 | 1,03246333 | 279 | 1,03799856 | 319 | 1,04356346 |
| 240 | 1,03260135 | 280 | 1,03813732 | 320 | 1,04370297 |
| 241 | 1,03273939 | 281 | 1,03827609 | 321 | 1,04384249 |
| 242 | 1,03287744 | 282 | 1,03841489 | 322 | 1,04398203 |
| 243 | 1,03301552 | 283 | 1,03855371 | 323 | 1,04412159 |
| 244 | 1,03315361 | 284 | 1,03869254 | 324 | 1,04426117 |
| 245 | 1,03329173 | 285 | 1,03883139 | 325 | 1,04440077 |
| 246 | 1,03342986 | 286 | 1,03897027 | 326 | 1,04454038 |
| 247 | 1,03356801 | 287 | 1,03910916 | 327 | 1,04468002 |
| 248 | 1,03370617 | 288 | 1,03924817 | 328 | 1,04481967 |
| 249 | 1,03384436 | 289 | 1,03938699 | 329 | 1,04495934 |
| 250 | 1,03398157 | 290 | 1,03952594 | 330 | 1,04509903 |
| 251 | 1,03412079 | 291 | 1,03966491 | 331 | 1,04523874 |
| 252 | 1,03425903 | 292 | 1,03980389 | 332 | 1,04537847 |
| 253 | 1,03439729 | 293 | 1,03994289 | 333 | 1,04551822 |
| 254 | 1,03453557 | 294 | 1,04008191 | 334 | 1,04565798 |
| 255 | 1,03467387 | 295 | 1,04022095 | 335 | 1,04579777 |
| 256 | 1,03481218 | 296 | 1,04036001 | 336 | 1,04593757 |
| 257 | 1,03495052 | 297 | 1,04049908 | 337 | 1,04607739 |
| 258 | 1,03508887 | 298 | 1,04063818 | 338 | 1,04621723 |
| 259 | 1,03522724 | 299 | 1,04077729 | 339 | 1,04635709 |
| 260 | 1,03536563 | 300 | 1,04091642 | 340 | 1,04649697 |
| 261 | 1,03550404 | 301 | 1,04105557 | 341 | 1,04663686 |
| 262 | 1,03564247 | 302 | 1,04119474 | 342 | 1,04677678 |
| 263 | 1,03578091 | 303 | 1,04133393 | 343 | 1,04691671 |
| 264 | 1,03591938 | 304 | 1,04147314 | 344 | 1,04705667 |
| 265 | 1,03605786 | 305 | 1,04161236 | 345 | 1,04719664 |

Days

| Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. | Days = t. | The Amounts of 1 *l.* &c. |
|---|---|---|---|---|---|
| 346 | 1,04733663 | 353 | 1,04831708 | 360 | 1,04929845 |
| 347 | 1,04747664 | 354 | 1,04845722 | 361 | 1,04943872 |
| 348 | 1,04761666 | 355 | 1,04859738 | 362 | 1,04957901 |
| 349 | 1,04775671 | 356 | 1,04873756 | 363 | 1,04971932 |
| 350 | 1,04789677 | 357 | 1,04887775 | 364 | 1,04985965 |
| 351 | 1,04803686 | 358 | 1,04901797 | 365 | 1,04999999 |
| 352 | 1,04817696 | 359 | 1,04915820 | 366 | 1,05 |

I think it is needlefs to fay any Thing of the Ufe of thefe *Tables*, becaufe I take it for granted, that whoever underftands the Work of the foregoing *Examples*, at 6 *per Cent.* cannot but know how to make Ufe of thefe *Tables* at 5 *per Cent.* as Occafion requires.

Thus far concerning *Annuities*, or *Leafes*, &c. that are *limited* by any affigned *Time*; and 'tis only fuch that can be computed by *Theorems* or certain *Rules*. However, it may not perhaps be *unacceptable*, to infert a brief Account of fome *Eftimates* that have been reafonably made, by *two* very ingenious *Perfons*, about the Proportion or Difference of *Mens Lives*, according to their feveral *Ages*; which may be of good Ufe in computing the *Values* of *Annuities*, or taking of *Leafes* for *Lives*, &c.

Sir *William Petty*, in his Difcourfe made before the *Royal Society* (*Anno* 1674) concerning the Ufe of DUPLICATE PROPORTION, in the Life of *Man* and its Duration, faith, that it's found by *Experience* there are more Perfons living of between 16 and 26 *Years Old*, than of any other *Age* or *Decade* of Years in the whole Life of *Man* (*viz.* 70 or 80 *Years.*) *His Reafon for that Affertion I fhall omit*; but fuppofing it true, he thence infers, that the *Roots* of every *Number* of *Mens Ages* under 16 (whofe *Root* is 4) compared with the faid Number 4, doth fhew the *Proportion* of the Likelihood of fuch Mens reaching the *Age* of 70 Years.

As for *Example*, 'tis 4 *Times* more likely, that one of 16 *Years Old* fhould live to 70, than a New-born *Babe*: 'Tis 3 *Times* more likely, that one of 9 *Years Old* fhould attain the *Age* of 70, than the faid *Infant*, &c.

On the other Hand, 'tis 5 to 4, that one of 25 *Years Old* will die before one of 16: And 6 to 5, that one of 36 will die before one of 25. And fo on according to the *Roots* of any other declining *Age*, compared with the (4,6) the *Root* of 21, which is the *Year* of *Perfection* according to the Senfe of our *Law*, and the *Age* for whofe Life a *Leafe* is moft *valuable*.

2. *The ingenious* and great *Mathematician*, Doctor *Edmund Halley* (*in Philosoph. Transact. Numb.* 196) doth, with great *Industry* and *Skill*, draw an *Estimate* of the *Proportion* of *Mens* Lives, from the *Monthly Tables* of the *Births* and *Funerals* in *Breslaw*, the *Capital City* of the *Province* of *Silesia*; or, as the *Germans* call it, *Schlesia*. Whence he proves that it's 80 to 1, a Person of 25 *Years Old* will not die in a *Year*: That it is 5½ to 1, that a Man of 40 will live 7 *Years*: That a Man of 30 *Years Old* may reasonably expect to live 27 or 28 *Years*, &c.

Now from these and the like *Proportions* (he justly infers) that the *Price* of *Insurance* upon *Lives* ought to be regulated, there being a great Difference between the Life of a Man of 20, and one of 50. For *Example*: 'Tis 100 to 1, that a Man of 20 dies not in a *Year*, and but 38 to 1, for a Man of 50 *Years* of *Age*. And upon these also depends the *Valuation* of *Annuities* for Lives; for it is plain, that the *Purchaser* ought to pay only such a Part of the *Value* of any *Annuity*, as he hath *Chances* that he is living.

And for that Purpose he hath taken the Pains (*which was not a little*) to compute the following *Table* (that shews the *Value* of *Annuities*) for every Fifth Year of *Age* to the 70th.

| *Age* | *Year's Purchase.* | *Age* | *Year's Purchase.* | *Age* | *Year's Purchase.* |
|---|---|---|---|---|---|
| 1 | 10,28 | 25 | 12,27 | 50 | 9,21 |
| 5 | 13,40 | 30 | 11,72 | 55 | 8,51 |
| 10 | 13,44 | 35 | 11,12 | 60 | 7,60 |
| 15 | 13,33 | 40 | 10,57 | 65 | 6,54 |
| 20 | 12,78 | 45 | 9,91 | 70 | 5,32 |

The same *ingenious Gentleman* proceeds on, and shews how to *estimate* or find the *Value* of *Two Lives*, and then of *Three Lives*, which being too long a Discourse to be recited here, I have, for Brevity's Sake, omitted it; and shall only add this serious Observation,

*Viz.* How unjustly we repine at the Shortness of our *Lives*, and think ourselves wrong'd if we attain not to *Old Age*; whereas it appears, that the *One Half* of those, that are BORN, *die* in Seventeen Years Time. For by the aforesaid Bills of Mortality at *Breslaw*, it was found, that 1238 were in that Time reduced to 616. So that, instead of murmuring at what we call a *Short Life*, we ought to account it as a great Blessing that we have *surviv'd*, perhaps by many Years, that *Period* of *Life* whereat the one Half of the whole Race of Mankind does not arrive.

## *Sect.* 4. *Of Purchasing* Free-hold, *or* Real Estates; *at Compound Interest.*

All *Free-hold* or *Real Estates*, are supposed to be purchased or bought to continue for *ever* (viz. *without any limited Time)*; therefore the Business of computing the true *Value* of such *Estates* is grounded upon a *Rank* or *Series* of *Geometrical Proportions* continually *decreasing*, *ad Infinitum.*

Thus, let *P*, *u*, *R*, denote the same *Data* as in the last *Section.* Then the *Series* will be, $\frac{u}{R}$, $\frac{u}{RR}$, $\frac{u}{R^3}$, $\frac{u}{R^4}$, $\frac{u}{R^5}$, and so on in $\div\!\!\div$ until the last *Term* $= 0$. Then will $P - 0$ (viz. $P$) be the sum of all the *Antecedents*. And $P - \frac{u}{R}$ will be the *Sum* of all the *Consequents*; therefore it will be $u : \frac{u}{R} :: P : P - \frac{u}{R}$ which produces $PR - u = P$.

This *Equation* affords the following *Theorems.*

*Theorem* 1. $PR - P = u$. *Theorem* 2. $\left\{ \frac{u}{R-1} = P. \right.$

*Theorem* 3. $\left\{ \frac{P+u}{P} = R. \right.$

Example. *Suppose a Free-hold Estate of* 75 *l. Yearly Rent were to be sold; what is it worth, allowing the Buyer* 6 per Cent. *&c. Compound Interest for his Money?*

In this *Question* there is given $u = 75$. $R = 1{,}06$ to find $P$. Per *Theorem* 2. Thus $R - 1 = 0{,}006)\ 75 = u\ (1250\ l. = P$. the *Answer* required. And so on for any of the rest, as Occasion requires. But if the *Rent* is to be paid, either by *Quarterly* or *Half Yearly Payments*;

Then $R = \surd\ 1{,}06$ for *Half Yearly* } *Payments at* 6 *per Cent.*
And $R = \surd : \surd\ 1{,}06$ for *Quarterly* }

Or { $R = 1{,}08$ for *Yearly*
{ $R = \surd\ 1{,}08$ for *Half Yearly* } *Payments at* 8 *per Cent.*
{ $R = \surd : \surd\ 1{,}08$ for *Quarterly*

The like is to be understood for any other proposed *Rate* of *Interest*, either *greater* or *less* than 6 *per Cent.*

The Application of these *Theorems* to Practice is so very easy, that it's needless to insert more *Examples.*

AN

# AN INTRODUCTION TO THE Mathematicks.

## PART III.

## CHAP. I.

### Of Geometrical Definitions, &c.

#### *Sect.* I. Of *Lines* and *Angles*.

A Point *hath no Parts:* That is, a *Geometrical Point* is not any *Quantity*, but only an *assignable Place* in any *Quantity*, denoted by a *Point:* As at *A.* and *B.* } *A.* *B.*

*Such a Place may be conceived so infinitely small, as to be void of Length, Breadth, and Thickness; and therefore a* Point may be said to have no *Parts.*

2. A Line is called a *Quantity* of one *Dimension*, because it may have any supposed *Length*, but no *Breadth* nor *Thickness*, being made or represented to the *Eye*, by the *Motion* of a *Point.*

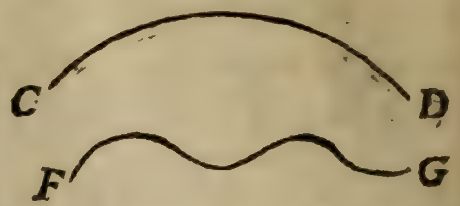
That is, if the *Point* at *A*, be *moved (upon the same Plane)* to the *Point* at *B*, it will describe a *Line* either *right* or *circular* (viz. *crooked)* according to its *Motion.*

Therefore the *Ends* or *Limits* of a *Line* are *Points.*

3. A Right Line, is that *Line* which lieth *even* or *straight* betwixt those *Points* that *limit* its *Length*, being the *shortest Line* that can be drawn between any *Two Points.* As the *Line A B.* } *A*———*B.*

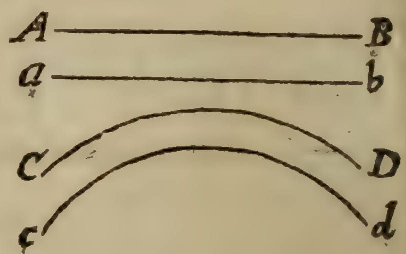
*Therefore, between any two Points, there can lie or be drawn but one right Line.*

4. A CIRCULAR, *crooked* or OBLIQUE *Line*, is that which *lies* bending between those *Points* which *limit* its *Length*, as the *Lines* *C D* or *FG*, &c.

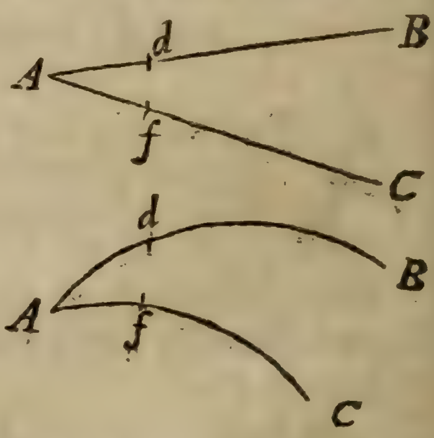



*Of these Kinds of Lines there are various Sorts; but those of the Circle, Parabola, Ellipsis, and Hyperbola are of most general Use in Geometry; of which a particular Account shall be given further on.*

5. PARALLEL LINES, are those that lie *equally distant* from one another in all their *Parts*, viz. such *Lines* as being *infinitely extended (upon the same Plane)* will never *meet:* As the *Lines* *A B* and *a b*: or *C D* and *c d*.




6. LINES not PARALLEL, but INCLINING (*viz. leaning*) one towards another, whether they are *Right Lines*, or *Circular Lines*, will (if they are *extended) meet and* make an *Angle*; the *Point* where they *meet* is called the *Angular Point*, as at *A*. And according as such *Lines* stand, *nearer* or *further* off each other, the *Angle* is said to be *lesser* or *greater*, whether the *Lines* that *include* the *Angle* be *long* or *short*. That is, the *Lines* *A d* and *A f* *include* the same *Angle* as *A B* and *A C* doth; notwithstanding that *A B* is *longer* than *A d*, &c.



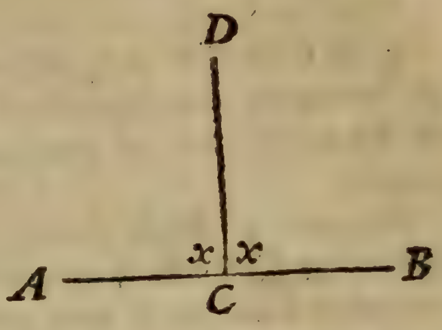

7. All ANGLES *including* between *Right Lines* are called *Right-lin'd Angles*; and those *included* between *Circular Lines* are called *Spherical Angles*. But all *Angles*, whether *Right-lin'd* or *Spherical*, fall under one of these *Three Denominations*.

*Viz.* { A **Right Angle.**<br>An **Obtuse Angle.**<br>An **Acute Angle.** }

8. A RIGHT-ANGLE is that which is *included* betwixt *Two Lines*, that *meet* one another *Perpendicularly*.

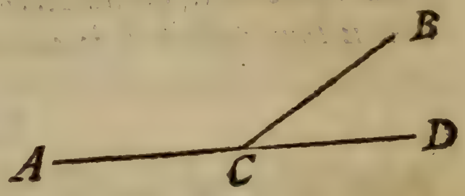
That

That is, when a *Right Line*, as *D C*, *meets* with another *Right-Line*, as *A B*, so directly as that it neither *inclines* nor *declines* to one *Side* more than the other, but make the *Angles* on both *Sides* of it *equal*, as at *x*, *x*; then are those *Angles* called *Right-Angles*; and the *Lines* so *meeting* are said to be *Perpendicular* to each other.




That is, *A C*, and *C B*, are *Perpendicular* to *D C*, as well as *D*, *C* is to *either* or both of them.

9. An OBTUSE ANGLE is that which is *greater* than a *Right Angle*. Such is the *Angle included* between the *Lines A C* and *C B*.




10. An ACUTE ANGLE is that which is *less* than a *Right Angle*: As the *Angle included* between the *Lines C B* and *C D*.

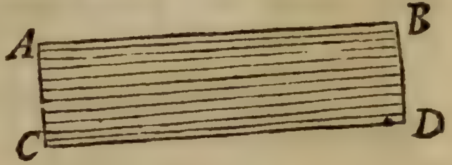
These *Two Angles* are generally called OBLIQUE *Angles*.

## *Sect.* 2. Of a Circle, &c.

*Before a Circle and its Parts are defined, it will be convenient to give a brief Account of Superficies in general.*

1. A SUPERFICIES or SURFACE is the Upper, or very *Out-side* of any *visible* Thing. But by *Superficies* in GEOMETRY, is meant only so much of the *Out-side* of any Thing as is *inclosed* within a *Line* or *Lines*, according to the Form or *Figure* of the Thing *designed*; and it is *produced* or *formed* by the *Motion* of a *Line*, as a *Line* is described by the *Motion* of a *Point*; thus:

Suppose the *Line A B* were *equally moved (upon the same Plane)* to *C D*; then will the *Points* at *A* and *B describe the two Lines A C* and *B D*; and by so doing they will form (and *inclose*) the SUPERFICIES or *Figure A B C D*, being a *Quantity of Two Dimensions*, viz. it hath *Length* and *Breadth*, but not *Thickness*. *Consequently the Bounds or Limits of a Superficies are Lines.*

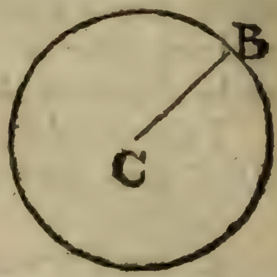



Note,

Note, *The Superficies of any Figure, is usually called its* AREA.

2. A CIRCLE is a *plain regular Figure*, whose *Area is bounded* or *limited* by one continued *Line*, called the CIRCUMFERENCE or PERIPHERY of the *Circle*, which may be thus *described* or *drawn.*

Suppose a *Right Line*, as *C B*, to have one of its *Extream Points*, as *C*, so fix'd upon any *Plane*, as that the other *Point* at *B* may *move* about it; then if the *Point* at *B* be *moved round* about *(upon the same Plane)* it will *describe a Line equally distant* in all its *Parts* from the *Point C*, which will be the *Circumference* or *Periphery* of that *Circle*; the *Point C* will be its CENTER, and the contained *Space* will be its *Area*, *and the Right Line C B, by which the Circle is thus described, is called* RADIUS.



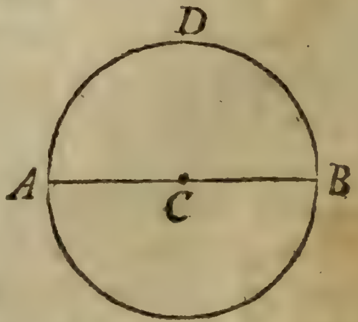

Confectary.

*From hence 'tis evident, that an infinite Number of Right Lines may be drawn from the Center of any Circle to touch its Periphery, which will be all equal to one another, because they are all Radius's. And with a little Consideration it will be easy to conceive, that no more than two equal Right Lines can be drawn from any Point within a Circle to touch its Periphery, but from the Center only.* (9. *e.* 3.)

3. EQUAL CIRCLES are those which have *equal Radius's*; for it's plain by the last *Definition*, that one and the same *Radius* (as *C B*) must needs *describe equal Circles*, how many soever they are.

4. The *Diameter* of a *Circle*, is *twice* its *Radius joined* into one *Right Line*; as *A B* drawn through the *Center C*, and *ending* at the *Periphery* on each *Side*.

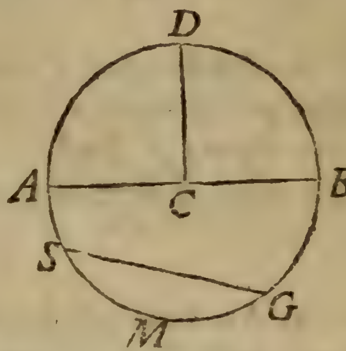
That is the *Diameter divides* the *Circle* into *Two* equal Parts.




5. A *Semicircle* (viz. *Half a Circle*) is a *Figure included* between the *Diameter*, and *Half* the *Periphery cut off* by the *Diameter*; as *A D B*.

6. A

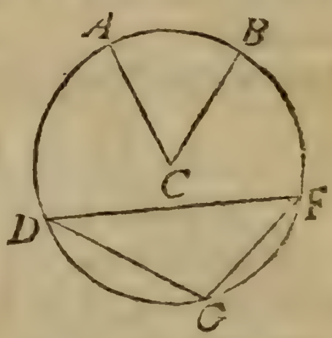
6. A QUADRANT is *Half* a *Semicircle*, viz. one *Quarter* of a *Circle*; and 'tis made by the *Radius* (as *D C*) *standing Perpendicular* upon the *Diameter* at the *Center C*, *cutting* the *Periphery* of the *Semicircle* in the *Middle*, as at *D*. *Therefore a Quadrant, or half the Semicircle, is the Measure of a Right Angle.*




7. A CHORD LINE, or the *Subtense* of an *Arch*, is any *Right Line that cuts* the *Circle* into *Two unequal Parts, as the Line S G*; and is always *less* than the *Diameter*.

8. A SEGMENT of a *Circle*, is a *Figure included* betwixt the *Chord* and that *Arch* of the *Periphery* which is *cut off* by the *Chord*: And it may either be *greater* or *less* than a *Semicircle*; as the *Figure S D G*, or *S M G*.

9. A SECTOR is a *Figure included* between *Two Radius*'s of the *Circle*, and that *Arch* of its *Periphery* where they *touch*, as the *Figure A C B*: And the *Arch A B* is the *Measure* of the *Angle* at *C*, *included* betwixt the *Radius*'s *A C* and *B C*.




Note, *All Angles of Sectors are called Angles at the Center of a Circle.*

10. An ANGLE in the *Segment* of a *Circle* is that which is *included* between *Two Chords* that *flow* from *one* and the *same Point* in the *Periphery*, as at *D*, and *meet* with the *Ends* of another *Chord Line*, as at *F* and *G*.

That is, the *Angles* at *D*, at *F* and at *G*, are called *Angles* at the *Periphery*, or *Angles standing* on the *Segment* of a *Circle*.

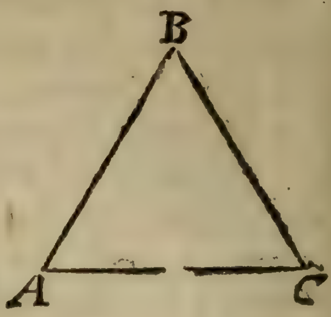
## *Sect.* 3. *Of* TRIANGLES.

*There are two Kinds of Triangles*, viz. *Plain and Spherical*; *but I shall not give any Definition of the Spherical, because they more immediately relate to Astronomy.*

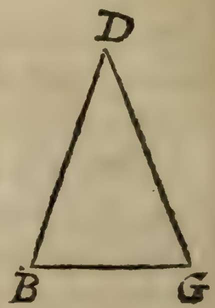
1. A PLAIN TRIANGLE is a *Figure* whose *Area* is *contained* within the *Limits* of *Three Right Lines* called *Sides*, *including Three Angles*: And it may be *divided*, and takes its *Name*, either according to its *Sides* or *Angles*.

1. By

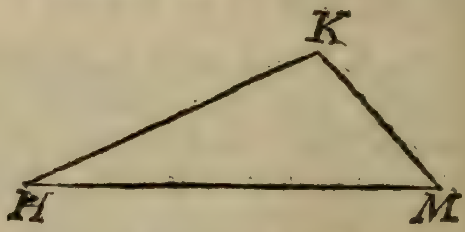
1. By its SIDES.




2. An EQUILATERAL TRIANGLE is that which hath all its *Three Sides* equal; as the *Figure A B C.*
That is, $AB = BC = AC$.

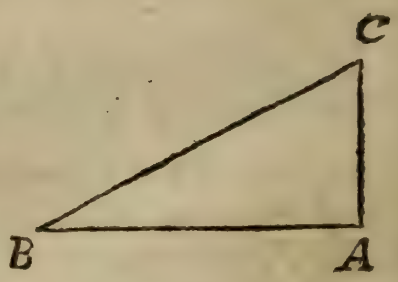



3. An ISOSCELES TRIANGLE, is that which hath *only Two* of its *Sides equal*, as the *Figure B D G:* That is, $BD = DG$; but the *Third Side B G* may be either *greater* or *lefs*, as Occafion requires.




4. A SCALENE TRIANGLE, is that which hath all its *Three Sides unequal*;
fuch as the *Figure H K M.*

2. By its ANGLES.




5. A RIGHT-ANGLED *Triangle*, is that which hath one *Right Angle*; that is, when *Two* of its *Sides* are *Perpendicular* to each other, as *C A* is fuppofed to be to *B A.* Therefore the *Angle* at *A*, is a *Right Angle*, per *Defin.* 8. *Sect.* 1.

Note, *The longeft Side of every Right-angled Triangle (as B C) is called the Hypothenufe, and the longeft of the other Two Sides which include the Right Angle (as B A) is called the Bafe: The Third Side (as C A) is called the Cathetus or Perpendicular.*

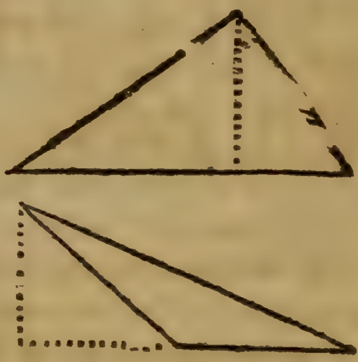
6. An OBTUSE-ANGLED *Triangle*, is that which hath *one* of its *Angles Obtufe*, and it's called an *Amblygonium Triangle.* Such is the *Third Triangle H K M.*

7. An ACUTE-ANGLED TRIANGLE, is that which hath all its *Angles Acute*, and it's called an *Oxygonium Triangle*; fuch are the *Firft* and *Second Triangles A B C* and *B D G.*

Note, *All Triangles that have not a Right Angle, whether they are Acute, or Obtufe, are, in general Terms, called Oblique Triangles,*

*gles, without any other Distinction, as before. And the longest Side of every oblique Triangle is usually called the* Base; *the other two are only called* Sides *or* Legs.

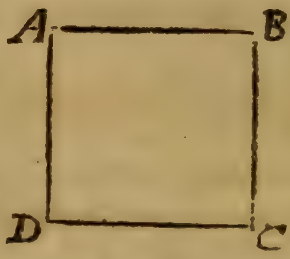
8. The ALTITUDE or HEIGHT of any *Plain Triangle*, is the *Length* of a *Right Line* let fall *perpendicular* from any of its *Angles*, upon the *Side* opposite to that *Angle* from whence it falls; and may be either within, or without the *Triangle*, as *Occasion requires*, being denoted by the *Two prick'd Lines*, in the *annexed Triangles.*



## *Sect.* 4. Of Four-sided Figures.

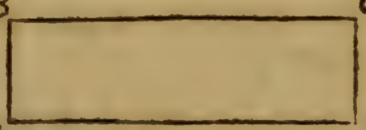
1. A SQUARE is a plain *regular Figure*, whose *Area* is limited by *Four* equal *Sides* all *perpendicular* one to another.

That is, when $AB = BC = CD = DA$, and the *Angles* $A$, $B$, $C$, $D$ are all *equal*, then it's usually called a *Geometrical Square.*

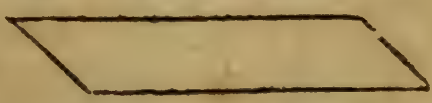



2. A RHOMBUS, or *Diamond-like Figure*, is that which hath *Four* equal *Sides*, but no *Right-angle.* That is, a *Rhombus* is a *Square* mov'd out of its *right Position*, as the *annexed Figure.*

3. A RECTANGLE, or a *Right-angled Parallelogram* (often called an *Oblong*, or *long Square)* is a *Figure* that hath *four Right-angles* and its *two opposite Sides* equal, *viz.* $BC = HD =$ and $BH = CD$.



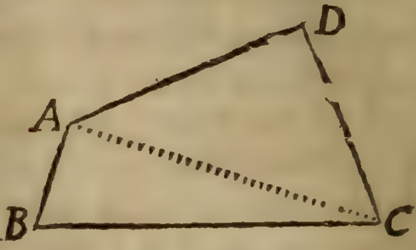

4. A RHOMBOIDES, is an *Oblique-angled Parallelogram*; that is, it is a *Parallelogram moved* out of its *right Position*, like the *annexed Figure.*



5. The ALTITUDE or *Height* of any *Oblique-angled Parallelogram*, *viz.* either of the *Rhombus* or *Rhomboides*, is a *Right-line* let fall *perpendicular* from any *Angle* upon the *Side opposite* to that *Angle*; and may either be within or without the *Figure:* As the *prick'd Lines* in the *annexed Figure.*

6. Every *Four-ſided Figure*, different from thoſe before-mentioned, is called a TRAPEZIUM.

That is, when it has neither *oppoſite Sides*, nor *oppoſite Angles equal*; as the *Figure A B C D*.




7. A *Right-line*, drawn from any *Angle* in a *Four-ſided Figure* to its *oppoſite Angle*, is called a DIAGONAL *Line*, and will *divide* the *Area* of the *Figure* into *two Triangles*, being *denoted* by the *prick'd Line A C* in the laſt *Figure*.

8. All *Right-lin'd Figures*, that have more than *four Sides*, are call'd *Polygons*, whether they be *regular* or *irregular*.

9. A REGULAR POLYGON is that which hath all its *Sides equal*, ſtanding at *equal Angles*, and is named according to the *Number* of its *Sides* (or *Angles)*. That is, if it have *five equal Sides*, it is called a PENTAGON; if *ſix equal Sides*, it is call'd a HEXAGON; if *ſeven*, 'tis a HEPTAGON; if *eight*, 'tis an OCTAGON, &c.

Note, *All* Regular Polygons *may be inſcrib'd in a Circle*; *that is, their Angular Points, how many ſoever they have, will all juſt touch the Circle's Periphery.*

10. An IRREGULAR POLYGON is that *Figure* which hath many *unequal Sides* ſtanding at *unequal Angles* (like unto the *annexed Figure*, or otherwiſe); and of ſuch Kind of *Polygons* there are *infinite Varieties*, but they may all be *reduced* to *regular Figures* by drawing *Diagonal Lines* in them; as ſhall be ſhew'd farther on.

Theſe are the moſt *general* and *uſeful Definitions* that concern *plain* or *ſuperficial Geometry*.

As for thoſe which relate to *Solids*, I thought it convenient to omit giving any Account of them in this Place, becauſe they would rather *puzzle* and *amuſe* the *Learner*, than *improve* him, until he has gain'd a *competent Knowledge* in the moſt uſeful *Theorems* concerning *Superficies*; for then thoſe *Definitions* may be more eaſily underſtood, and will help him to form a *clearer Idea* of their *reſpective Solids*, than 'tis *poſſible* to conceive of them before; and therefore I have reſerv'd thoſe *Definitions* until we come to the Fifth Part.

2 Sect.

## *Sect.* 5. *Of such* Terms *as are generally used in Geometry.*

Whatsoever is proposed in *Geometry* will either be a PROBLEM or a THEOREM.

Both which *Euclid includes* in the *general Term* of *Proposition.*

A PROBLEM is that which *proposes* something to be *done*, and relates more immediately to *practical* than *speculative Geometry*; That is, it's generally of such a *Nature*, as to be performed by some known or *Commonly-receiv'd* Rules, without any Regard had to their *Inventions* or *Demonstrations.*

A THEOREM is when any *Commonly-receiv'd* Rule, or any *New Proposition* is required to be *demonstrated*, that so it may from thence forward become a *certain* Rule, to be *rely'd* upon in Practice when Occasion requires it. And therefore several Rules are often call'd *Theorems*, by which *Operations* in *Arithmetick*, and *Conclusions* in *Geometry*, are *perform'd.*

Note, *By* DEMONSTRATION *is understood the highest Degree of* Proof *that human Reason is capable of attaining to, by a Train of Arguments deduced or drawn from such plain Axioms, and other Self-evident Truths, as cannot be denied by any one that considers them.*

A COROLLARY, *or* CONSECTARY, is some *Consequent Truth* drawn or gain'd from any *Demonstration.*

A LEMMA is the *Demonstration* of some *Premises* laid down or proposed as *preparative* to obviate and shorten the Proof of the *Theorem* under Consideration.

A SCHOLIUM is a brief *Commentary* or *Observation* made upon some *precedent Discourse.*

N. B. *I advise the young Geometer to be very perfect in the* Definitions, *viz. Not to rest satisfied with a bare Remembrance of them; but, that he endeavour to gain a clear* Idea *or* Understanding *of the Things defined; and for that Reason I have been fuller in every* Definition *than is usual.*

*And, that he may know from whence most of the following* Problems *and* Theorems *contain'd in the Two next Chapters are collected, I have all along cited the* Proposition *and* Book *of* Euclid's Elements *where they may be found.*

*As for Instance; at* Problem I. *there is* (3 *e.* I.) *which shews that it is the* Third Proposition *in* Euclid's First Book. *The like must be understood in the* Theorems.

## CHAP. II.

### The First Rudiments, or *Leading* and *Preparatory* Problems, in Plain Geometry.

*IN order to perform the following* Problems, *the young Geometer ought to be provided with a thin ſtreight* Ruler, *made either of* Braſs *or* Box-wood, *and two Pair of very good* Compaſſes, *viz. one Pair call'd* Three-pointed Compaſſes, being very uſeful for *drawing of* Figures *or* Schemes, *either with Black Lead or Ink*; *and one Pair of* plain Compaſſes *with very fine Points, to meaſure and ſet off Diſtances*; *alſo he ſhould have a very good Steel* Drawing Pen: *And then he may proceed to the Work with this Caution*; *that he ought to make himſelf Maſter of one* Problem *before he undertakes the next: That is, he ought to underſtand the* Deſign, *and, as far as he can, the Reaſon of every* Problem, *as well as how to do it*; *and then a little Practice will render them very eaſy, they being all grounded upon theſe following* Poſtulates.

### Poſtulates or Petitions.

1. That a *Right-line* may be *drawn* from any one given *Point* to another.

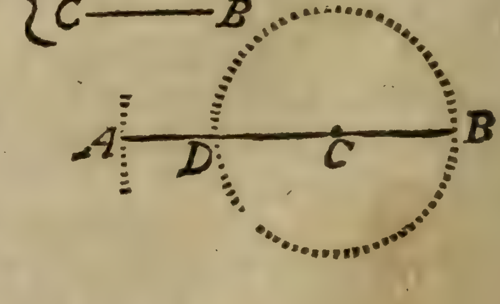
2. That a *Right-line* may be *produced*, *increaſed*, or made *longer* from either of its Ends.

3. That upon any given *Point* (or *Center*) and with any given *Diſtance* (*viz.* with any RADIUS) a *Circle* may be deſcribed.

### PROBLEM I.

*Two Right-lines being given, to find their Sum and Difference.* (3. *e.* 1.)

Let the given *Lines* be



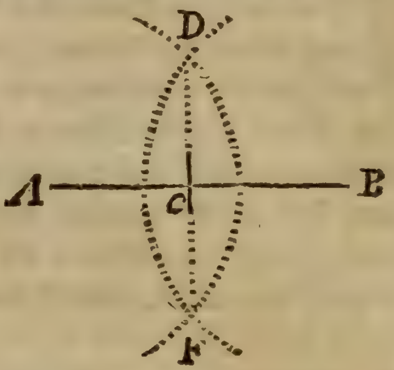

Make the *ſhorteſt Line*, as *C B*, *Radius*, and with it *deſcribe a Circle*: From its Center *C* ſet off the other *Line A C*, and *join A C B* with a *Right-line*. Then will $AB = AC + CB$; and $AD = AC - CB$; as was required.

PRO-

## PROBLEM II.

*To bisect, or divide a Right-line given (as A B) into two equal Parts* (10. *e.* 1.)

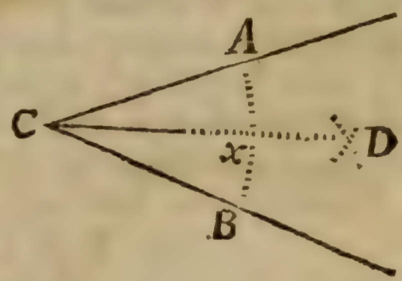
From both *Ends* of the given *Line* (*viz. A* and *B*) with any *Radius greater* than *half* its *Length*, *describe Two Arches* that may *cross* each other in *two Points*, as at *D* and *F*; then *join* those *Points D F* with a *Right-line*, and it will *bisect* the *Line A B* in the Middle at *C*; *viz.* it will make *A C* = *C B*; as was required.




## PROBLEM III.

*To bisect a Right-lin'd Angle given, into two equal Angles.* (9. *e.* 1.)

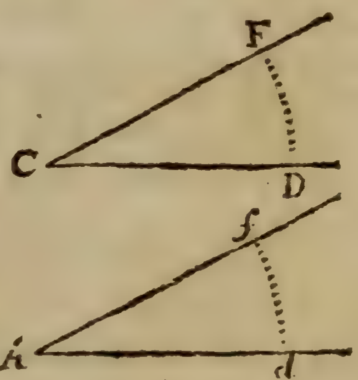
Upon the *Angular Point*, as at *C*, with any *convenient Radius*, *describe* an *Arch* as *A B*; and from those *Points A* and *B*, *describe two* equal *Arches crossing* each other, as at *D*; then *join* the *Points C* and *D* with a *Right-line*, and it will *bisect* the *Arch A B*, and consequently the *Angle*; as was requir'd.




## PROBLEM IV.

*At a Point A, in a Right-line given A B, to make a Right-lin'd Angle equal to a Right-lined Angle given C.* (23. *e.* 1.)

Upon the given *Angular Point C* describe an *Arch*, as *F D*, (making *C D* any *Radius* at *Pleasure*) and with the same *Radius describe* the like *Arch* upon the given *Point A*, as *f d*; that is, make the *Arch f d* equal to the *Arch F D*; Then *join* the *Points A* and *f* with a *Right-line*, and it will form the Angle requir'd.

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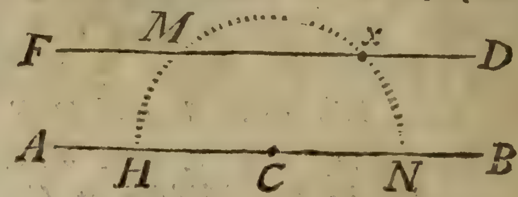
## PROBLEM V.

*To draw a Right-line, as F D, parallel to a given Right-line A B, that shall pass thro' any assign'd Point, as at x,* viz. *at any Distance required.* (31. *e.* 1.)

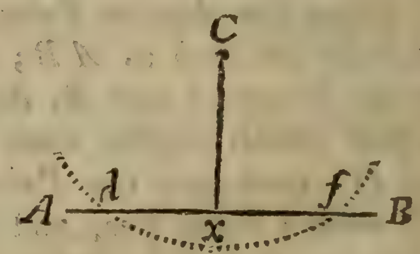
Take any convenient Point in the given *Line*, as at *C*, (the farther off *x* the better ;) make *C x Radius*, and with it upon the *Point C*, describe a *Semicircle*, as *H M x N*; then make the *Arch H M* equal to the *Arch x N*; thro' the *Points M* and *x* draw the *Right-line F D*, and it will be *parallel* to the *Line A C*, as was requir'd.




## PROBLEM VI.

*To let fall a* Perpendicular, *as C x, upon a given Right-line A B, from any assign'd Point that is not in it, as from C.* (12. *e.* 1.)

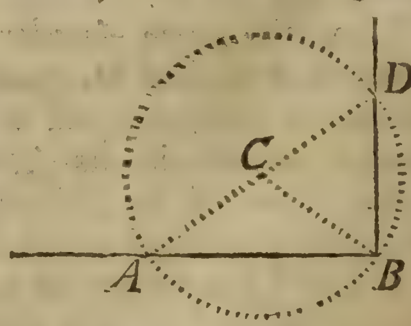
Upon the given Point *C describe* such an *Arch* of a *Circle* as will cross the given Line *A B* in *two Points*, as at *d* and *f*; Then *bisect* the *Distance* between those *two Points d f* (per Probl. 2.) as at *x*. *Draw* the *Right-line C x*, and it will be the *Perpendicular* requir'd.




## PROBLEM VII.

*To erect or raise a* Perpendicular *upon the End of any given Right-line, as at B; or upon any other Point assign'd in it.* (11. *e.* 1.)

Upon any *Point (taken at an Adventure)* out of the given *Line*, as at *C*, *describe* such a *Circle* as will *pass* through the *Point* from whence the *Perpendicular* must be *raised*, as at *B*, (viz. *make C B Radius)* : And from the *Point* where the *Circle cuts* the given *Line, as at A, draw* the *Circle's Diameter A C D*; then from the *Point D* draw the *Right-line D B*, and it will be the *Perpendicular* as was requir'd.

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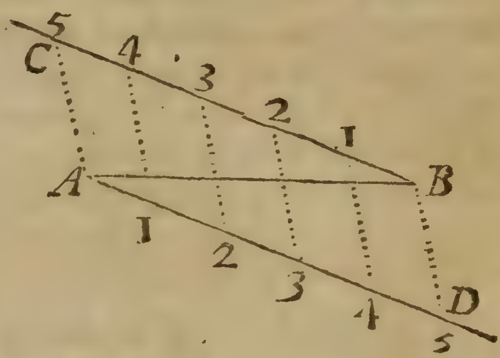
## PROBLEM VIII.

*To divide any given Right-line, as A B, into any proposed Number of equal Parts.* (10. *e.* 6.)

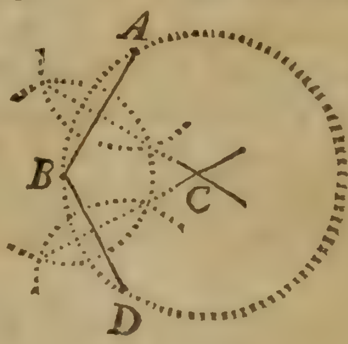
At the extream Points (or Ends) of the given Line, as at *A* and *B*, make *two* equal *Angles* (by *Prob.* 4.) continuing their Sides *A D* and *B C* to any sufficient *Length*; then upon those Sides, beginning at the Points *A* and *B*, set off the proposed *Number* of equal Parts *(suppose 'em 5.)* If *Right-lines* be drawn *(cross the given Line)* from one Point to the other, as in the annexed Figure, those *Lines* will divide the given *Line A B* into the *Number* of equal Parts required.




## PROBLEM IX.

*To describe a Circle that shall pass (or cut) thro' any Three Points given, not lying in a Right-line, as at the Points A B D.*

*Join* the Points *A B* and *B D* with *Right-lines*; then bisect both those *Lines (per Problem* 2.*)* the Point where the bisecting Lines meet, as at *C*, will be the *Center* of the *Circle* required.




The Work of this Problem being well understood, 'twill be *easy* to perform the two following, without any *Scheme*, viz.

1. *To find the Center of any Circle given.* (1. *e.* 3.)

By the last Problem 'tis plain, that if *three* Points be any where taken in the given *Circle's Periphery*, as at *A*, *B*, *D*, the *Center* of that Circle may be found as before.

2. *If a Segment of any Circle be given, to compleat or describe the whole Circle.*

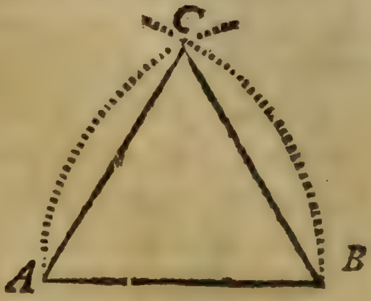
This may be done by taking any *three Points* in the given *Segment's* Arch, and then proceed as before.

PRO-

## PROBLEM X.

*Upon a Right-line given, as A B, to defcribe an Equilateral Triangle.* (1. *e.* 1.)

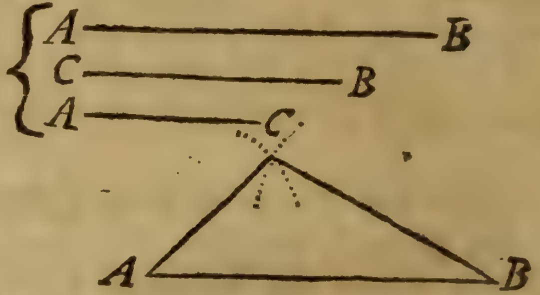
Make the given Line *Radius*, and with it, upon each of its *extream Points* or *Ends*, as at *A* and *B*, defcribe an *Arch*, *viz.* *A C* and *B C*; then join the *Points* *A C* and *B C* with *Right-lines*, and they will make the *Triangle* requir'd.




## PROBLEM XI.

*Three Right-lines being given, to form them into a Triangle, (provided any two of them, taken together, be longer than the Third)* (22. *e.* 1.)

Let the given *Lines* be

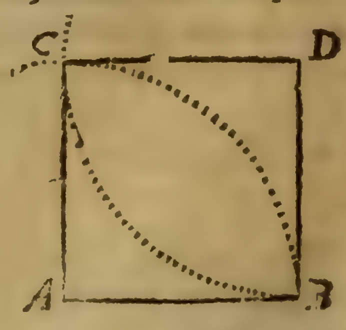



Make either of the *fhorter Lines* (as *A C*) *Radius*, and upon either End of the *longeft Line* (as at *A*) defcribe an Arch; then make the other Line *C B* *Radius*, and upon the other End of the *longeft Side* (as at *B*) defcribe another Arch, to crofs the Firft Arch (as at *C*): Join the *Points* *C A* and *C B* with *Right-lines*, and they will form the *Triangle* required.

## PROBLEM XII.

*Upon a given Right-line, as A B, to form a Square.* (46. *e.* 1.)

Upon one End of the given *Line*, as at *B*, erect the *Perpendicular* *B D*, equal in Length with the given *Line*, *viz.* make *B D* = *A B*; that being done, make the given Line *Radius*, and upon the *Points* *A* and *D* defcribe equal Arches to crofs each other, as at *C*; then join the *Points* *C A* and *C D* with *Right-lines*, and they will form the *Square* required.

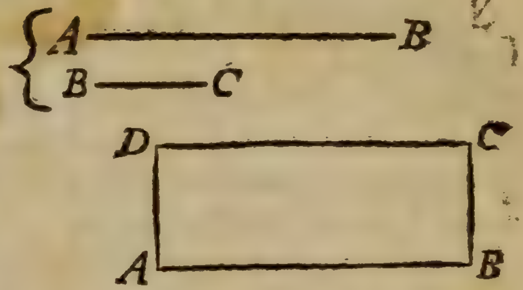



PRO-

## PROBLEM XIII.

*Two unequal Right-lines being given, to form or make of them a Right-angled* Parallelogram.

Let the given *Lines* be




Upon one End of the *longest Line*, as at *B*, erect a *Perpendicular* of the same Length with the *shortest Line B C*; then from the *Point C* draw a Line parallel and of the same Length, to *A B*; *viz.* make *D C = A B*: Join *D A* with a *Right-line*, and it will form the *Oblong* or *Parallelogram* required.

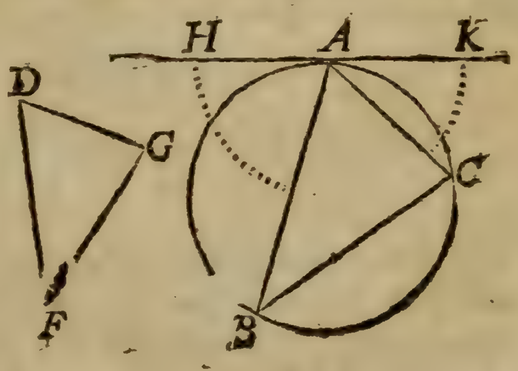
As for *Rhombus*'s and *Rhomboides*'s, to wit, *Oblique-angled* Parallelograms, they are made, or describ'd, after the same Manner with the two last Figures; only instead of erecting the Perpendiculars, you must set off their given Angles, and then proceed to draw their Sides parallel, &c. as before.

## PROBLEM XIV.

*In any given Circle, to inscribe or make a Triangle, whose Angles shall be equal to the Angles of a given Triangle; as the Triangle F D G,* (2. *e.* 4.)

Note, *Any Right-lin'd Figure is said to be inscrib'd in a Circle, when all the Angular Points of that Figure do just touch the Circle's Periphery.*

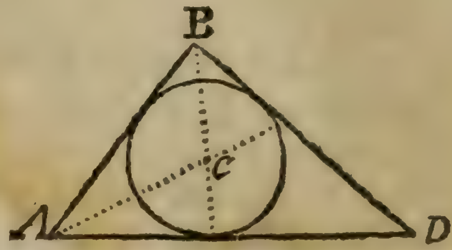
Draw any *Right-line* (as *H K*) so as just to touch the Circle, as at *A*; then make the Angle *K A C* equal to any one Angle of the given Triangle, as *D F G*; and the Angle *H A B* equal to another Angle of the Triangle, as *D G F*; then will the Angle *B A C* be equal to the Angle *F D G*. Join the *Points B* and *C* with a Right-line, and 'twill form the Triangle required.



## PROBLEM XV.

*In any given Triangle, as A B D, to describe a Circle that shall touch all its Sides.* (4. *e.* 4.)

Bisect any two Angles of the Triangle, as *A* and *B*, and where the bisecting Lines meet (as at *C*) will be the Center of the Circle required; and its *Radius* will be the nearest Distance to the Sides of the Triangle.




## PROBLEM XVI.

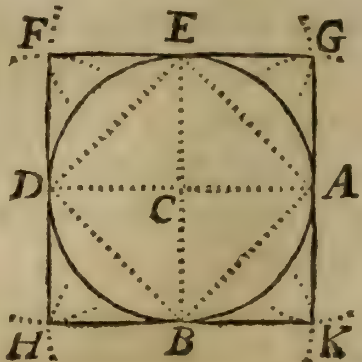
*To describe a Circle about any given Triangle.* (5. *e.* 4.)

This *Problem* is perform'd in all Respects like the Ninth, *viz.* by bisecting any *Two Sides* of the given Triangle; the Point, where those bisecting Lines meet, will be the Center of the Circle required.

## PROBLEM XVII.

*To describe a Square about any given Circle.* (7. *e.* 4.)

Draw two *Diameters* in the given Circle (as *D A* and *E B*) crossing at Right Angles in the Center *C*; and, with the Circle's *Radius C A*, describe from the *extream Points* of those Diameters, *viz. A*, *B*, *D*, *E*, cross Arches, as at *F*, *G*, *H*, *K*; then join those *Points* where the Arches cross with *Right-lines*, and they will form the *Square* required.




## PROBLEM XVIII.

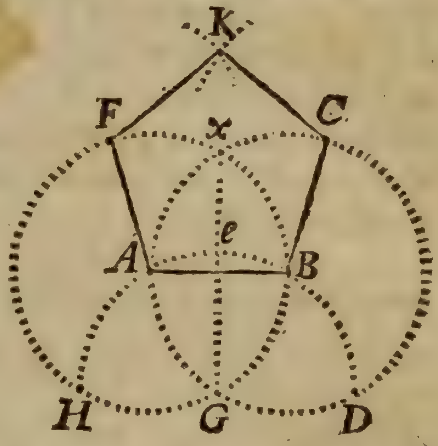
*In any given Circle, to describe the largest Square it can contain.* (6. *e.* 4.)

Having drawn the *Diameters*, as *D A* and *E B*, bisecting each other at Right-angles in the Center *C*, (as in the last Scheme); then join the *Points A*, *B*, *D*, and *E*, with *Right-lines*, viz. *A B*, *B D*, *D E*, *E A*, and they will be Sides of the *Square* required.

## PROBLEM XIX.

*Upon any given Right-line, as A B, to describe a regular* Pentagon, *or Five-sided* Polygon.

Make the given Line *Radius*, and upon each End of it describe a Circle; and through those Points where the Circles cross each other (as at *G x*) draw the Right-line *G e x:* Upon the Point *G* with the same *Radius* describe the Arch *H A e B D*, and laying a Ruler upon the Points *D*, *e*, mark where it *crosses* the other Circle, as at *F*. Again, lay the Ruler upon the Points *H*, *e*, and mark where it *crosses* the other Circle, as at *C:* Then from the Points *F* and *C* *(with the same Radius as before)* describe cross Arches, as at *K:* Join the Points *A F*, *F K*, *K C*, and *C B*, with Right-lines, and they will form the *Pentagon* required, *viz.* $AF = FK = KC = CB = AB$; and the Angles at *A*, *B*, *C*, *K*, *F* will be equal.

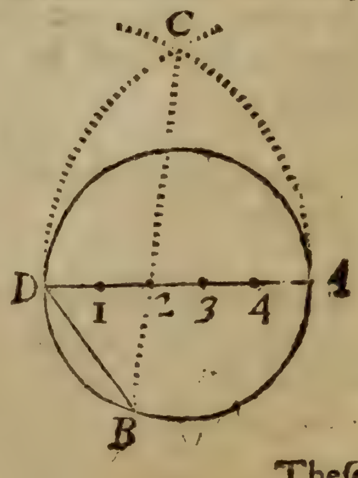



## PROBLEM XX.

*In any given Circle, to describe a* regular Pentagon. (11. *e.* 4. & 10. *e.* 3.)

*Or, in general Terms, to describe any* regular Polygon *in a Circle.*

Draw the Circle's Diameter *D A*, and divide it into as many equal Parts as the proposed *Polygon* hath Sides; then make the whole Diameter a *Radius*, and describe the two Arches *C A* and *C D*. If a Right-line be drawn from the Point *C*, through the Second of those equal Parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's *Periphery*, as at *B*. Join *D B* with a Right-line, and it will be the Side of the *Pentagon* required.






These

Thefe Twenty *Problems* are fufficient to exercife the young Practitioner, and bring his Hand to the right Management of a *Ruler* and *Compaffes*, wherein I would advife him to be very ready and exact.

As to the Reafon why fuch Lines muft be fo drawn as directed at each *Problem*, that, I prefume, will fully and clearly appear from the following *Theorems*; and therefore I have *(for Brevity's Sake)* omitted giving any *Demonftrations* of them in this Chapter, defiring the Learner to be fatisfied with the bare Knowledge of doing them only, until he hath fully confidered the Contents of the next Chapter; and then I doubt not but all will appear very plain and eafy.

## CHAP. III.

### *A Collection of moft ufeful* Theorems *in plain Geometry* Demonftrated.

Note, *In order to fhorten feveral of the following Demonftrations, it will be neceffary to premife, that*

1. THE *Periphery* (or Circumference) of every Circle *(whether great or fmall)* is fuppos'd to be divided into 360 equal Parts, called *Degrees*; and every one of thofe *Degrees* are divided into 60 *equal Parts*, called *Minutes*, &c.

2. All *Angles* are meafured by the Arch of a Circle defcrib'd upon the *Angular Point (See Defin.* 9. *Page* 287.) and are efteem'd *greater* or *lefs*, according to the Number of Degrees contain'd in that *Arch*.

3, A *Quadrant*, or *Quarter-part* of any Circle, is always 90 Degrees, being the *Meafure* of a *Right-angle (Defin.* 6. *P.* 287.) and a *Semicircle* is 180 Degrees, being the *Meafure* of *two Right-angles*.

4. *Equal Arches* of a *Circle*, or of equal Circles, *meafure* equal *Angles*.

To thofe *five* general *Axioms* already laid down in *Page* 146, *(which I here fuppofe the Reader to be very well acquainted with)* it will be convenient to underftand thefe following, which begin their *Number* where the other ended.

Axioms

## Axioms.

6. *Every* whole Thing is GREATER than its PART.
That is, the whole Line *A B* is *greater* than its *Part A c*, &c. } *A* ——————— | *c* ——— *B*
The ſame is to be underſtood of *Superficies*'s and *Solids*.

7. *Every* Whole is EQUAL to all its PARTS taken together.
That is, the whole Line *A B* is *equal* to its *Parts A C* + *c d* + *d e* + *e B*. } *A*— | *c* — | *d* — | *e* —*B*
The ſame is alſo true in *Superficies*'s and *Solids*.

8. Thoſe *Things* which being laid one upon another, do *agree* or *meet* in all their Parts, are equal one to the other.

But the Converſe of this *Axiom*, to wit, that *equal Things* being laid one upon the other will meet, is only *true* in *Lines* and *Angles*, but not in *Superficies*'s, unleſs they be alike, *viz.* of the ſame *Figure* or Form: As for Inſtance, a Circle may be *equal* in *Area* to a Square; but if they are *laid* one upon the other, 'tis plain they cannot *meet* in all their *Parts*, becauſe they are *unlike Figures*. Alſo, a *Parallelogram* and a *Triangle* may be equal in their *Area*'s one to another, and both of them may be equal to a *Square*; but if they are *laid* one upon the other, they will not *meet* in all their *Parts*, &c.

Note, *Beſides the* Characters *already explain'd in* Part I, *and in other Places of this Tract, theſe following are added.*

Viz. ∠ denotes an Angle in general, and ∠∠ ſignifies Angles; △ ſignifies a Triangle; □ ſignifies a Square, and ▭ denotes a Parallelogram. And when an Angle is denoted by any *three Letters* (as, *A B C*) the middle *Letter* (as *B*) always denotes the Angular Point; and the other *two Letters* (*as A B* and *B C*) denote the *Lines* or *Sides* of the Triangle which includes that Angle.

Theſe Things being *premiſed*, the young *Geometer* may proceed to the *Demonſtrations* of the following *Theorems*; wherein he may perceive an abſolute Neceſſity of being well *verſed* in ſeveral Things that have been already deliver'd: And alſo it will be very *advantageous* to ſtore up ſeveral uſeful *Corollaries* and *Lemma*'s, as they become *diſcover'd Truths*: For it often happens, that a *Propoſition* cannot be clearly demonſtrated *a priori*, or of itſelf, without a great Deal of Trouble; therefore it will be uſeful to have *Recourſe* to thoſe *Truths* that may be aſſiſting in the *Demonſtrations* then in Hand.

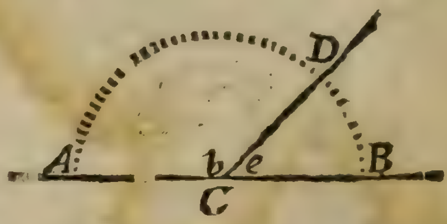
THEO.

## THEOREM I.

*If a Right-line stand upon (or meet with) another Right-line, and make Angles with it, they will either be two Right-angles, or two Angles equal to two Right-angles.* (13. *e.* 1.)

### Demonstration.

Suppose the Lines to be *A B* and *D C*, meeting in the *Point* at *C:* Upon *C* describe any Circle at pleasure; then will the Arch *A D* be the *Measure* of the ∠ *b*, and the Arch *D B* the *Measure* of ∠ *e*; but the Arches *A D* + *D B* = 180°, *viz.* they compleat the *Semicircle.*




Consequently the ∠ *b* + ∠ *e* = 180°. *Which was to be prov'd.*

*Corollaries.*

1. Hence it follows, that if the ∠ *b* = 90° then ∠ *e* = 90°; but if ∠ *b* be *obtuse*, then the ∠ *e* will be *acute*, &c.

From hence it will be easy to conceive, that if several Right-lines stand upon, or meet with any Right-line at one and the same Point, and on the same Side, then all the Angles taken together will be = 180°, *viz.* Two Right-angles.

## THEOREM II.

*If two Angles intersect* (i. e. *cut or cross) each other, the two opposite Angles will be equal.* (15. *e.* 1.)

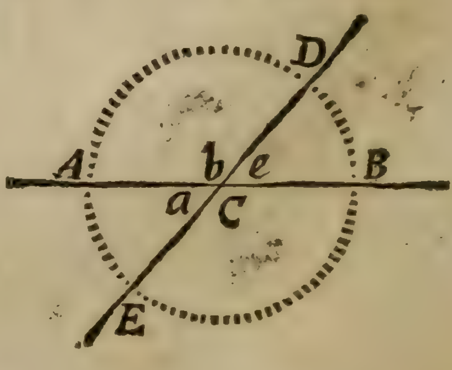
### Demonstration.

Let the *two Lines* be *A B* and *D E*, intersecting each other in the Center *C*.

Then ∠ *b* + ∠ *e* = 180° } *per* last.
And ∠ *b* + ∠ *a* = 180° }

Consequently ∠ *b* + ∠ *e* = ∠ *b* + ∠ *a*, *per Axiom* 5.

Subtract ∠ *b* on both Sides of the *Æquation*, and it will leave ∠ *e* = ∠ *a*.




Again, ∠ *b* + ∠ *e* = 180°, as before; and ∠ *e* + ∠ *C* = 180°, consequently ∠ *e* + *C* = ∠ *b* + ∠ *e*. *Sultract* ∠ *e*, and then ∠ *C* = ∠ *b*. Q. E. D,

*Corol-*

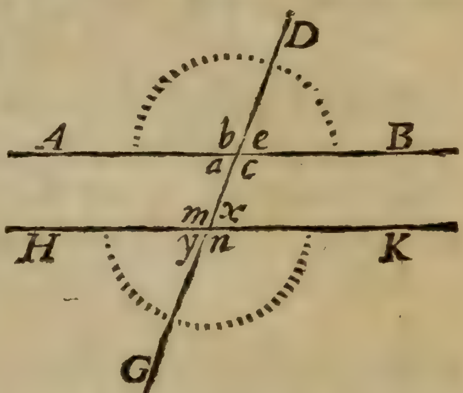
*Corollary.*

From hence it is evident, that if two Lines interfect each other, they will make four Angles; which, being taken together, will always be equal to Four Right-angles.

## THEOREM III.

*If a Right-line cut (or crofs) two parallel Lines, it will make the oppofite Angles equal one to another.* (29. *e.* 1.)

Suppofe the two Lines *AB* and *HK* to be parallel, and the Right line *D G* to cut them both at *C* and *n*: Upon the Point *C* (with *any Radius)* defcribe a Semicircle; and with the fame *Radius*, upon the Point at *n*, defcribe another Semicircle oppofite to the firft, as in the Figure. Then 'tis plain, and I fuppofe very eafy to conceive, that if the Center *C* were mov'd along upon the Line *D G*, until it came to the Center at *n*, the two Lines *A B* and *H K* would *meet* and concur, *viz.* become one Line *(for parallel Lines are as it were but one broad Line)*. Confequently the two Semicircles would alfo meet, and become one entire Circle, like to that in the laft *Demonftration.*




And therefore the $\angle y = \angle x = \angle a = \angle e$ } { as before, *per*
And $\angle m = \angle n = \angle b = \angle c$ } { *laft Theorem.*

Q. E. D.

*Corollary.*

Hence it follows, that if *three*, *four*, or ever fo many Parallel-lines, are cut or crofs'd by one Right-line, all their oppofite Angles will be equal.

## THEOREM IV.

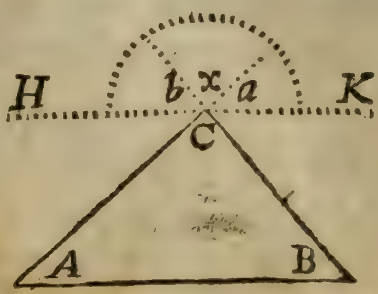
*The three Angles of every plain Triangle are equal to two Right-angles.* (32. *e.* 1.)

*Confequently, any two Angles of any plain Triangle muft needs be lefs than two Right-angles.* (17. *e.* 1.)

Demon-

## Demonstration.

Let the $\triangle$ *A B C* be propos'd; draw the Right-line *H K* parallel to the Side *A B*, just touching the Vertical Angle *C*; and upon the same Angular Point *C* describe any Semicircle, and produce the Sides *A C* and *B C* to its *Periphery*. Then will $\angle b = \angle B$, $\angle a = \angle A$, and $\angle x = \angle C$, *per* last *Theorem*. But $\angle b + \angle a + \angle x = 180°$, or two Right-angles: Consequently $\angle B + \angle A + \angle C = 180°$ per *Axiom* 5. Q. E. D.




*Corollary.*

Hence it follows, that the two *acute* Angles of every Right-angled Triangle are equal to a Right-angle, or 90°.

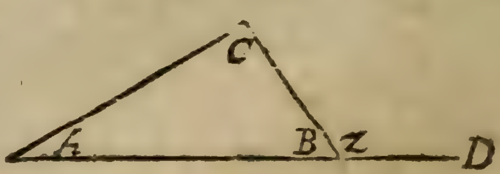
Consequently, if one of the *acute* Angles be given, the other is also given, *viz.* 90°—the given $\angle$ leaves the other $\angle$.

## THEOREM V.

*If one Side of any plain Triangle be continued or produced beyond, or out of the Triangle, the outward Angle will always be equal to the two inward opposite Angles.* (32. *e.* 1.)

## Demonstration.

Let the Side *A B* of the $\triangle$ *A B C* be produced out of the $\triangle$, suppose to *D*, &c. as in the Figure. Then $\angle z = \angle A + \angle C$, for the $\angle B + \angle z = 180°$ per *Theorem* 1. and the $\angle B + \angle A + \angle C = 180°$, *per* last *Theorem*. Therefore $\angle B + \angle z = \angle B + \angle A + \angle C$, per Axiom 5. *Subtract* $\angle B$ on both Sides the *Æquation*, and it will leave $\angle z = \angle A + \angle C$ *(per Axiom* 2.) Q. E. D.




*Consequently, the outward Angle (at z) of any plain Triangle, must needs be greater than either of the inward opposite Angles,* viz. *greater than* $\angle$ *A, or* $\angle$ *C* (16. *e.* 1.).

*Corollary.*

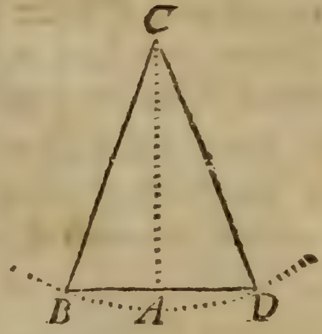
Hence it follows, that if one Angle of any plain Triangle be given, the *Sum* of the other two Angles is also given; for 180°—the given $\angle$ = the other two $\angle\angle$.

THEO-

## THEOREM VI.

*In every plain Triangle, equal Sides ſubtend* (viz. *are oppoſite to) equal Angles.* (5. e. 1.)
*Conſequently, equal Angles are ſubtended by equal Sides* (6. *e.* 1.)

### Demonſtration.




Suppoſe the $\triangle$ *B C D* to be an *Iſoſceles* $\triangle$; that is, let $BC = CD$. Biſect the $\angle C$, or *(which is all one)* make *C A perpendicular* to *B D*; then will the $\angle\ \angle$ on each Side of it *(viz.* $\angle BAC$ and $\angle DAC$*)* be Right-angles.

Therefore $\left\{ \begin{array}{l} \frac{1}{2} \angle C + \angle B = 90^\circ \\ \frac{1}{2} \angle C + \angle D = 90^\circ \end{array} \right\}$ per *Corol.* to *Theorem* 4.

Conſequently, $\frac{1}{2} \angle C + \angle B = \frac{1}{2} \angle C + \angle D$, per *Axiom* 5. *Subtract* $\frac{1}{2} \angle C$ from both Sides of the *Æquation*, and it will leave $\angle B = \angle D$, per *Axiom* 2. Q. E. D.

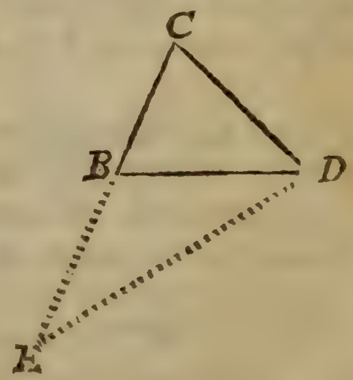
*Corollary.*

From hence it follows, that the *three Angles* of an *Equilateral Triangle* are equal one to another.

## THEOREM VII.

*In every plain Triangle, the longeſt Side ſubtends the greateſt Angle.* (18. *e.* 1.)
*Conſequently, the greateſt Angle of any plain Triangle is ſubtended by the longeſt Side.*

This *Theorem* is evident by Inſpection only: For, let one of the Sides of any *plain Triangle* (as *C B*) be produced, ſuppoſe to *E*; join *D E* with a *Right-line*; then 'tis evident, that becauſe *C E* is now made longer than the Side *B C*, therefore the $\angle$ at *D* is become *larger* than it was before by the $\angle$ *B D E*: And it's plain, the longer the Side *C E* had been made, the $\angle$ at *D* would have been the more enlarged.



## THEOREM VIII.

*If the Sides of two Triangles are equal, the Angles opposite to those equal Sides will be equal.* (8. *e.* 1.)

The Truth of this *Theorem* is evident by the two included *Triangles* in the 6th *Theorem*, for they have their respective Sides equal, *viz.* $BC = CD$, $BA = DA$, and $CA$ common to both *Triangles*. And it is there prov'd, that the $\angle$ opposite to those equal Sides are equal, &c. which needs no further *Proof*.

Note, *The Converse of this* Theorem *holds not true; for the Angles of two Triangles may be equal, and their opposite or subtending Sides unequal; as will appear at* Theorem XII.

*Corollary.*

Hence it follows, that *Triangles mutually equilateral* are also *mutually equiangular*; and,

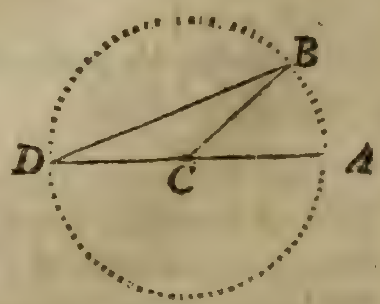
That *Triangles mutually equilateral* are equal one to another. (4. & 26. *e.* 1.)

## THEOREM IX.

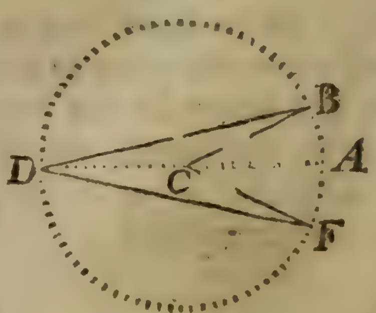
*An Angle at the Center of any Circle is always double to the Angle at the Periphery, when both the Angles stand upon the same Arch.* (20. *e.* 3.) This *Theorem* hath three Varieties or Cases.

### Demonstration.

*Case* 1. Let the Diameter $DA$, and the Line $DB$, be the two Lines which form the $\angle D$ at the *Periphery*; draw the *Radius* $BC$, then $\angle BCA$ is the $\angle$ at the *Center*. But $\angle BCA = \angle D + \angle B$, per *Th.* 5. and because $DC = BC$, therefore $\angle D = \angle B$, per *Theorem* 6. consequently $\angle BCA = 2 \angle D$.




*Case* 2. Suppose the $\angle BCF$ at the Center to be within the $\angle BDF$ at the *Periphery*, (*as in the annexed Figure.*) Draw the Diameter $DA$;
then the $\angle BCA = 2\angle BDA$ and the $\angle FCA = 2\angle FDA$ } per *Case* 1.
add these two *Æquations* together.




Then

Then will $\angle BCA + \angle FCA = 2\angle BDA + 2\angle FDA$, per *Ax.* 1. But $\angle BCA + \angle FCA = \angle BCF$, and $2\angle BDA + 2\angle FDA = 2\angle BDF$. Consequently $\angle BCF = 2\angle BDF$.

*Case* 3. Again, suppose the $\angle BCF$ at the Center to be out of the $\angle BDF$ at the Periphery. From the Angular Point $D$ at the Periphery draw the Diameter $DA$.


Then $\angle FCA = 2\angle FDA$ and $\angle BCA = 2\angle BDA$ } per Case 1.

*Subtract* this last *Æquation* from the other, and it will leave $\angle FCA - \angle BCA = 2\angle FDA - 2\angle BDA$, per *Axiom* 2. But $\angle FCA - \angle BCA = \angle FCB$, and $2\angle FDA - 2\angle BDA = 2\angle FDB$: Consequently $\angle FCB = 2\angle FDB$. Q. E. D.

*Corollary.*

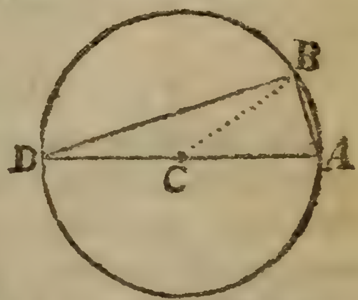
Hence 'tis evident, that all Angles at the Periphery, which stand on the same *Segment* or *Arch* of a Circle, or upon equal Arches, are equal one to another. (21. *e.* 3.)

## THEOREM X.

*An Angle in a Semicircle is a Right-angle.* (31. *e.* 3.) That is, *if the Diameter of any* Circle *be the Side of a Triangle, and the Angle opposite to that Side be any where in the* Circle's *Periphery, it will be a Right-angle.*

### Demonstration.

Let $DA$ be the Diameter, and $DBA$ the Triangle, then, $\angle B = 90^\circ$. Draw the Radius $BC$, then is the $\angle DBA = \angle D + \angle A$. For $\angle CBD = \angle D$, and $\angle CBA = \angle A$, per *Theorem* 6. Therefore $\angle DBA = \angle CBD + \angle CBA$, per *Axiom* 5. Again $\angle DBA + \angle D + \angle A = 180^\circ$, per *Theorem* 4. Consequently, $\angle DBA = 90^\circ$ or a Right-angle. Q. E. D.



*Corollaries.*

1. Hence it will be eaſy to conceive, that an Angle made in any Segment leſs than a Semicircle will be *obtuſe*, or greater than a Right-angle.

2. And an Angle, made in any Segment greater than a Semicircle, muſt conſequently be *acute*.

## THEOREM XI.

*In any Right-angled Triangle, the Square which is made of the Hypothenuſe, or Side ſubtending the Right-angle, is equal to both the Squares which are made of the Sides including the Right-angle.* (47. *e.* 1.)

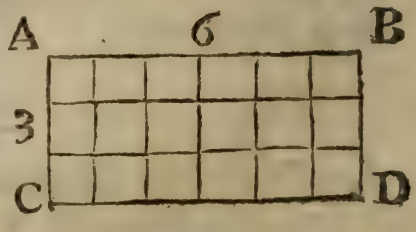
There are ſeveral Ways of demonſtrating this noble and uſeful *Theorem*, but, I preſume, none more eaſy to be underſtood by a *Learner* than that which I ſhall here propoſe: And, in order thereto, 'twill be neceſſary to premiſe the following *Lemma's*.

### Lemma 1.

A *Right-line* is ſaid to be *multiply'd* with a *Right-line*, when either a *Square*, or other *Right-angled Parallelogram*, is made of the *two Lines*.

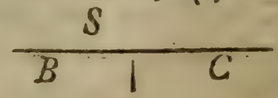
That is, the *Area* of any *Right-angled Parallelogram* is equal to the *Product* of thoſe *Numbers* which expreſs the *Meaſure* of its Sides.

Thus, if $AB = 6$ Inches and $AC = 3$ Inches: Then $AB \times AC = 6 \times 3 = 18$ ſquare Inches; which is the *Area* of the Parallelogram *ABCD*.




### Lemma 2.

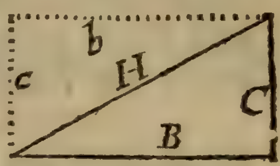
If a *Right-line* be any way cut into *two Parts*, the *Square* of the whole *Line* will be equal to the *Squares* of each Part, and a double *Rectangle* or *Parallelogram* made of both the Parts, (4. *e.* 2.) that is, if the *Line S* be cut into the two Parts *B* and *C*; then is $S = B + C$: But if both the Sides of the *Æquation* be involv'd, it will be $SS = BB + 2BC + CC$.




Lemma

## Lemma 3.

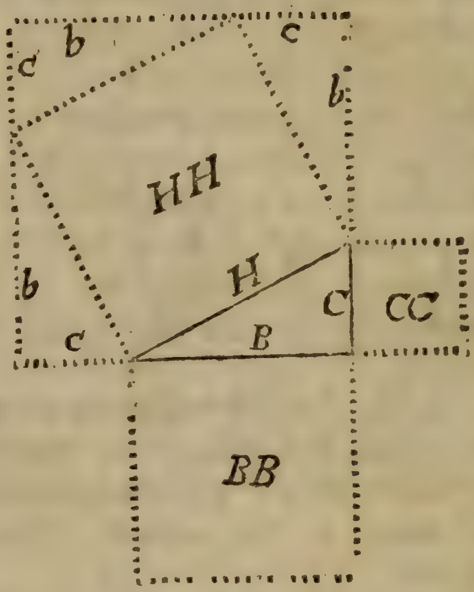
The *Area* of every *Right-angled Triangle* is half the *Parallelogram* made of its Baſe and Perpendicular.




For $B \times C =$ the *Area* of the whole *Parallelogram*, by the firſt *Lemma*. And $\triangle\, B\,C\,H + \triangle\, b\,c\,H =$ the *Parallelogram*; but $B = b$, and $C = c$. Therefore $\frac{1}{2}\, B \times C =$ the *Area* of each $\triangle$, *viz.* $\frac{1}{2}\, B \times C + \frac{1}{2}\, b \times c\; B \times C$.

Theſe Things being premiſed, let us ſuppoſe the Triangle $B\,C\,H$ to be a *Right-angled Triangle*, *viz.* the Side $C$ perpendicular to the Side $B$; then will $B\,B + C\,C = H\,H$.

## Demonſtration.




Make a *Square* whoſe Side is $= B + C$, and draw the included *Square* whoſe Side is $= H$, as in the Scheme: Then will the *Area* of the *great Square* be equal to the *Area* of the four *Triangles* $+ HH$; but the *Area* of each $\triangle = \frac{1}{2}\, B\,C$, or $B \times C$, per *Lemma* 3. Therefore the 4 $\triangle$'s $= \frac{1}{2}\, B\,C \times 4 = 2\, B\,C$, conſequently, the *Area* of the great *Square* is $HH + 2\, BC$. *Involve* $B + C$, and it will be $BB + 2\, B\,C + C\,C =$ the *Area* of the great *Square*; per *Lemma* 3. Conſequently, $HH + 2\, B\,C = BB + 2\, B\,C + CC$, per *Axiom* 5. Subtract $2\, B\,C$ from both Sides of the *Æquation*, and there will *remain* $H\,H = B\,B + C\,C$.

To illuſtrate this *Theorem* by *Numbers*, let us

Suppoſe $C = 3$. $B = 4$. and $H = 5$.
Then will $CC = 9$. $B\,B = 16$. and $H\,H = 25$.
Conſequently, $B\,B + C\,C = H\,H = 16 + 9 = 25$.

*Conſectary.*

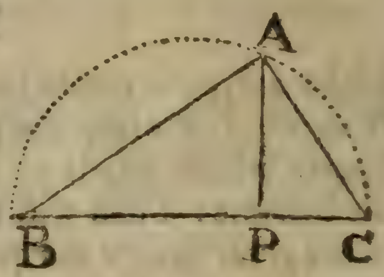
From this admirable *Theorem (ſaid to be firſt invented by* Pythagoras) is deduced the *Method* of adding and ſubtracting Squares, Parallelograms, Circles, *&c.*

THEO-

## THEOREM XII.

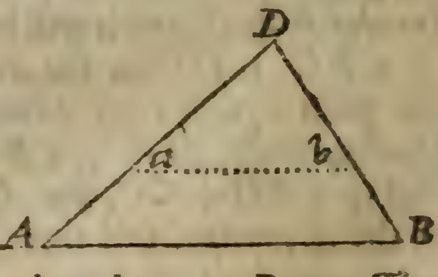
*In any Right-angled Triangle, a Perpendicular being let fall from the Right-angle upon the Hypothenuſe will divide the Triangle into two Right-angled Triangles, which will be both ſimilar (or alike) to the firſt Triangle, and to each other.* (8. *e.* 6.)

Note, *All plain Triangles are ſaid to be ſimilar (viz. alike) when each ſingle Angle in one of the Triangles is equal to each ſingle Angle of the other; but if any two ſingle Angles of one Triangle are equal to two ſingle Angles of the other, the third Angle will be equal.* Per Theo. 4.




1. In the Right-angled △ *BAC*, let *AP* be ſuppoſed *perpendicular* to the *Hypothenuſe BC*; then ∠ *BAP* = ∠ *C*. For ∠ *BAP* + ∠ *B* = 90°, and ∠ *B* + ∠ *C* = 90°, per *Corollary* to *Theorem* 4. Therefore the ∠ *BAP* = ∠ *C*, per *Axiom* 5. again, ∠ *PAC* + ∠ *C* = 90, and ∠ *B* + ∠ *C* = 90°. Therefore ∠ *PAC* = ∠ *B*, &c. Conſequently the △ *BAP* is alike to the △ *ACP*; and each is like to the whole △ *BAC*.

2. Or if a *Right-line* be drawn parallel to one of the Sides of any *plain Triangle*, (*viz.* within it) it will cut off a Triangle ſimilar or alike to the whole Triangle. Thus:




In the △ *ABD* draw the *Right-line a b* parallel to the Side *A B*; then will the included △ *a D b* be like the △ *A D B*: For ∠ *a* = ∠ *A* and ∠ *b* = ∠ *B*, per *Theorem* 3; and ∠ *D* is common to both the Triangles; *Ergo*, &c.

## THEOREM XIII.

*If two Triangles are alike, their like Sides will be proportional.*

That is, thoſe Sides which ſubtend the equal Angles, as alſo thoſe Sides which are *about* the equal Angles, will be *proportional* to each other; and conſequently, if any *two Triangles* have their Sides proportional, their Angles are equal. (4, 5, 6, 7. *e.* 6.)

Demon-

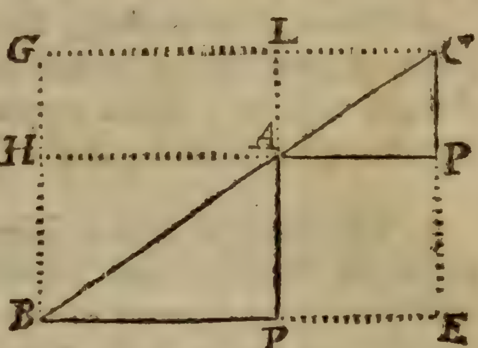
## Demonstration.

Let the *similar Triangles* in the *Scheme* of the last *Theorem* be here proposed again.

Then it will be $BP : AP :: AP : CP$, according to this *Theorem*. *Ergo* $BP \times CP = AP \times AP$.

*First.*

Let us suppose the aforesaid *Right-angled* $\triangle BAC$ cut through the Perpendicular $AP$, and there open'd until the Sides $BA$ and $CA$ become one *Right-line*. Let the Sides $BP$ and $CP$ be continued until they meet in $E$; then compleat the Parallelograms by drawing the parallel Lines $GLC$, $HAP$, $GHB$, and $LAP$, as in the *Figure*.

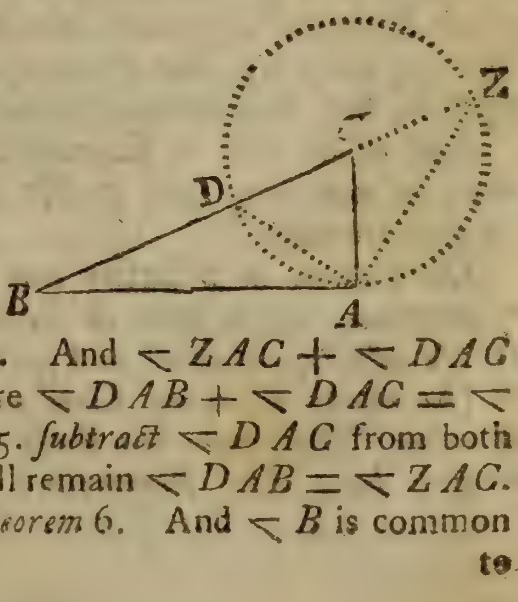



Then it is evident, that the $\triangle BHA = \triangle BPA$, and the $\triangle CPA = \triangle CLA$; also that the $\triangle BEC = \triangle BGC$, because all their respective Sides are *equal*.

But the $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA, + \triangle CPA + \square APEP$. Now if from both *Sides* of this *Æquation* there be subtracted the equal *Triangles*, there will remain $\square HGLA = \square APEP$. But $\square HGLA = BP \times CP$, and $\square APEP = AP \times AP$. Consequently $BP : AP :: AP : CP$. Which was to be prov'd.

Or otherwise, thus:

Suppose the $\triangle BAC$ to be *Right-angled* at $A$: Upon the $\angle$ Point $C$, with the *Radius* $CA$ describe a Circle, and continue the *Hypothenuse* $BC$ to $Z$; join $ZA$ and $AD$ with *Right-lines*; then will the $\triangle BAD$ be like to the $\triangle BZA$. For $\angle DAB + \angle DAC = 90°$, by *Construction*. And $\angle ZAC + \angle DAC = 90°$, by *Theorem* X. Therefore $\angle DAB + \angle DAC = \angle ZAC + \angle DAC$. By *Axiom* 5. *subtract* $\angle DAC$ from both Sides of the *Æquation*, and there will remain $\angle DAB = \angle ZAC$. But $\angle ZAC = \angle CZA$, by *Theorem* 6. And $\angle B$ is common



to both Triangles. Therefore $\angle BDA = \angle BAZ$, by *Theorem* 6, consequently $\triangle BAD$ is like to $\triangle BZA$.

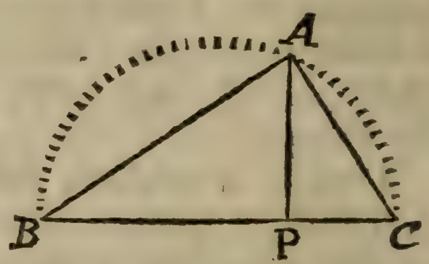
Let the *Sides* $\left\{ \begin{array}{l} BA = b \\ BC = h \\ CA = c \end{array} \right\}$ Then $bb + cc = hh$, by *Theorem* 11. Consequently $bb = hh - cc$, which gives the following Analogy, Viz. $b : h + c :: h - c : b$; that is, $BA : BZ :: BD : BA$.

Q. E. D.

*Corollaries.*

1. Hence it is evident, that, in any *Right-angled Triangle*, a *Perpendicular*, being let fall from the *Right-angle* upon the *Hypothenuse*, will be a *Mean* proportional between the *Segments* of the *Hypothenuse*: That is, $BP : PA :: PA : PC$.

2. The Base *(B A)* is a *Mean* proportional between the *Hypothenuse (B C)* and that *Segment* of the *Hypothenuse* next to the Base, *(viz. B P)* that is, $BC : BA :: BA : BP$.




3. The *Cathetus (A C)* is a *Mean* proportional between the *Hypothenuse (B C)* and that *Segment* of the *Hypothenuse* next to the *Cathetus*, (viz. *P C)*: That is, $BC : AC :: AC : PC$.

*Scholium.*

I have been more large upon this most excellent *Theorem*, in giving a double *Demonstration* of it, because it is so *universally useful* in all Parts of the *Mathematicks*: For the Business of *Trigonometry (both Plain and Spherical)* wholly depends upon it; and therefore one may truly say, that *Astronomy*, *Dialling*, *Navigation*, *Surveying*, *Opticks*, &c. depend upon a due Application of it.

And of its Use in *Geometry*, *Des-Cartes* takes particular Notice; as you may find in Dr. *Pell*'s Algebra, *Pag.* 65, whose Words are these:

*Des-Cartes*, in a Letter not yet printed, writes thus: " In " searching the *Solution* of *Geometrical Questions*, I always make " use of Lines parallel and perpendicular, as much as is possible, " [*he means as many Lines as are useful*] and I consider no " other Theorems but these two, [*the Sides of like* Triangles *have like Proportion*]. And [*in Rectangle Triangles* " *the*

" *the Square of the greatest Side is equal to the Squares of the two other* " *Sides.*] And I am not afraid to ſuppoſe many unknown Quanti- " ties, that I may reduce the propos'd Queſtion to ſuch Terms, as " to depend on no other Theorems but theſe Two."

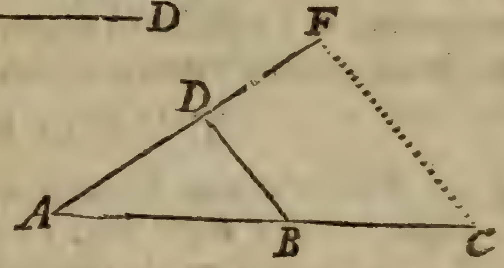
This I thought convenient to inſert, that the young *Learner* may ſee how the great *Des-Cartes* eſteem'd theſe two Theorems, *viz.* the laſt, and Theorem 11; for, in Truth, all the precedent Theorems are only (as it were) Preparatives to theſe Two.

This laſt Theorem demonſtrates the *Reaſon* of the *Method* uſed in finding out *Proportional Lines*; as in the Three following *Problems.*

## PROBLEM I.

*Two Right-lines being given to find a Third in Proportion to them.* (11. *e.* 6.)

Let theſe two *Lines* be { *A*——*B* / *A*——*D*

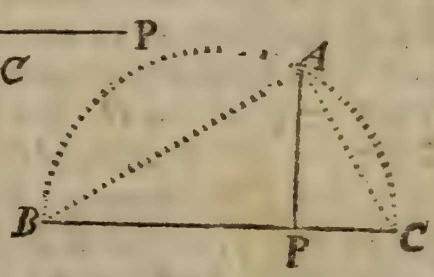



Set the Two given *Lines* at any Angle in the Point *A*, and produce the Line *A B* to *C*, making *B C* = *A D*; join the Points *B D* with a *Right-line*, and draw *C F* parallel to *B D*; then will the △ *A B D* be like the △ *A C F*. Therefore *A B* : *B C* (= *A D*) :: *A D* : *D F*, which is the third *Proportional* requir'd.

## PROBLEM II.

*Two Right-lines being given, to find a Mean proportional Line between them.* (13. *e.* 6.)

Let the given *Lines* be { B——P / P——C




Join the two given *Lines* into one, *viz.* make *B C* = *B P* + *P C*, and upon *B C*, as *Diameter*, deſcribe a *Semicircle*; then upon the Point *P*, where the two *Lines* meet, erect a Perpendicular to touch the

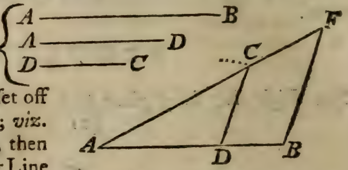
Circle's *Periphery*, as *P A*, and it will be the *Mean proportional* requir'd, viz. $BP : AP :: AP : PC$.

By this *Problem* 'tis eaſy to conceive how to make a *Square* equal to any given *Parallelogram*. (14. *e.* 6.)

For if *B P* be the *Length*, and *P C* be the *Breadth* of the given *Parallelogram*, then will *A P* be the Side of the *Square*, equal in *Area* to that *Parallelogram*.

## PROBLEM III.

*Three Right-lines being given, to find a fourth Proportional Line.* (12. *e.* 6.)

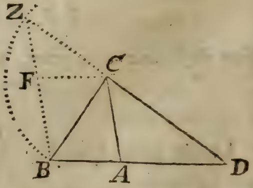
Suppoſe the three Lines 


Upon the longeſt Line *A B* ſet off the next longeſt Line *A D*; *viz.* make $DB = AB - AD$; then upon the Point *D* ſet the other Line *D C* at an *Angle*, either *right* or *oblique*, and draw the *Right-line AC* continuing it a ſufficient Length; make *B F parallel* to *D C*, and it will be the fourth *Proportional* requir'd; that is, $AD : DC :: AB : BF$.

## THEOREM XIV.

*If any Angle of a plain Triangle be biſected* (viz. *divided into two equal Angles*) *with a Right-line*, (viz. *as C A is ſuppos'd to do the Angle B C D*) *it will cut the oppoſite Side* (viz. *B D*) *in Proportion to the other two Sides of the Triangle* (3. *e.* 6.) i. e. $BA : BC :: AD : CD$.

### Demonſtration.



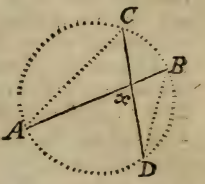

Produce the Side *D C*, until $CZ = CB$: join the Points *Z B* with a *Right-line*, and draw the Line *F C* parallel to *B D*; whence the $\angle Z = \angle CBZ$ per *Theorem* 6. and $\angle Z + \angle CBZ$, or $2 \angle CBZ = \angle BCD$, per *Theorem* 5; or, dividing both Sides of the Æquation by 2, $\angle CBZ = \frac{1}{2} \angle BCD$. But $\frac{1}{2} \angle BCD = \angle ACB = \angle ACD$ by the *Hypotheſis*, therefore $\angle ACB = \angle CBZ$ per *Axiom* 5: Whence *A C* is parallel to *B Z* per *Theorem* 3, and the Triangles *B D Z*, *A D C*, and *F C Z* are ſimilar by the ſecond Figure to *Theorem* 12, conſequently $BA (= FC) : BC (= ZC) :: AD : CD$. Q. E. D.

THEO-

## THEOREM XV.

*If two Right-lines (howsoever drawn) within a* Circle *do cut each other, the Rectangle made of the Segments (or Parts) of the one Line, will be equal to the Rectangle made of the Segments (or Parts) of the other Line.* (35. *e.* 3.)

That is, if two Lines (as $AB$ and $CD$) do cut each other in any Point, as at $x$, then will $Ax \times Bx = Dx \times Cx$.



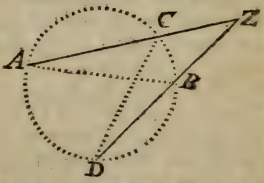

### Demonstration.

Join the Points of $AC$ and $BD$ with *Right-lines*, then will the $\triangle CxA$ be like to $\triangle BxD$: For $\angle B = \angle C$ and $\angle A = \angle D$. by Corollary to Theorem 9. and $\angle AxC = \angle BxD$. by *Theorem* 2. Therefore it will be $Ax : D :: Cx : Bx$. by *Theorem* 13. Consequently $Ax \times Bx = Dx \times Cx$. Q. E. D.

## THEOREM XVI.

*If two Right-Lines are so drawn within a* Circle, *as, being continued, they will meet in a Point out of the* Circle's *Periphery, the Rectangle made of the one whole Line, and its Part out of the* Circle, *will be equal to the Rectangle of the other whole Line, and its Part out of the* Circle. (36, 37. *e.* 3.)

That is, if the Lines $AC$ and $DB$ be continued unto the Point Z; then will $AZ \times CZ = DZ \times BZ$.




### Demonstration.

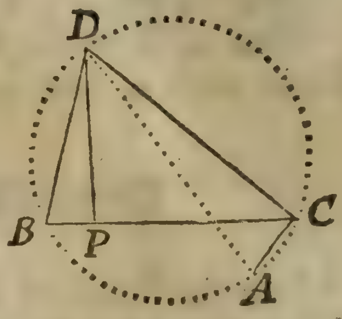
Draw the Lines $AB$ and $CD$, then will $\triangle CZD$ be like to the $\triangle BZA$; for $\angle A = \angle D$, and $\angle Z$ is common to both *Triangles*, consequently, $\angle ABZ = \angle DCZ$, by *Theorem* 4. therefore $AZ : BZ :: DZ :: CZ$. *Ergo*, $AZ \times CZ = DZ \times BZ$.

## THEOREM XVII.

*If from any Angle of a plain Triangle inscribed in a* Circle *there be let fall a Perpendicular upon the opposite Side, as* $DP$; *as that Perpen-*

*dicular is in Proportion to one of the Sides including the Angle, so is the other Side including the Angle to the Diameter of the* Circle.

### Demonstration.



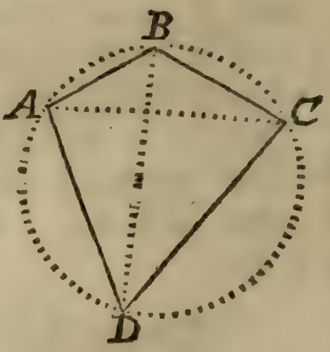

Let $BCD$ be the proposed *Triangle.* From the $\angle$ at $D$ draw the Diameter $DA$; then will $\angle A = \angle B$, because they both stand upon the same Arch $DC$, and $\angle DCA = 90°$, by *Theorem* 10. consequently the $\angle ADC = \angle BDP$, by *Theorem* 4. Therefore $\triangle DCA$ is like to the $\triangle DPB$; and therefore, $DP : DB :: DC : DA$; or $DP : DC :: DB : DA$. Q. E. D.

## THEOREM XVIII.

*If any Quadrangle (that is, a* Trapezium) *be inscrib'd within a* Circle, *the two opposite Angles, taken together, are equal to two Right-Angles,* viz. 18° (22. *e.* 3.)

That is, in the *Quadrangle* $ABCD$ the $\angle A + \angle C = 180°$. And the $\angle B + \angle D = 180°$.

### Demonstration.




Draw the *two* Diagonals $AC$ and $BD$; then will the $\angle BDA = \angle BCA$, and the $\angle BDC = \angle BAC$ by Corollary to *Theorem* 9. But $\angle ABC + \angle BCA + \angle BAC = 180°$. by *Theorem* 4. and the $\angle BDA + \angle BDC = \angle ADC$. Therefore the $\angle ABC + \angle ADC = 180°$. and by the same Way of arguing it may be prov'd, that the $\angle BAD + \angle BCD = 180°$. Q. E. D.

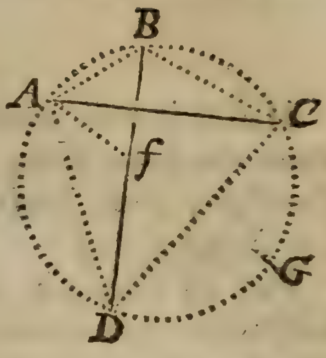
## THEOREM XIX.

*If in any Quadrangle inscrib'd within a* Circle *there be drawn two Diagonals, as $AC$ and $BD$, the Rectangle made of the two Diagonals will be equal to both the Rectangles made of the opposite Side of the Quadrangle.*

That is, $AC \times BD = AB \times CD + AD \times BC$.

Demon-

## Demonstration.




Make the Arch $DG =$ Arch $BC$, and from the Points $AG$ draw the Line $Af$, and it will form the $\triangle AfD$, like to the $\triangle ABC$: For the $\angle fAD = \angle BAC$, because the Arches $DG$ and $BC$ are equal.

Again, the $\angle fDA = \angle BCA$, because they both stand upon the Arch $AB$: Consequently, the $\angle AfD = \angle ABC$, by *Theorem* 4. Therefore it will be $AC : BC :: AD : Df$, by *Theorem* 13. *Ergo* $\frac{BC \times AD}{AC} = Df$.

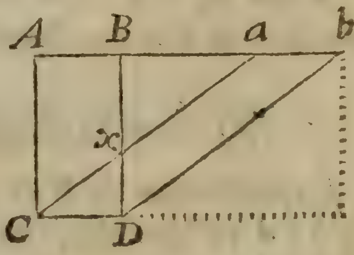
Again, the $\triangle BAf$ and $\triangle ACD$ are alike: For $\angle ABf = \angle ACD$, and $\angle BAf = \angle CAD$, because the $\angle fAD = \angle BAC$, and the $\angle CAf$ is common to both *Triangles*. Consequently, the $\angle AfB = \angle ADC$. Therefore $AC : CD :: AB : Bf$, by *Theorem* 13. *Ergo* $\frac{CD \times AB}{AC} = Bf$. But $Df + Bf = BD$. Consequently, $BC \times AD + CD \times AB = BD \times AC$. Q. E. D.

### THEOREM XX.

*All Parallelograms (whether Right or Oblique angled) that stand upon the same* Base, *or upon equal* Bases, *and betwixt the same Parallels, are equal to one another.* (35. & 36. *e.* 1.)

That is, ▭ $ABCD =$ ▭ $abCD$.

## Demonstration.




Because $AB = CD = ab$, by Supposition, therefore $Aa = Bb$; for $Ba$ is common to both. And because $AC = BD$, and the $\angle A = \angle B$, therefore the $\triangle ACa = \triangle BDb$: And if from both Triangles there be taken the $\triangle Bxa$ common to both, there will remain the *Trapezium* $ABxC = abxD$, per *Axiom* 5.

But

But the *Trapezium* $ABxC + \triangle CxD = \sqsubset ABCD$. and the *Trapezium* $abxD + \triangle CxD = \sqsubset abCD$. consequently $\sqsubset ABCD = \sqsubset abCD$. Q. E. D.

*Corollary.*

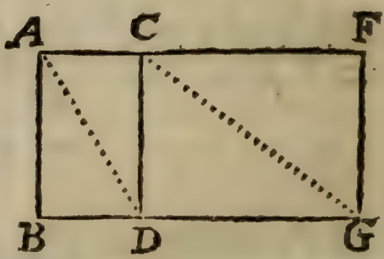
Hence it will be easy to conceive, that all *Triangles* which stand upon the same Base, or upon equal Bases, and between the same Parallels, (viz. *having the same Height)* are equal one to another. (37 & 38 *e.* 1.)

For all *Triangles* are the Halfs of their circumscribing *Parallelograms*; and therefore, if the Wholes be equal, their Halfs will also be equal.

### THEOREM XXI.

Parallelograms *(and consequently* Triangles) *which have the same Heighth, have the same Proportion one to another as their* Bases *have.* (1. *e.* 6.)

### Demonstration.




Draw $AF$ parallel to $BG$, and draw $AB$, $CD$, $FG$ Perpendiculars to them. Then will $BD \times AB = \sqsubset ABCD$. And because $CD = AB$, therefore $DG \times AB = \sqsubset CDFG$, but $BD : DG :: BD \times AB : DG \times AB$. And consequently $\triangle ABD : \triangle CDG :: BD : DG$, &c.

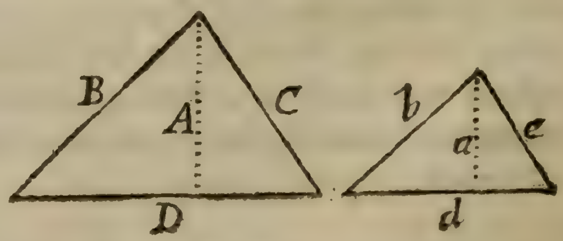
Q. E. D.

### THEOREM XXII.

*Like Triangles are in a duplicate* Ratio *to that of their homologous Sides.* (19. *e.* 6.)

That is, the *Area*'s of like *Triangles* are in *Proportion* one to another as are the *Squares* of their like Sides.

### Demonstration.




Suppose the $\triangle BCD$ and $\triangle bcd$ to be alike, and their like Sides to be those mark'd with the same Letters.

Let

Let $A$ and $a$ be Perpendiculars to the two Bases $D$ and $d$.

Then $\frac{1}{2}\,DA =$ the Area of $\triangle\, BCD$ } By *Lemma* 3, *Page* 303.
And $\frac{1}{2}\,d\,a =$ the Area of $\triangle\, b\,c\,d$ }

| | | |
|---|---|---|
| But | 1 | $B : b :: D : d$ } &c. By *Theorem* 13. |
| And | 2 | $B : b :: A : a$ } |
| Conseq. | 3 | $D : d :: A : a$ |
| 3 ∴ | 4 | $D\,a = d\,A$ |
| $4 \times \frac{1}{2}\,D\,d$ | 5 | $\frac{1}{2}\,D\,D\,d\,a = \frac{1}{2}\,D\,d\,d\,A$. By *Axiom* 3. |
| 5, Hence | 6 | $D\,D : d\,d :: \frac{1}{2}\,DA : \frac{1}{2}\,d\,a$. And so for other Sides. |

Q. E. D.

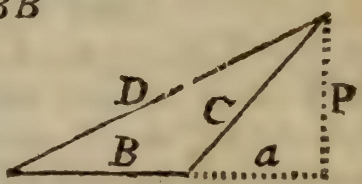
## THEOREM XXIII.

*In every Obtuse-angled Triangle (as BCD) the Square of the Side subtending the obtuse Angle (as D) is greater than the Squares of the other two Sides (B and C) by a double Rectangle made out of one of the Sides (as B) and the Segment or Part of that Side produced (as a) until it meet with the Perpendicular (P) let fall upon it.* (12 *e.* 2.)

That is, $D\,D = B\,B + C\,C + 2\,B\,a$.

### Demonstration.

| | | |
|---|---|---|
| First | 1 | $DD = PP + aa + 2\,B\,a + BB$ |
| And | 2 | $CC = PP + aa$ |
| 1 — 2 | 3 | $DD - CC = 2\,Ba + BB$ |
| 1 + CC | 4 | $DD = BB + CC + 2\,B\,a$ |




Q. E. D.

*Corollary.*

Hence it is evident, that, if the Sides of any *Obtuse-angled Triangle* are given, the *Segment (a)* of the Side produced (or the Perpendicular *P)* may be easily found.

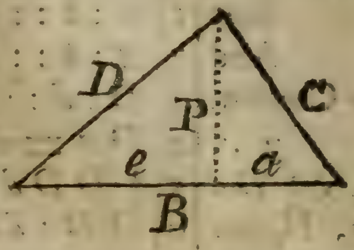
## THEOREM XXIV.

*If a* Perpendicular *(as P) be let fall into any Acute-angled Triangle (as BCD), the Square of either of the Two Sides (as D) is less than the Squares of the other Side, and that Side upon which the* Perpendicular *falls* (viz. *C and B) by a double Rectangle made of the Side B, and that Segment or Part of it* (viz. *a) which lies next to the Side C.* (13. *e.* 2.)

That is, $DD + 2\,Ba = BB + CC$.

## Demonstration.

| | | |
|---|---|---|
| First | 1 | $DD = PP + ee$ } By *Theo.* 11. |
| And | 2 | $CC = PP + aa$ } By *Theo.* 11. |
| But | 3 | $B - a = e$, by Figure. |
| 3 ⊛ 2 | 4 | $BB - 2\,Ba + aa = ee.$ |
| 4 — $aa$ | 5 | $BB - 2\,Ba = ee - aa.$ |
| 1 — 2 | 6 | $DD - CC = ee - aa.$ |
| 5, 6 | 7 | $DD - CC = BB - 2\,Ba.$ |
| 7 ± | 8 | $DD + 2\,Ba = BB + CC.$ |




Q. E. D.

*Corollary.*

Hence it follows, that, if the Sides of any *Acute-angled Triangle* be known, the Perpendicular $P$, and the Segments of the Side whereon it falls *(viz. a, e.)* may be easily found.

---

## CHAP. IV.

*The* **Solution** *of several Easy* **Problems** *in plain Geometry, whereby the Learner may (in Part) perceive the Application or Use of the foregoing Theorems.*

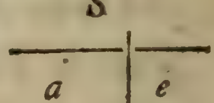
NOTE, *when a Line, or the Side of any plain Triangle, is any Way cut into two or more Parts, either by a Perpendicular Line let fall upon it, or otherwise, those Parts are usually call'd* Segments; *and so much as one of those Parts is longer than the other, is call'd the* Difference of the Segments.

*And when any Side of a Triangle, or any Segment of its Side is given, 'tis usually mark'd with a small Line cross it, thus:* ——|—— *and those Sides or Parts of Sides, that are sought, are marked with four Points, thus:* —∷—

### PROBLEM I.

*To cut or divide a given Right-line (as S) into Extreme and Mean Proportion.* (11. *e.* 2.)

That is, to *divide a Line* so, that the *Square* of the *greater Segment* (or *Part*) $a$, may be equal to the *Rectangle* made of the whole Line $S$, and the lesser Segment $e$.



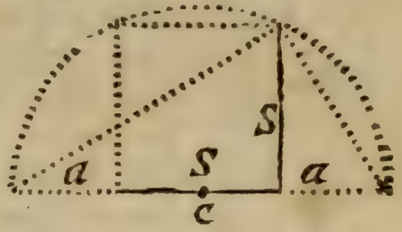

| | | |
|---|---|---|
| *Viz.* | 1 | $Se = aa$, by the *Problem.* |
| And | 2 | $S - a = e$, for $S = a + e$. |

1 ÷ S

| | | |
|---|---|---|
| 1 ÷ $S$ | 3 | $\frac{aa}{S} = e$ |
| 2 and 3 | 4 | $\frac{aa}{S} = S - a$. By *Axiom* 5. |
| 4 × $S$ | 5 | $aa = SS - Sa$ |
| 5 + $Sa$ | 6 | $aa + Sa = SS$ |
| 6, folved | 7 | $a = \sqrt{SS + \frac{1}{4}SS} - \frac{1}{2}S$. See *Pages* 195, 196. |

Note, *The last* Problem *cannot be truly anfwered by Numbers, but Geometrically it may be performed, thus :*

1. Make a Square whofe Side is = $S$ the given Line, and bifect one of its Sides in the Middle, as at $C$; upon the Point $C$ defcribe fuch a Semicircle as will pafs through the remoteft Points of the Square, and compleat its Diameter.




2. Then will either Part of the Diameter, on each End of the Side $S$, be = $a$, the *greater Segment* fought.

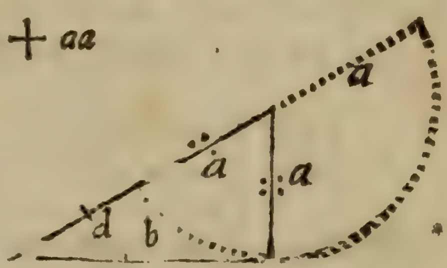
But $a + S : S :: S : a$. By *Theorem* 13.

*Ergo*, $aa + Sa = SS$. Which was to be done.

## PROBLEM II.

*The* Bafe *of any Right-angled Triangle, and the Difference between the* Hypothenufe *and* Cathetus *being given, to find the* Cathetus, &c.

| | | |
|---|---|---|
| Let | 1 | $b = 72$ |
| | 2 | $d = 32$ |
| And | 3 | $a =$ *Cathetus* fought |
| Then | 4 | $bb + aa = dd + 2da + aa$ By *Theorem* 11. |
| 4 − $aa$ | 5 | $bb = dd + 2da$ |
| 5 − $dd$ | 6 | $2da = bb - dd$ |
| 6 ÷ $2d$ | 7 | $a = \frac{bb - dd}{2d} = 65$ |
| Or, | 8 | $b : d + 2a :: d : b$. By *Theorem* 13. |
| 8 ∵ | 9 | $bb = dd + 2da$. As before at the 5th Step. |

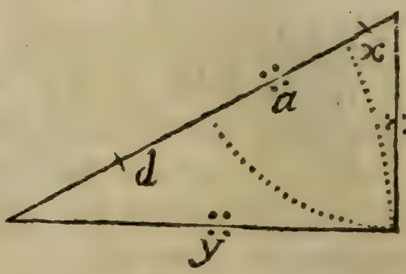


Here you see that either Way raises the same *Æquation*; neither is there any constant *Method* or *Road* to be observed in solving *Geometrical Problems*, but every one makes Use of such Ways and Theorems as happen to come first into their Mind, the Result being every Way the same.

## PROBLEM III.

*The Difference between the* Base *and* Hypothenuse *of any Right-angled Triangle, and the Difference between the* Cathetus *and* Hypothenuse *being both given, to find the Triangle.*

| | | |
|---|---|---|
| Let | 1 | $d = 32$ |
| | 2 | $x = 25$ |
| And | 3 | $d + x + a =$ the *Hypot.* |
| Then | 4 | $d + a = y$ } by the *Probl.* |
| | 5 | $x + a = e$ } |
| 4 $\odot^2$ | 6 | $dd + 2da + aa = yy$ |
| 5 $\odot^2$ | 7 | $xx + 2xa + aa = ee$ |
| 3 $\odot^2$ | 8 | $dd + 2dx + 2da + 2xa + xx + aa = \square$ *Hypothenuse.* |
| 6+7 | 9 | $dd + 2da + 2xa + xx + 2aa = yy + ee.$ |



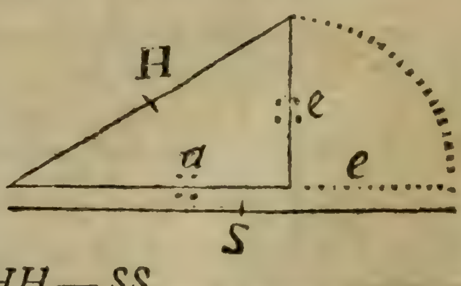

The two last Steps are equal, by *Theorem* 11. Consequently, if those Things that are equal in both be taken away, the Remainders will be equal. By *Axiom* 2.

| | | |
|---|---|---|
| That is | 10 | $aa = 2dx = 1600$ |
| 10 $\text{ɯ}^2$ | 11 | $a = \sqrt{2dx} = 40$ |
| 1 + 11 | 12 | $d + a = 72 = y$ *The* Base. |
| 2 + 11 | 13 | $x + a = 65 = e$ *The* Cathetus. |
| 1+2+11 | 14 | $d + x + a = 97$ *The* Hypothenuse. |

## PROBLEM IV.

*The* Hypothenuse, *and the Sum of the other two Sides, of any Right-angled Triangle, being given, thence to find the Sides.*

| | | |
|---|---|---|
| Let | 1 | $H = 97$ |
| And | 2 | $a + e = S = 137$ |
| By *Fig.* | 3 | $aa + ee = HH$ |
| 2 $\odot^2$ | 4 | $aa + 2ae + ee = SS$ |
| 4 — 3 | 5 | $2ae = SS - HH$ |
| 3 — 5 | 6 | $aa - 2ae + ee = 2HH - SS$ |
| 6 $\text{ɯ}^2$ | 7 | $a - e = \sqrt{2HH - SS}$ |

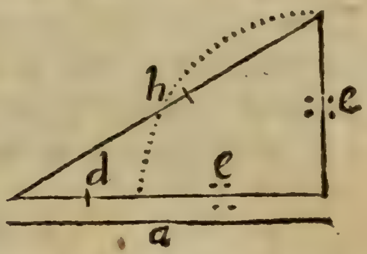



2 + 7

| | | |
|---|---|---|
| 2 + 7 | 8 | $2a = S + \sqrt{2HH - SS} = 144$ |
| 8 + 2 | 9 | $a = \frac{S + \sqrt{2HH - SS}}{2} = 72$ *The* Base *required.* |
| 2 — 9 | 10 | $e = \frac{S - \sqrt{2HH - SS}}{2} = 65$ *The* Cathetus. |

## PROBLEM V.

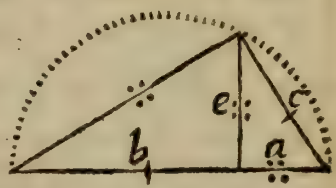
*The* Hypothenuse, *and the Difference of the other two Sides of any Right-angled Triangle being given, to find the Sides.*




| | | |
|---|---|---|
| Let | 1 | $h = 97$ As before. |
| And | 2 | $a - e = d = 7$ Quere $a$ |
| By *Fig.* | 3 | $aa + ee = hh$ |
| 2 ⊛² | 4 | $aa - 2ae + ee = dd$ |
| 3 — 4 | 5 | $2ae = hh - dd$ |
| 3 + 5 | 6 | $aa + 2ae + ee = 2hh - dd$ |
| 6 ш² | 7 | $a + e = \sqrt{2hh - dd}$ |
| 2 + 7 | 8 | $2a = d + \sqrt{2hh - dd} = 144$ |
| 8 ÷ 2 | 9 | $a = 72$ |
| 7 — 2 | 10 | $2e = \sqrt{2hh - dd} : - d = 130$ |
| 1 ÷ 2 | 11 | $e = 65$ |

## PROBLEM VI.

*In any Right-angled Triangle, either the* Base, *or* Cathetus, *and the alternate Segment of the* Hypothenuse *made by a Perpendicular let fall from the Right-angle, being given, to find the other Segment.*

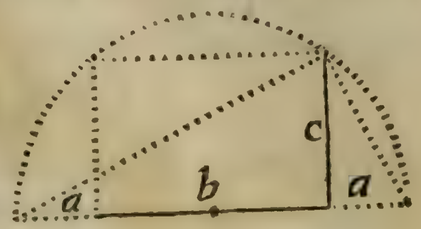



| | | |
|---|---|---|
| Let | 1 | $e = 45$ *The* Cathetus |
| And | 2 | $b = 48$ *The* alternate *Segm.* |
| Then | 3 | $b : e :: e : a$ Quere $a$ |
| 3 ∵ | 4 | $ba = ee$ |
| Again, | 5 | $cc - aa = ee$. By *Theor.* 11. |
| 4, 5 | 6 | $ba = cc - aa$ |
| 6 + *aa* | 7 | $aa + ba = cc$ |
| 7, C □ | 8 | $aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb$ |
| 8 ш² | 9 | $a + \frac{1}{2}b = \sqrt{cc + bb}$ |
| 9 — $\frac{1}{2}b$ | 10 | $a = \sqrt{cc + \frac{1}{4}bb} : - \frac{1}{2}b = 27$ And so on for $e$, &c. |

 I shall

I shall now shew the *Geometrical Construction* (or *Solution)* of the three Cases of *Quadratick Æquations* promised in 202. Let the first Example be that above, *viz.* $aa + ba = cc$. Case 1.

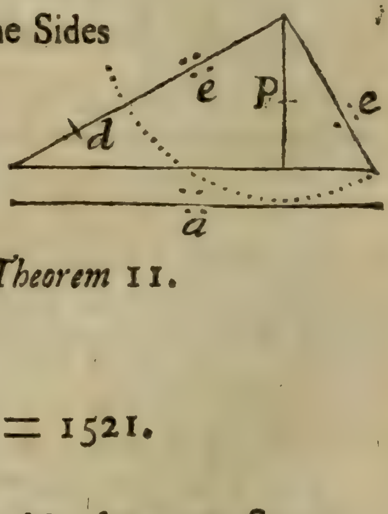
Make the Co-efficient $b$, and the *Root* of the *Resolvend* (which is here) $c$, into a *Right-angled Parallelogram.* And upon the middle Point of the Side $= b$, describe such a Semicircle, as will pass through the remotest Points or Angles of the *Parallelogram*, compleating its Diameter, as in the annexed *Scheme.*




Then will either Part of the Diameter, on each End, be equal to $a$; the other Part will be $a + b$, and the Side $c$ will be a *mean Proportional* between them: That is, $a + b : c :: c : a$. By *Theorem* 13, consequently $aa + ba = cc$. Which was to be done.

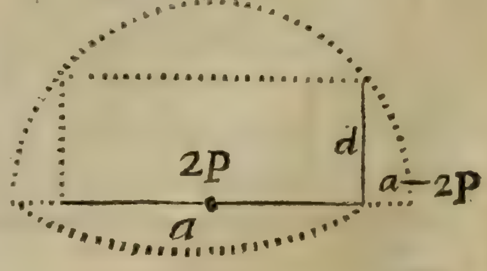
## PROBLEM VII.

*The Difference between the* Base *and* Cathetus *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuse, *being given; thence to find the* Hypothenuse, *&c.*




| | | |
|---|---|---|
| Let | 1 | $d = 15$ The Difference of the Sides |
| And | 2 | $p = 36$ |
| Quere $a$ | 3 | $a =$ *The* Hypothenuse. |
| By *Fig.* | 4 | $d + e : p :: a : e$ |
| 4 ∵ | 5 | $de + ee = pa$ |
| Again, | 6 | $dd + 2de + 2ee = aa$. By *Theorem* 11. |
| $5 \times 2$ | 7 | $dde + 2ee = 2pa$ |
| $6 - 7$ | 8 | $dd = aa - 2pa$. Case 2. |
| 8 C □ | 9 | $aa - 2pa + pp = dd + pp = 1521$. |
| 9 ш² | 10 | $a - p = \sqrt{dd + pp} = 39$ |
| $10 + p$ | 11 | $a = p + \sqrt{dd + pp} = 75$, &c. for $e$. per *Step* 5. |

*The Geometrical Construction of this* Case 2, viz. $aa - 2pa = dd$ may be performed in the very same Manner as the *last Case* was; that is, by making a Right-angled Parallelogram of the Co-efficient $2p$ and the $\sqrt{dd}$, viz. $d$, &c. As in the annexed Figure.

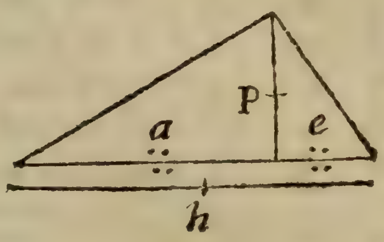



Then

Then will the *greater* Part of the Diameter to one End of the *Parallelogram* be $= a$, and the *leſſer* Part will be $a - 2p$. For $a : d :: d : a - 2p$ by *Theorem* 13. Conſequently, $aa - 2pa = dd$. Which was to be done.

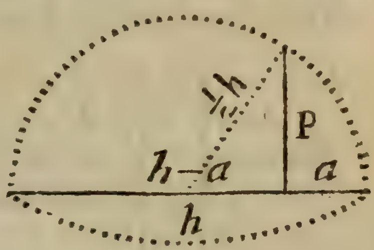
## PROBLEM VIII.

*The* Hypothenuſe *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuſe, *being given, to find the greater Segment of the* Hypothenuſe, *&c.*




| | | |
|---|---|---|
| Let | 1 | $h = 75$ *The* Hypothenuſe |
| And | 2 | $p = 36$ |
| Then | 3 | $a + e = h$ Quere $a$ |
| per *Fig.* | 4 | $a : p :: p : e$ |
| 4 ∴ | 5 | $\frac{pp}{a} = e$ |
| 3 — $a$ | 6 | $h - a = e$ |
| 5, 6 | 7 | $h - a = \frac{pp}{a}$ |
| 7 × $a$ | 8 | $ha - aa = pp$ Caſe 3. |
| 8 + | 9 | $aa - ha = -pp$ |
| 9 $\overline{C}$ □ | 10 | $aa - ha + \frac{1}{4}hh = \frac{1}{4}hh - pp = 110,25$ |
| 10 $\omega^2$ | 11 | $a - \frac{1}{2}h = \sqrt{\frac{1}{4}hh - pp} = 10,5$ |
| 11 + $\frac{1}{2}h$ | 12 | $a = \frac{1}{4}h \pm \sqrt{\frac{1}{2}hh - pp} = 48$. Or, $a = 27$. |

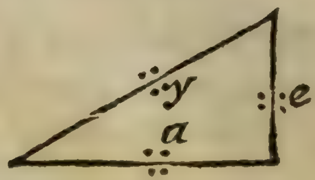
The Geometrical Conſtruction of *Caſe* 3, viz. $ha - aa = pp$, may be thus performed: Draw a *Right-line* (of any convenient Length at Pleaſure) and near its Middle erect a Perpendicular $= p$, *viz.* of the ſame Length with the *Root* of the *Reſolvend.* From the top Point or upper End of that Perpendicular, ſet off Half the Length of the Co-efficient, *viz.* $\frac{h}{2}$ and upon the Point where $\frac{h}{2}$ juſt touches the firſt Line (with the ſame Diſtance) deſcribe a Semicircle; then will its Diameter $h$ be cut by the Perpendicular $p$ into two Segments, which are the two Values of the Root $a$, *viz.* the *greater* and *leſſer Roots*, both taken together, being always equal to the Co-efficient: (*vide* Page 201.) For $h - a : p :: p : a$ by *Theorem* 13. Ergo, $ha - aa = pp$. Which was to be done.



## PROBLEM IX.

*The* Perimeter, *i. e. the Sum of all the three Sides of any Right-angled Triangle, and its* Area, *being given, thence to find each Side.*

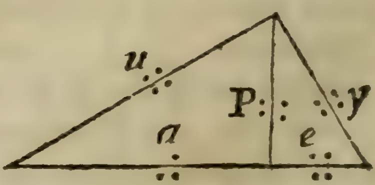
| | | |
|---|---|---|
| *Viz.* Let | 1 | $a + e + y = s = 234$ The *Sum* of the *Sides.* |
| And | 2 | $\frac{1}{2} ae = A$ The *Area* $= 2340$ |
| Again, | 3 | $aa + ee = yy$ By Figure |
| 2 × 4 | 4 | $2ae = 4A$ |
| 3 + 4 | 5 | $aa + 2ae + ee = yy + 4A$ |
| 1 — $y$ | 6 | $a + e = s - y$ |
| 6 ⊛$^{2}$ | 7 | $aa + 2ae + ee = ss - 2sy + yy$ |
| 5, 7 | 8 | $yy + 4A = ss - 2sy + yy$ |
| 8 ± | 9 | $2sy = ss - 4A = 45396$ |
| 9 ÷ $2s$ | 10 | $y = \frac{ss - 4A}{2s} = \frac{1}{2}s - \frac{2A}{s} = 97$ *The* Hypothenuse. |
| 6, 10 | 11 | $a + e = s - y = 137$ |
| 3 — 4 | 12 | $aa - 2ae + ee = yy - 4A = 49$ |
| 12 ɯ$^{2}$ | 13 | $a - e = \sqrt{49} = 7$ |
| 11 + 13 | 14 | $2a = 137 + 7 = 144$ |
| 13 ÷ 2 | 15 | $a = 72$ *The* Base. |
| 11 — 15 | 16 | $e = 137 - 72 = 65$ *The* Cathetus. |




## PROBLEM X.

*In any Right-angled Triangle a* Perpendicular *being let fall from the Right-angle upon the* Hypothenuse; *if the Sum of each Segment, when added to its adjacent or next Side, be given, thence to find each Side, and the Segments.*

| | | |
|---|---|---|
| *Viz.* If | 1 | $a + u = s = 108$ |
| And | 2 | $e + y = z = 72$ |
| To find | | $a, e, u, y,$ and $p$ |
| 1 — $a$ | 3 | $u = s - a$ |
| 3 ⊛$^{2}$ | 4 | $uu = ss - 2sa + aa$ |
| 4 — $aa$ | 5 | $uu - aa = ss - 2sa = pp$ |
| 2 — $e$ | 6 | $z - e = y$ |
| 6 ⊛$^{2}$ | 7 | $zz - 2ze + ee = yy$ |
| 7 — $ee$ | 8 | $zz - 2ze = yy - ee = pp$ |
| 5, 8 | 9 | $zz - 2ze = ss - 2sa$ |
| By *Fig* | 10 | $a : p :: p : e$ |
| 10 ∴ | 11 | $ae = pp$ |
| 5, 11 | 12 | $ae = ss - 2sa$ |



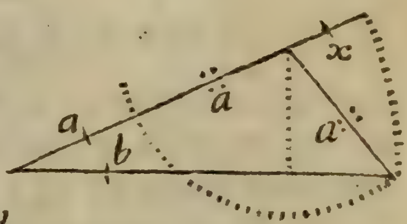

12 ÷ $a$

| | | |
|---|---|---|
| 12 $\div$ $a$ | 13 | $e = \frac{ss - 2sa}{a}$ |
| 13 $\times$ $2z$ | 14 | $2ze = \frac{2zss - 4zsa}{a}$ |
| 9 + 14 | 15 | $zz = ss - 2sa + \frac{2zss + 4zsa}{a}$ |
| 15 $\times$ $a$ | 16 | $zza = ssa - 2saa + 2zss - 4zsa$ |
| 16 $\pm$ | 17 | $2saa + zza + 4zsa - ssa = 2zss$ |
| 17 $\div$ $2s$ | 18 | $aa + \frac{zza}{2s} + 2z - \frac{1}{2}sa = zs$ |
| Substitute | 19 | $2x = \frac{zz}{2s} + 2z - \frac{1}{2}s = 114$ |
| Then | 20 | $aa + 2xa = zs = 7776$ |
| 20 C □ | 21 | $aa + 2xa + \overline{xx = zs} + xx = 11025$ |
| 21 $\omega^2$ | 22 | $a + x = \sqrt{zs + xx} = 105$ |
| 22 $-$ $x$ | 23 | $a = \sqrt{zs + xx} : - x = 48$ |
| 1 $-$ 23 | 24 | $u = 60 =$ *The* Base. |
| per 13 | 25 | $e = \frac{ss}{a} - 2s = 27$ |
| 2 $-$ 25 | 26 | $y = 45 =$ *the* Cathetus. |
| 23 + 25 | 27 | $a + e = 75 =$ *the* Hypothenuse. |

## PROBLEM XI.

*The Difference of the Sides of any Oblique-angled plain Triangle, the Difference of the Segments of the* Base, *and the Difference between the greater Side and the* Base, *being given, to find the* Base, *&c.*

| | | |
|---|---|---|
| Let | 1 | $d =$ the Difference of the Sides $= 405$ |
| | 2 | $b =$ the Difference of the Segments $= 495$ |
| | 3 | $x = 165$ the Differ. of the greater Side and Base |
| And | 4 | $a =$ the least Side |
| Then | 5 | $d + a + x =$ the Base |
| And | 6 | $d + a + x : d + 2a :: d : b$ By *Theorem* 16. |
| 6 ∵ | 7 | $db + ba + bx = dd + 2da$ |
| 7 $\pm$ | 8 | $2da - ba = db + bx - dd$ |
| 8 $\div$ $\overline{2d - b}$ | 9 | $a = \frac{db + bx - dd}{2d - b} = \frac{118125}{315} = 375$ |
| 1 + 9 | 10 | $d + a = 780 =$ *the* greatest Side. |
| 3 + 10 | 11 | $d + a + x = 945 =$ *the* Base. |

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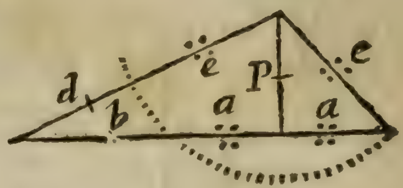
## PROBLEM XII.

*The Difference of the Sides of any plain Triangle, the Difference of the Segments of the* Base, *and the* Perpendicular *let fall from the Vertical Angle, being given, thence to find all the Sides.*



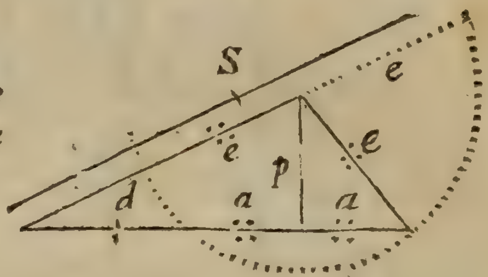

| | | |
|---|---|---|
| Let | 1 | $d = 405$ } as before |
| | 2 | $b = 495$ } as before |
| And | 3 | $p = 300$ |
| *Quere* | 4 | $a =$ the lesser *Segment.* |
| Then | 5 | $b + 2a : d + 2e :: d : b.$ |
| 5 ∵ | 6 | $bb + 2ba = dd + 2de$ |
| 6 $-$ $dd$ | 7 | $bb - dd + 2ba = 2de$ |
| Substitute | 8 | $2x = bb - dd = 81000$ |
| 7, 8 | 9 | $2x + 2ba = 2de$ |
| 9 $\div$ $2d$ | 10 | $\frac{x + ba}{d} = e$ |
| But | 11 | $pp + aa = ee$ By *Theorem* 11. |
| 10 $⊙^2$ | 12 | $\frac{xx + 2xba + bbaa}{dd} = ee$ |
| 11, 12 | 13 | $\frac{xx + 2xba + bbaa}{dd} = pp + aa$ |
| 13 $\times$ $dd$ | 14 | $xx + 2xba + bbaa = ppdd + ddaa$ |
| 14 $\pm$ | 15 | $bbaa - ddaa + 2xba = ppdd - xx$ |
| 8, 15 | 16 | $2xaa + 2xba = ppdd - xx$ |
| 16 $\div$ $2x$ | 17 | $aa + ba = \frac{ppdd}{2x} - \frac{1}{2}x$ |
| 17 C □ | 18 | $aa + ba + \frac{1}{4}bb = \frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x$ |
| 18 $\sqrt{\ }$ | 19 | $a + \frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x}$ |
| 19 $-$ $\frac{1}{2}b$ | 20 | $a = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x} : -\frac{1}{2}b = 225$ |
| 20 $\times$ 2 | 21 | $2a = 450$ |
| 2 $+$ 21 | 22 | $b + 2a = 945$ *the* Base. |
| 10, *Num.* | 23 | $e = 375 =$ *the* lesser Side. |
| 1 $+$ 23 | 24 | $d + e = 780 =$ *the* greater Side. |

## PROBLEM XIII.

*The Sum of the two Sides of any plain Triangle, the Difference of the Segments of the* Base, *and the* Perpendicular *let fall from the Vertical*

*Vertical Angle upon the* Base, *being given, thence to find the* Base *and the Sides.*

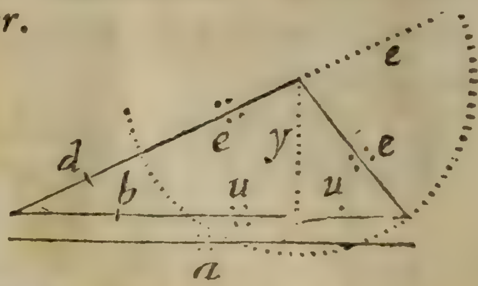
| | | |
|---|---|---|
| Let | 1 | $s = 1155$ the *Sum* of the *Sides.* |
| | 2 | $d = 495$ the *Difference* of the *Segments.* |
| | 3 | $p = 300$ the *Perpendicular.* |
| Put | 4 | $a =$ the least *Segment.* |
| | 5 | $e =$ the least *Side.* |
| Then | 6 | $d + 2a =$ the *Base.* |
| And | 7 | $s - 2e =$ the *Difference* of the *Sides.* |
| Per *Fig.* | 8 | $d + 2a : s :: s - 2e : d$ |
| | 9 | $aa - + pp = ee$ |
| 9 $\text{w}^2$ | 10 | $\sqrt{aa + pp} = e$ |
| 8 ∵ | 11 | $dd + 2da = ss - 2se$ |
| 11 $\pm$ | 12 | $2se = ss - dd - 2da$ |
| Suppose | 13 | $2x = ss - dd$ |
| Then | 14 | $2se = 2x - 2da$ |
| 14 ÷ 2s | 15 | $e = \frac{x - da}{s}$ |
| 10, 15 | 16 | $\frac{x - da}{s} = \sqrt{aa + pp}$ |
| 16 $\odot^2$ | 17 | $\frac{xx - 2xda + ddaa}{ss} = aa + pp$ |
| 17 × ss | 18 | $xx - 2xda + ddaa = ssaa + sspp$ |
| 18 $\pm$ | 19 | $ssaa - ddaa + 2xda = xx - sspp$ |
| 13 19 | 20 | $2xaa + 2xda = xx - sspp$ |
| 20 ÷ 2x | 21 | $aa + da = \frac{1}{2}x - \frac{sspp}{2x}$, &c. as before. |
| 21, hence | 22 | $a = 225$ |
| 22 × 2 | 23 | $2a = 450$ |
| 2 + 23 | 24 | $d + 2a = 945$ the *Base.* |
| 10, *Num.* | 25 | $e = 375$ the lesser Side. |
| 1 − 25 | 26 | $s - e = 780$ the greater Side. |




## PROBLEM XIV.

*The Area of any Oblique-angled plain* Triangle, *the Difference of the Sides, and the Difference of the Segments of the* Base, *being given, thence to find the* Base, *&c.*

| | | |
|---|---|---|
| Let | 1 | $A = 141750 =$ the *Area.* |
| | 2 | $d = 405$ |
| | 3 | $b = 495$ |




| | | |
|---|---|---|
| Put | 4 | $y =$ the *Perpendicular.* |
| | 5 | $a =$ the *Base.* |
| Then | 6 | $\frac{1}{2} ya = A$ |
| Per *Fig.* | 7 | $a : d + 2e :: d : b$ |
| 7 ∵ | 8 | $ba = dd + 2de$ |
| 8 — $dd$ | 9 | $ba - dd = 2de$ |
| 9 $\circledcirc^2$ | 10 | $bbaa - 2ddba + dddd = 4ddee$ |
| Per *Fig.* | 11 | $\frac{a - b}{2} = u$ the lesser Segment of the *Base.* |
| 11 $\circledcirc^2$ | 12 | $\frac{aa - 2ba + bb}{4} = uu$ |
| 6 × 2 | 13 | $ya = 2A$ |
| 13 ÷ $a$ | 14 | $y = \frac{2A}{a}$ |
| 14 $\circledcirc^2$ | 15 | $yy = \frac{4AA}{aa}$ |
| Per *Fig.* | 16 | $yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$ |
| 10 ÷ $4dd$ | 17 | $\frac{bbaa - 2ddba + dddd}{4dd} = ee$ |
| 16, 17 | 18 | $\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$ |
| 18 × $aa$ | 19 | $\frac{bba^4 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bba^2}{4}$ |
| 19 × $4dd$ | 20 | $bba^4 - 2ddba^3 + d^4aa = 16AAdd + dda^4 - 2ddba^3 + ddbba^2$ |
| 20 ± | 21 | $bba^4 - dda^4 + d^4a^2 - ddbba^2 = 16AAdd$ |
| 21 ÷ | 22 | $aaaa - ddaa = \frac{16AAdd}{bb - dd}$ |
| 22 C □ | 23 | $aaaa - ddaa + \frac{1}{4}dddd = \frac{16AADD}{bb - dd} + \frac{1}{4}d^4$ |
| 23 $\omega^2$ | 24 | $aa - \frac{1}{2}dd = \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}d^4}$ |
| 24 + $\frac{1}{2}dd$ | 25 | $aa = \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}d^4}$ |
| 25 $\omega^2$ | 26 | $a = \sqrt{:} \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}d^4} = 945$ |

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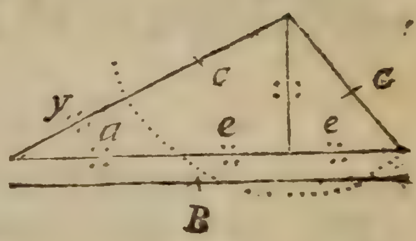
## PROBLEM XV.

*There is an Oblique-angled plain Triangle, wherein a* Perpendicular *is let fall from the Vertical Angle upon the* Base; *the least Side and the* Base *are given; and the Rectangle of the Difference of the Sides into the least Side is equal to the Square of the Difference of the Segments of the* Base; *'Tis requir'd to find the Segments of the* Base, *&c.*

| | | |
|---|---|---|
| Let | 1 | $c = 56$ = the least Side. |
| | 2 | $B = 92$ = the *Base*. |
| And | 3 | $a + 2e = B$ |
| Put | 4 | $y$ = the Difference of the *Sides*. |
| Then | 5 | $cy = aa$ by the Question. |
| By *Figure* | 6 | $B : 2c + y :: y : a$, for $B = a + 2e$ |
| 6 ∵ | 7 | $Ba = 2cy + yy$ |
| 5 × 2 | 8 | $2cy = aa$ |
| 7 — 8 | 9 | $Ba - 2aa = yy$ |
| 5 $\circledcirc^2$ | 10 | $ccyy = aaaa$ |
| 10 ÷ $cc$ | 11 | $yy = \frac{aaaa}{cc}$ |
| 9, 11 | 12 | $Ba - 2aa = \frac{aaaa}{cc}$ |
| 12 × $cc$ | 13 | $ccBa - 2ccaa = aaaa$ |
| 13 ÷ $a$ | 14 | $ccB - 2cca = aaa$ |
| 14 + $2cca$ | 15 | $aaa + 2cca = ccB$ |
| 15 in *Num.* | 16 | $aaa + 6272a = 288512$ |



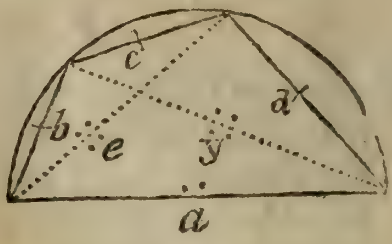

The *Value* of $a$, in this *Æquation*, may be found as in the Examples *Page* 238, *viz.* by putting $r + e = a$, &c. as in those Examples you will find $a = 37{,}55502$, &c.

## PROBLEM XVI.

*The three Chords or Subtenses of three Arches compleating a* Semicircle *being each given, thence to find the Diameter of that* Circle. That is,

*Any* Trapezium *being inscrib'd in a* Semicircle, *if one of its Sides be the Diameter, and the other three Sides be given, thence to find the Diameter or fourth Side.*

| | | |
|---|---|---|
| Let | 1 | $b = 3$ the 3 Sides. |
| | 2 | $c = 4$ |
| | 3 | $d = 5$ |
| *Quære* | 4 | $a =$ the *Diam.* sought |
| | | Draw the two *Diagonals* $e$ and $y$ |
| Then | 5 | $ca + bd = ey$. By *Theorem* 19. |
| And | 6 | $aa - bb = yy$ By *Theorem* 10 and 11. |
| | 7 | $aa - dd = ee$ |
| 5 $\circledS^2$ | 8 | $ccaa + 2bdca + bbdd = eeyy$ |
| 6 $\times$ 7 | 9 | $aaaa - bbaa - ddaa + bbdd = eeyy$ |
| 8, $=$ 9 | 10 | $aaaa - bbaa - ddaa = ccaa + 2bdca$ |
| 10 $-$ $a$ | 11 | $aaa - bba - dda - cca = 2bdc$ |
| 11 $-$ $cca$ | 12 | $aaa - bba - dda - cca = 2bdc$ |
| 12, *Numb.* | 13 | $aaa - 50a = 120$ |

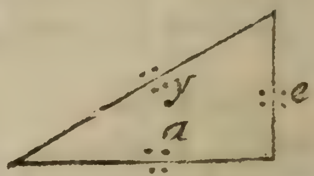



This *Æquation* being solv'd as in *Example* 2, *Page* 240, you will find $a = 8{,}05581$, &c.

## PROBLEM XVII.

*In any Right-angled Triangle, the* Area *and the Sum of the* Hypothenuse, *when added to either Side, being given, thence to find the Sides*, &c.

| | | |
|---|---|---|
| Suppose | 1 | $\frac{ae}{2} = A = 1350$ the *Area.* |
| | 2 | $y + e = s = 120$ the *Sum*, &c. |
| | 3 | Quære $a$, $e$, and $y$ |
| 1 $\times$ 2 | 4 | $ae = 2A$ |
| 4 $\div$ $a$ | 5 | $e = \frac{2A}{a}$ |
| Per *Fig.* | 6 | $aa + ee = yy$ |
| 2 $-$ $e$ | 7 | $y = s - e$ |
| 5, 7 | 8 | $y = s - \frac{2A}{a}$ |
| 8 $\circledS^2$ | 9 | $yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$ |
| 5 $\circledS^2$ | 10 | $ee = \frac{4AA}{aa}$ |
| 10 $+$ $aa$ | 11 | $aa + ee = \frac{4AA}{aa} + aa$ |



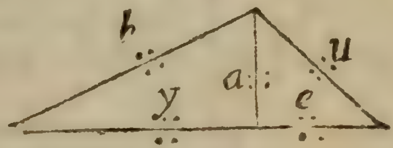
| | | |
|---|---|---|
| 6, 9, 11 | 12 | $\frac{4AA}{aa} + aa = yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$ |
| 12, That is | 13 | $aa = ss - \frac{4sA}{a}$ |
| 13 $\times$ $a$ | 14 | $aaa = ssa - 4sA$ |
| 14 $+$ | 15 | $ssa - aaa = 4sA$ |
| 15, in *Num.* | 16 | $14400a - aaa = 648000$ |

The *Value* of $a$, in this *Æquation*, may be found as in the third Example, *Page* 241; that is, by making $r + e = a$, &c. it will be found that $a = 60$.

## PROBLEM XVIII.

*There is an Oblique-angled plain* Triangle, *wherein a* Perpendicular *is let fall from the* Vertical Angle *upon the* Base; *the Sum of each Segment of the* Base, *when added to its adjacent or next Side, and the* Area *of the* Triangle, *are given, to find the* Perpendicular *and each Side.*

| | | |
|---|---|---|
| Let | 1 | $y + b = z = 1500$ } Quære $y$, $b$, $e$, and $u$ |
| | 2 | $e + u = s = 600$ |
| | 3 | $A =$ the *Area* $= 141750$ |
| And | 4 | $a =$ the *Perpendicular* sought. |
| Then | 5 | $\overline{y+e} \times \frac{1}{2}a = A$ |
| $5 \times 2 \div a$ | 6 | $y + e = \frac{2A}{a}$ |
| Per *Fig.* | 7 | $yy + aa = bb$ |
| | 8 | $ee + aa = uu$ |
| 1 $-$ $y$ | 9 | $b = z - y$ |
| 2 $-$ $e$ | 10 | $u = s - e$ |
| 9 $\odot^2$ | 11 | $bb = zz - 2zy + yy$ |
| 10 $\odot^2$ | 12 | $uu = ss - 2se - ee$ |
| 7, 11 | 13 | $zz - 2zy = aa$ |
| 8, 12 | 14 | $ss - 2se = aa$ |
| 13 $\pm$ | 15 | $zz - aa = 2zy$ |
| 14 $\pm$ | 16 | $ss - aa = 2se$ |
| 15 $\div$ 2z | 17 | $\frac{zz - aa}{2z} = y$ |
| 16 $\div$ 2s | 18 | $\frac{ss - aa}{2s} = e$ |
| 17 $\div$ 18 | 19 | $\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = y + e$ |




Having found the Value of $a$ from the 24th Step, $e$ and $y$ will be easily found by these two Steps, and $b$, $u$, by the 9th and 10th Step.

6, 19

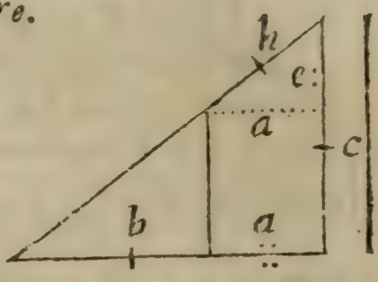
| | | |
|---|---|---|
| 6, 19 | 20 | $\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = \frac{2A}{a}$ |
| 20 × 2z | 21 | $zz - aa + \frac{zzs - zaa}{s} = \frac{4za}{a}$ |
| 21 × s | 22 | $zzs - saa + zss - zaa = \frac{4zA}{a}$ |
| 22 × a | 23 | $zzsa - saaa + zssa - zaaa = 4zAs$ |
| 23, *Numb.* | 24 | $900000a - aaa = 243000000$ |

Here $a = 300$ found as in the last *Problem.*

## PROBLEM XIX.

*There is a Right-angled Triangle, wherein a Right-line is drawn parallel to the* Cathetus; *there is given the* Cathetus, *that Segment of the* Hypothenuse *next to the* Cathetus, *and the alternate Segment of the* Base; *thence to find the* Base, *&c.*

| | | |
|---|---|---|
| viz. Let | 1 | $b = 20 \,.\, c = 24 \,.$ and $h = 15$ |
| Then | 2 | $b + a =$ the *Base.* Quære $a$ |
| Here | 3 | $b + a : c :: a : e$ per *Figure.* |
| And | 4 | $aa + ee = hh$ per *Figure.* |
| 3 ∴ | 5 | $\frac{ca}{b + a} = e$ |
| 5 ©² | 6 | $\frac{ccaa}{bb + 2ba + aa} = ee$ |
| 4 − aa | 7 | $hh - aa = ee$ |
| 6, 7 | 8 | $\frac{ccaa}{bb + 2ba + aa} = hh - aa$ |
| 8 × | 9 | $ccaa = hhbb - bbaa + 2hhba - 2ba^3 + hhaa - a^4$ |
| 9 ± | 10 | $a^4 + 2baaa + ccaa + bbaa - hhaa - 2hhba = hhbb$ |
| That is, | 11 | $aaaa + 40aaa + 751aa - 9000a = 90000$ |




For a *Solution* of this *Æquation*, let it be made

$aaaa + baaa + caa - da = G$ *Viz.* $\begin{cases} b = 40 \;.\; c = 751 \\ d = 9000 \;.\; G = 90000 \end{cases}$

Put $r + e = a$

Then $\left.\begin{cases} r + 4rrre + 6rree = a^4 \\ brrr + 3brre + 3bree = baaa \\ crr + 2cre + cee = caa \\ - dr - de = - da \end{cases}\right\} = G = 90000$

Let $r = 10$

Then

$$\text{Then}\left\{\begin{array}{l} +10000+4000e+600ee \\ +40000+12000e+1200ee \\ +75100+15020e+751ee \\ -90000-9000e \end{array}\right\}=G=90000$$

That is, $35100+22020e+2551ee=90000$

Hence it will be $22020e+2551ee=54900$

Consequently, $8{,}63e+ee=21{,}52=D$

And $\frac{D}{8{,}63+e}=e$

*Operation*, 8,63) 21,52 (2,1 = *e*
+ *e* = 2,1   20

1. *Divisor* = 10   1,52   First $r=10$
2. *Divisor* = 10,7   1,07   $+e=2{,}1$

45 &c. $r+e=12{,}1=r$ for a second Operation, which being involv'd, and multiply'd into the *Co-efficients*, as before, will produce these *Numbers*:

$$\left.\begin{array}{l} +21435{,}8881+7086{,}24e+878{,}46ee \\ +70862{,}4400+17569{,}20e+1452{,}00ee \\ +109953{,}9100+18174{,}20e+751{,}00ee \\ -108900{,}0000-9000{,}00e \end{array}\right\}=C.$$

*Viz.* $93352{,}2381-33829{,}64e+3081{,}46ee=90000$

Here, because $93352{,}2381>90000$ therefore $12{,}1>a$, and therefore it must be made $r-e=a$, which will produce the same *Numbers*, only all the second Signs must be changed.

Thus, $93352{,}2381-33829{,}64e+3081{,}46ee=90000$ from whence will arise this *Æquation*:

$$+33829{,}64e-3081{,}46ee=3352{,}2381$$

Consequently, $10{,}9784e-ee=1{,}08787332=D$

*Operation*, 10,9784) 1,08787332 (0,0999 = *e*
− *e* = ,0999   9792

1. *Divisor* 10,88   108673   Last $r=12{,}1$
2. *Divisor* 10,879   97911   $-e=0{,}0999$
3. *Divisor* 10,8785   1076232   $r-e=12{,}0001=e$

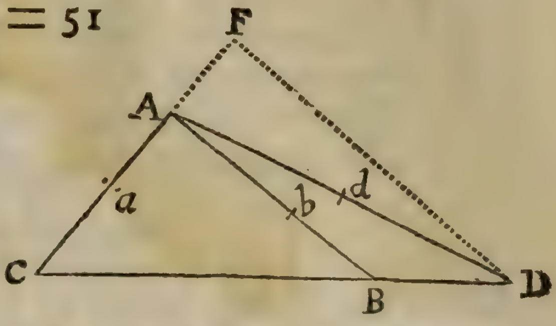
979065

&c.

## PROBLEM XX.

*In the Oblique-angled Triangle CAD, there is given the Side AD, and the Sum of the Sides AC + CD; also within the Triangle is given the*

*the Line AB perpendicular to the Side CA; thence to find the Side CA,* &c.




| | | |
|---|---|---|
| Let | 1 | $CA + CD = s = 51$ |
| | 2 | $AD = d = 32$ |
| | 3 | $AB = b = 21$ |
| And | 4 | $CA = a$ sought. |
| Then | 5 | $s - a = CD$ |
| Suppose | | the *Line DF* *parallel* to $AB$; $CA$ being produced to $F$ |
| Then | | $\triangle CAB$, and $\triangle CFD$ will be alike. |
| And | 6 | $BC : CA :: DC : CF$ |
| But | 7 | $BC = \sqrt{bb + aa}$. Let $AF = e$, and $FD = y$ |
| 6, 7 | 8 | $\sqrt{bb + aa} : a :: s - a : a + e$ |
| 8 ∵ | 9 | $\dfrac{sa - aa}{\sqrt{:bb + aa}} = a + e$ |
| 5 $\circledcirc^2$ | 10 | $ss - 2sa + aa = \square CD$ |
| Per *Fig.* | 11 | $ss - 2sa + aa = aa + 2ae + ee + yy = \square CF + \square FD$ |
| 11 — $aa$ | 12 | $ss - 2sa = 2ae + ee + yy$ |
| But | 13 | $dd = ee + yy = \square AF + \square FD$ |
| 12 — 13 | 14 | $ss - 2sa - dd = 2ae$ |
| Let | 15 | $2x = ss - dd$ |
| 14, 15 | 16 | $x - sa = ae$ |
| 16 ÷ $a$ | 17 | $\dfrac{x - sa}{a} = e$ |
| 17 + $a$ | 18 | $\dfrac{x - sa + aa}{a} = a + e$ |
| 9 $\circledcirc^2$ | 19 | $\dfrac{ssaa - 2saaa + a^4}{bb + aa} = \square\, \overline{a + e}$ |
| 18 $\circledcirc^2$ | 20 | $\dfrac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} = \square\, \overline{a + e}$ |
| 19, 20 | 21 | $\dfrac{ssaa - 2saaa + a^4}{bb + aa} = \dfrac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa}$ |

This *Æquation* being brought out of the *Fractions*, and into *Numbers*, will become $- 2018a^4 + 125409a^3 - 2464230{,}25a^2 + 35468307a = 274183922{,}25$; which being divided by 2018, the *Co-efficient* of the higheſt *Power* of $a$, will be $- a^4 + 62{,}1456a^3 - 1221{,}125a^2 + 17575{,}9697a = 135869{,}138875$, &c.

2 And

And from hence the *Value* of *a* may be found, as in the *last Problem*, due Regard being had to the *Signs* of every *Term*.

This Work of *reducing*, or preparing *Æquations* for a *Solution* by *Division*, hath always been taught both by *ancient* and *modern* Writers of *Algebra*, as a Work so necessary to be done, that they do not so much as give a Hint at the *Solution* of any *adfected Æquation* without it.

Now it very often happens, that, in *dividing* all the *Terms* of an *Æquation*, some of their *Quotients* will not only run into a long *Series*, but also into imperfect *Fractions* (as in this *Equation* above) which renders the *Solution* both tedious and *imperfect*.

To remedy that *Imperfection*, I shall here shew how this *Æquation* (and consequently any other) may be *resolv'd* without such *Division* or *Reduction*.

Let $b = 2018$. $c = 125409$. $d = 2464230{,}25$

$f = 35468307$. And $G = 274183922{,}25$

Then the *precedent Equation* will stand thus:

$$-baaaa + caaa - daa + fa = G$$

Put $r + e = a$ as before.

$$\text{Then will}\left\{\begin{array}{llll} -br^4 & -4brrre & -6brree & = -ba^4 \\ +cr^3 & +3crre & +3cree & = +ca^3 \\ -drr & -2dre & -dee & = -daa \\ +fr & +fe & \ldots\ldots\ldots & = +fa \end{array}\right\} = G$$

This is plain and easily conceived. The next Thing will be, how to estimate the first *Value* of *r*; and, to perform that, let *G* be *divided* by *b*, only so far as to determine how many *Places* of whole *Numbers* there will be in the *Quotient*; consequently, how many *Points* there must be *(according to the Height of the Æquation.)*

Thus $b = 2018)$ $G = 274183922{,}25$ $(130000$

2018

7238, &c.

Now from hence one may as easily guess at the *Value* of *r*, as if all the *Terms* had been *divided*. That is, I suppose $r = 10$, which being *involved*, &c. as the *Letters above direct*, will be

$$\left.\begin{array}{l} -\ 20880000 - 8072000e - 121080000ee \\ +\ 125409000 + 37622700e + 3762270ee \\ -\ 246423025 - 49284605e - 2464230,25ee \\ +\ 354683070 + 35468307e \end{array}\right\} = G$$

*Viz.* $213489045 + 15734402e + 87239{,}75ee = 2741839$ &c.

Hence $15734402e + 87239{,}75ee = 60694877{,}25$

*Consequently*, $180{,}3e + ee = 695{,}72 = D$

And $\frac{D}{180{,}3 + e} = e$

*Operation.* $180{,}3)\ 695{,}72\ (3{,}7 = e$

| | | | |
|---|---|---|---|
| $+ e =$ | 3,7) | 549 | |
| 1. *Divisor* = | 183 | 146,72 | First $r = 10$ |
| 2. *Divisor* = | 184,0 | 128,80 | $+ e = 3{,}7$ |
| | | &c. | $r + e = 13{,}7 = r$ for a |

second *Operation*, with which you may proceed, as in the last *Problem*, and so on to a third *Operation*, if Occasion require such Exactness. But this may be sufficient to shew the Method of resolving any *adfected Æquation*, without reducing it; which is not only very exact, but also very ready in Practice, as will fully appear in the last *Chapter* of this *Part*, concerning the *Periphery* and *Area* of the *Circle*, &c. wherein you will find a farther Improvement in the *Numerical Solution* of *High Equations* than hath hitherto been publish'd.

## CHAP. V.

### *Practical* Problems, *and* Rules *for finding the* Superficial Contents, *or* Area's *of Right lin'd Figures.*

BEfore I proceed to the following *Problems*, it may be convenient to acquaint the *Learner*, that the *Superficies* or *Area* of any *Figure*, whether it be *Right-lin'd* or *Circular*, is *compos'd* or made up of *Squares*, either *greater* or *less*, according to the different *Measures* by which the *Dimensions* of the *Figures* are taken or *measur'd*.

That is, if the *Dimensions* are taken in *Inches*, the *Area* will be *compos'd* of *square Inches*; if the *Dimensions* are taken in *Feet*, the *Area* will be *compos'd* of *square Feet*; if in *Yards*, the *Area* will be *square Yards*; and if the *Dimensions* are taken by *Poles or Perches* (*as in surveying of Land*, &c.) then the *Area* will be *square Perches*, &c.

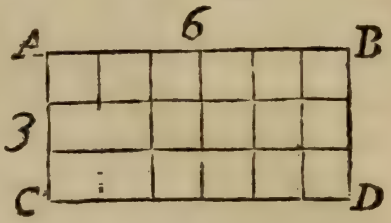
&c. These Things being understood, and the *Definitions* in the 283d and 284th *Pages* well consider'd, will help to render the following *Rules* very easy.

## PROBLEM I.

### *To find the Superficial Content, or Area of a* Square ; *or of any Right-angled* Parallelogram.

RULE. { *Multiply the Length into its Breadth, and the Product will be the Area requir'd.* (*See* Lemma 1. Page 302.)

*Example.* Suppose the *Line* $AB = 6$ *Yards*, and the *Breadth* $AC$ or $BD$ $= 3$ *Yards*, then $AB \times AC = 6 \times 3$ $= 18$ will be the Number of *square Yards* contain'd in the *Area* of the *Parallelogram* $ABCD$. This is so evident by the *Figure* only, that it needs no *Demonstration*.

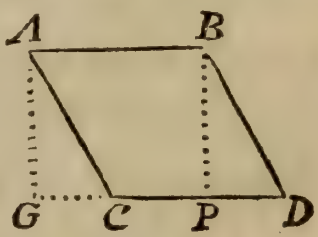



## PROBLEM II.

### *To find the Area of any Oblique-Triangled Parallelogram*, viz. *either of a* Rhombus *or* Rhomboides.

RULE. { *Multiply the Length into its perpendicular Height (or Breadth) and the Product will be the Area requir'd.*

That is, the *Side* $AB \times BP =$ the *Area* of the *Rhombus* $ABCD$. For if $BP$ be drawn *perpendicular* to $CD$, and $AG$ be made *parallel* to $BP$, then will $GC = PD$ and $GP = CD$. Consequently $\triangle AGC = \triangle BPD$, and ▭ $ABGP =$ *Rhombus* $ABCD$. But $AB \times BP =$ ▭ $ABGP$. Therefore $AB \times BP$, or $CD \times BP =$ the *Area* of the *Rhombus* $ABCD$.




*Example.* Suppose the Side $AB = 23$ *Inches*, and the *Perpendicular* $BP = 17{,}5$ *Inches (being the shortest or nearest Distance between the two Sides*, $AB$ *and* $CD$.) then $AB \times BP = 23 \times 17{,}5$ $= 402{,}5$ *square Inches*, being the *Area* of the *Rhombus* required.

The like may be done for any *Rhomboides* whose *Length* and *perpendicular Breadth* is given.

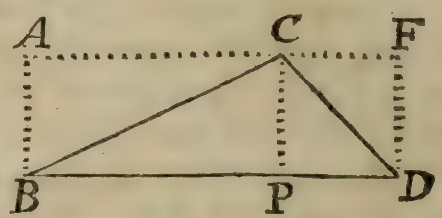
## PROBLEM III.

### *To find the Superficial Content, or Area of any plain* Triangle.

*Every plain Triangle is equal to half its circumscribing Parallelogram,* (41. *e.* 1.) *which affords the following Rule:*

RULE. { *Multiply the* Base *of the given Triangle into half its perpendicular Height, or half the* Base *into the whole Perpendicular, and the Product will be the Area.*

That is, $BD \times \frac{1}{2} CP$, or $\frac{1}{2} BD \times CP$ = *Area of* $\triangle BCD$. For $AC = BP$, $AB = CP$, *and* $BC$ is common to both $\triangle\triangle$; therefore $\triangle ABC = \triangle BCP$, and for the like Reasons $\triangle CFD = \triangle CPD$. Therefore $\triangle BCP + \triangle CPD = \frac{1}{2} \square ABCD$. Consequently $\frac{1}{2} BD \times CP$, or $BD \times \frac{1}{2} CP$ will be the *Area* of $\triangle BCD$.




*Example.* Suppose the *Base* $BD = 32$ *Inches*, and the *perpendicular Height* $CP = 14$ *Inches*.

Then $\frac{1}{2} BD \times CP = 16 \times 14 = 224$. Or $BD \times \frac{1}{2} CP = 32 \times 7 = 224$. Or thus, $32 \times 14 = 448$. Then 2) 448 (224 = the *Area* of the *Triangle* $BCD$ in *square Inches*.

## PROBLEM IV.

### *To find the Superficies, or Area of any* Trapezium.

First, *divide* the given *Trapezium* into *two Triangles*, by drawing a *Diagonal* from one of its acute *Angles* to the opposite *Angle*; and let fall two *Perpendiculars (from the other two Angles)* upon the *Diagonal*, as in the following *Figure*. Then

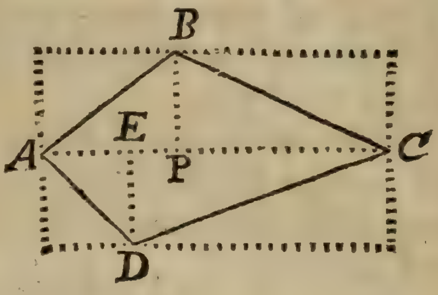
RULE. { *Multiply half the Diagonal into the Sum of the two Perpendiculars, or half the Sum of the Perpendiculars into the Diagonal, and the Product will be the Area.*

That is, $\frac{1}{2} AC \times \overline{BP + ED}$. Or $AC \times \overline{\frac{1}{2} BP + \frac{1}{2} ED}$ = *Area* of the *Trapezium* $ABCD$.

For the $\triangle ABC$ is *Half* its *circumscribing Parallelogram*; and the $\triangle ACD$ is also *Half* of its *circumscribing Parallelogram*, as hath been prov'd at the last *Problem*.

*Consequently*,

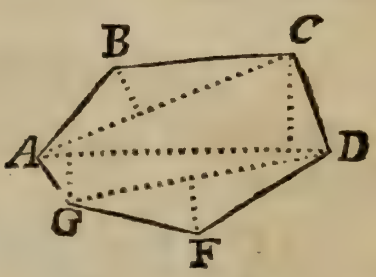
*Consequently*, $\overline{BP+ED}\times\frac{1}{2}AC$, or $\overline{\frac{1}{2}BP+\frac{1}{2}ED}\times AC$ will be the *Area* of the *Trapezium*, as above.




*Example.* Suppose the *Diagonal* $AC=33$ *Feet*, and the *Perpendicular* $BP=15$ *Feet*, and the *Perpendicular* $ED=14$ *Feet*. Then $BP+ED=29$ *Feet*, and $\overline{BP+ED}\times\frac{1}{2}AC=29\times16{,}5$ $=478{,}5$. Or $AC\times\overline{\frac{1}{2}BP+\frac{1}{2}ED}=33\times\frac{29}{2}=478{,}5$. Or thus, $29\times33=957$. Then 2) 957 (478,5 any of these *Products* are the *Area* of the *Trapezium ABCD*.

## PROBLEM V.

*To find the Superficial Content or Area of any irregular Polygon*, or *many-sided Figure, which by some Authors is call'd a Triangulate, because (as I suppose) it must be divided into Triangles*, as in the *annexed Figure ABCDFG*; by which it is evident, that the *Sum* of the *Area*'s of all those *Triangles*, found as in the last *Problem*, &c. will be the *Area* of their *circumscribing Polygon*.



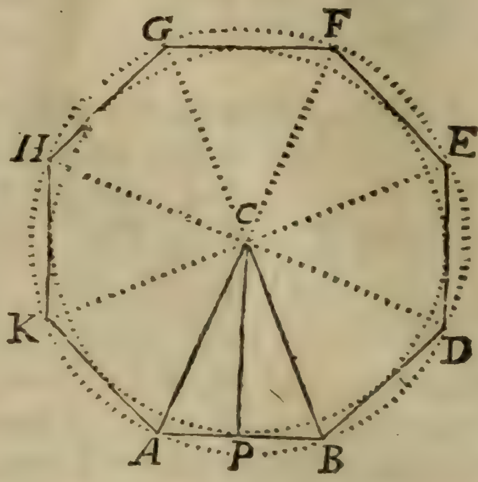

## PROBLEM VI.

*To find the Superficies, or Area of any regular Polygon*, viz. *of any regular* **Pentagon, Hexagon, Heptagon, Octagon**, &c.

*General* RULE. { *Multiply half the Sum of its Sides into the Radius of the inscrib'd Circle, or half the said Radius into the Sum of the Sides, and the Product will be the Area required.*

That is, $\frac{AB+BD+DE+EF+FG+GH+HK+KA}{2}:\times CP$ = the *Area* of the *annexed Octagon*; wherein it is evident, that its *Area* is compos'd of so many *equal Isosceles Triangles* as there are *Numbers* of *Sides in the Polygon*, viz. of *eight Isosceles Triangles*, whose *Bases* are the *Sides* of the *Octagon*, viz. $AB=BD=DE$, &c. And the *Sides* of those *Triangles*, *CA*, *CB*, *CD*, &c. are the *Radius's* of the *circumscribing Circle*; and their *perpendicular Heights*, viz. *PP*, is the *Radius* of the *inscrib'd Circle*.

But the *Area* of any one of those *Triangles* is $\frac{1}{2}$ $AB \times CP$ by *Problem* 3. Consequently the *Sum* of all their *Area*'s will be $CP$ into *half* the *Sum* of all their *Bases*, as above.




This, being *equally* evident in all *regular Polygons* whatsoever, makes the *Rule general* for finding their *Area's*.

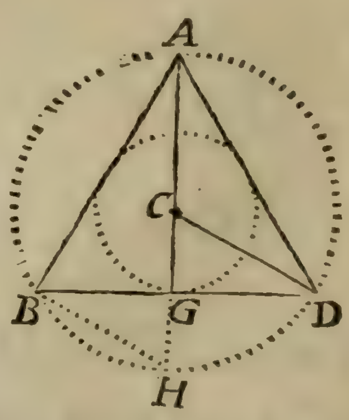
Now, because it is requir'd to have the *Radius* of the propos'd *Polygon*'s *inscrib'd Circle*, I shall here insert *(and demonstrate)* the *Proportions* that are between the *Sides* of several *regular Polygons* and the *Radius's* both of their *inscrib'd* and *circumscribing Circles*; the one will help to *delineate* or *project* the *Polygon (if Occasion require it)* and the other will help to find its *Area.*

## And *First*, Of an Equilateral Triangle.

The Side of any *Equilateral plain Triangle* is in *Proportion* to the *Radius* of

its { *Circumscribing Circle*, / *Inscrib'd Circle* / *Perpendicular Height*, } As 1 : To { 0,57735027 &c. / 0,28867513 &c. / 0,86602540 &c. }

i. e. $\begin{cases} AB : CD :: 1 : 0{,}57735027 \\ AB : CG :: 1 : 0{,}28867513 \\ AB : AG :: 1 : 0{,}86602540 \end{cases}$




### Demonstration.

Let $AB = BD = 1$, then will $BG = GD = 0{,}5$; but $\square\, AB - \square\, BG = \square\, AG$ by *Theorem* 11. That is, $1 - 0{,}25 = 0{,}75 = \square\, AG$, consequently, $\sqrt{0{,}75} = 0{,}86602540 = AG$: Then $AG : AB :: AB : AH$, by *Theorem* 13, that is, $0{,}8660254 : 1 :: 1 : 1{,}15470054$ &c. $= AH$, then $\frac{1}{2}\, AH = 0{,}57735027 = AC$. Again, $AG : DG :: DG : CG$, that is, $0{,}8660254 : 0{,}5 :: 0{,}5 : 0{,}28867513 = CG$. Q. E. D.

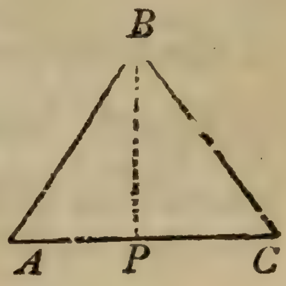
Now, by the Help of the *First* of these *Proportions*, it will be easy to *resolve* the following *Problem*.

PRO-

## PROBLEM VII.

*The Side of any Equilateral plain Triangle being given, to find its Area.*

*Example*, Suppose the Side of the propos'd *Triangle* $ABC$ to be 25 *Inches*, viz. $AB = BC = CA = 25$ First $1 : 0,866254 :: AB = 25 : 21,650635 = BP$ by *Theorem* 13. Then $AP$ ($= \frac{1}{2} CA$) $\times BP =$ the *Area* of $\triangle ABC$ by *Rule* to *Problem* 3, that is, $12,5 \times 21,650635 = 270,6329$ the *Area* in *square Inches.*



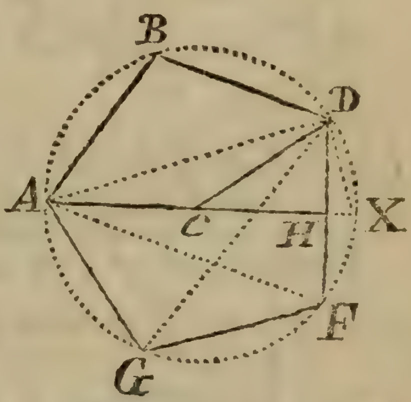

Or this *Problem* may be otherwise *resolv'd* thus: Let $b = AP = AC$. Then $2b = AB$. But $\square AB - \square AP = \square BP$. By *Theorem* 11. That is, $4bb - bb = 3bb = \square BP$. Consequently, $\sqrt{3bb} = BP$. Then $b \sqrt{3bb} = BP \times \frac{1}{2} AC$. viz. $\sqrt{3bbbb} \times \sqrt{3} =$ the *Area* of the *Triangle.*

### *Secondly*, For a Pentagon.

*The Side of any Pentagon is in Proportion to the Radius of*

*its* { *Circumscribing Circle*, / *Inscrib'd Circle*, / *Perpendicular Height*, } As 1 : To { 0,85065080 &c. / 0,68819096 &c. / 1,53884176 &c. }

*Viz.* { $AB : AC :: 1 : 0,85065080$ / $AB : CH :: 1 : 0,68819096$ / $AB : AH :: 1 : 1,53884176$ }




### Demonstration.

Let $AB = 1$. And draw the *Diagonals* $AD$, $AF$, and $DG$, which will be *equal* to one another. Then will $AG \times DF + AD \times GF = AF \times DG$ by *Theorem* 19. Consequently, $AG \times DF = AF \times DG : - AD \times GF$, that is, $\square AB = \square AD : - AD \times GF = 1$ (because $AB = AG = DF$, and $AD = AF = DG$) hence it will be $AD = 1,61803398$, then $\square AD - \square DH = \square AH$ by *Theor.* 11. But $DH = \frac{1}{2} AB$, therefore $\sqrt{\square AD - \frac{1}{4} \square AB} = AH = 1,53884176$. Again, $AH : AD :: AD : AX = 2 AC$. For $\triangle AHD$ and $\triangle ADX$ are alike.

Ergo

Ergo $\frac{\Box AD}{AH} = 2AC = 1,70130161$. Hence $AC = 0,85065080$

But $AH - AC = CH = 0,68819096$, &c. Q. E. D.

From hence it will be *easy* to *resolve* the following *Problem*.

## PROBLEM VIII.

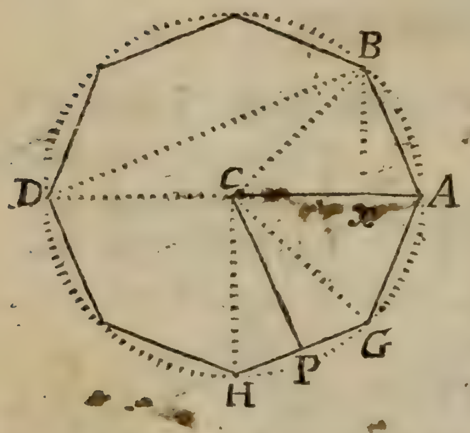
*The Side of any regular Pentagon being given, to find its Area.*

*Example.* Suppose the given *Side* to be 15 *Inches* long, then it will be, as 1 : 1,53884176 : : 15 : 22,0826264 the *perpendicular Height*; and by the general *Rule* $22,0826264 \times \frac{15}{2} = 165,619698$ the *Area* requir'd.

## *Thirdly*, For an Octagon.

*The Side of any regular Octagon is in Proportion to the Radius of its* { *Circumscribing Circle*, As 1 : to 1,30656296, &c. / *Inscrib'd Circle*, As 1 : to 1,20710678, &c.

*Viz.* { $BA : CA :: 1 : 1,30656296$ / $BA : CP :: 1 : 1,20710678$




### Demonstration.

Draw the *Right Line* $DB$, and from the *Point* $B$ let fall the *Perpendicular* $Bx$ upon the *Diameter* $DA$.

Then will $\triangle DBA$ and $\triangle DxB$ be alike, by *Theorem* 10 and 12.

Let { $b = BA = 1 \cdot a = CA$ / $e = BD$, and $y = Bx$

| | | |
|---|---|---|
| Then | 1 | $2a : b :: e : y$. viz. $DA : BA :: DB : Bx$ |
| 1 ∴ | 2 | $\frac{2ay}{b} = e = DB$ |
| 2 ©$^2$ | 3 | $\frac{4aayy}{bb} = ee = \Box DB$ |
| But | 4 | $4aa - \frac{4aayy}{bb} = bb$ |
| That is | 5 | $\Box DA - \Box DB = \Box BA$. By *Theorem* 11. |
| $4 \times bb$ | | $4bbaa - 4aayy = bbbb$ |
| Again | 6 | { $\frac{1}{2}aa = yy$. For $Cx = Bx$ / and $\Box Cx + \Box Bx = \Box CB = aa$ |

5, 6

| | | |
|---|---|---|
| 5, 6 | 7 | $4bbaa - 2a^4 = b^4$. Or $2a^4 - 4bbaa = -b^4$ |
| $7 \div 2$ | 8 | $aaaa - 2bbaa = -\frac{1}{2}b^4$ |
| 8 C □ | 9 | $a^4 - 2bbaa + b^4 = b^4 - \frac{1}{2}b^4 = \frac{1}{2}b^4$ |
| 9 ɯ² | 10 | $aa - bb = \surd\frac{1}{2}b^4$ |
| 10 + bb | 11 | $aa = bb + \surd\frac{1}{2}b^4$ |
| 11 ɯ² | 12 | $a = \surd: bb + \surd\frac{1}{2}b^4 = 1{,}30656296$, &c. $= CA$ |
| Then | 13 | $aa - \frac{1}{4}bb = \square CP$, viz. $\square CH - \square HP = \square CP$ |
| 13 ɯ² | 14 | $\surd\overline{aa - \frac{1}{4}bb} = 1{,}20710678$, &c. $= CP$. |

From hence 'twill be easy to find the *Area* of any *Octagon*.

## PROBLEM IX.

*The Side of any regular* Octagon *being given, to find its* Area:

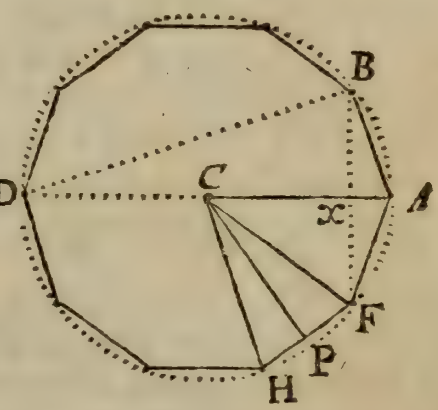
*Example.* Suppose the Side given to be 12 Inches long; *First*, as 1 : 1,20710678 : : 12 : 14,48528136 = the *Radius* of its inscrib'd Circle; then $12 \times 4 = 48$ is half the Sum of its Sides, and $48 \times 14{,}48528136 = 695{,}2935$ the *Area* required.

### *Fourthly*, For a Decagon.

*The Side of any regular* Decagon *(viz. a* Polygon *of ten equal Sides) is in Proportion to the* Radius *of*

*Its* { *Circumscribing Circle*, as 1 : to 1,61803398, &c.
*Inscrib'd Circle* as 1 : to 1,53884176, &c.

*Viz.* { $BA : CA :: 1 : 1{,}61803398$
$BA : CP :: 1 : 1{,}53884176$




### Demonstration.

Let { $b = BA = 1 . a = CA$
$c = DB$, and $y = Bx$

| | | |
|---|---|---|
| Then | 1 | $2a : b :: e : y$ |
| That is, | | $DA : BA :: DB : Bx$ |
| 1 ∵ | 2 | { $2ay = be$ and $2y = \frac{be}{a}$ |
| But | 3 | $2y : e :: 1 : 1{,}61803398$. See *Pentagon*. |
| 3 ∵ | 4 | $\frac{1e}{1{,}61803398} = 2y = \frac{be}{a} = \frac{1e}{a}$ |
| $4 \div 1e$ | 5 | $1{,}61803398 = a = CA$ |
| Again | 6 | { $aa - \frac{1}{4}bb = \square CP$. viz. $\square CF = \square PF = \square CP$. By *Theorem* 11. |
| That is, | 7 | $\surd\, 2{,}61803398 - 0{,}25 - 1{,}53884176 = CP$ |

## PROBLEM X.

*The Side of any regular* Decagon *being given, to find its* Area.

*Example.* Let the given Side be 14 Inches long; then, as 1 : 1,53884176 : : 14 : 21,543784 = the *Radius* of the inscrib'd Circle; and 14 × 5 = 70 is half the Sum of its Sides. Lastly, 21,543784 × 70 = 1508,06488 the *Area* requir'd.

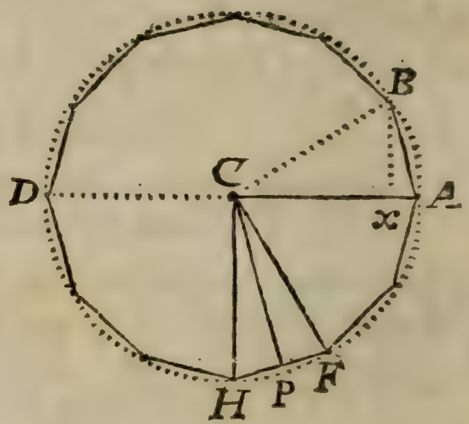
### *Fifthly*, For a Dodecagon.

*The Side of any regular* Dodecagon (viz. *a* Polygon *of twelve equal Sides*) *is in Proportion to the* Radius *of*

*Its* { *Circumscribing Circle*, as 1 : to 1,93185165, &c.
*Inscrib'd Circle* as 1 : to 1,86632012, &c.

*Viz.* { $BA : CA :: 1,03185165$
$BA : CP :: 1,86632012$

### Demonstration.




Let $b = BA = 1$ . $a = CA$ as before
And $e = xA$; then $a - e = Cx$

| | | |
|---|---|---|
| First | 1 | $bb - \square Bx = ee$ By Figure. |
| But | 2 | $Bx = \frac{1}{2} CA = \frac{1}{2} a$ |
| 2 $\circledcirc^2$ | 3 | $\square Bx = \frac{1}{4} aa$ |
| 1, 3 | 4 | $bb - \frac{1}{4} aa = ee$ |
| 4 $\omega^2$ | 5 | $\sqrt{bb - \frac{1}{4} aa} = ee$ |
| Again | 6 | $aa - \frac{1}{4} aa = aa - 2ae + ee$ |
| *Viz.* | | $\square CB - \square Bx = \square Cx$ |
| 5 × 2*a* | 7 | $2a \sqrt{bb - \frac{1}{4} aa} = 2ae$ |
| 4 − 7 | 8 | $bb - {}^{1} aa - 2a \sqrt{bb - \frac{1}{4} aa} = ee - 2ae$ |
| 7, 8 | 9 | $aa - \frac{1}{4} aa = aa + bb - {}_{4} aa - 2a \sqrt{bb - \frac{1}{4} aa}$ |
| 9 ± | 10 | $2a \sqrt{bb - \frac{1}{4} aa} = bb$ |
| 10 $\circledcirc^2$ | 11 | $4bbaa - aaaa = b^4$ |
| 11 ± | 12 | $aaaa - 4bbaa = - b^4$ |
| 13, C □ | 13 | $aaaa - 4bbaa + 4b^4 = 3b^4 = 3$ |
| 1 $\omega^2$ | 14 | $aa - 2bb = \sqrt{3} = 1,7320508075$ |
| 14 + 2*bb* | 15 | $aa = 2bb + \sqrt{3} = 3,7320508075$ |
| 15 $\omega^2$ | 16 | $a = \sqrt{3,7320508075} = 1,93185165 = CA$ |
| Again | 17 | $aa - \frac{1}{4} bb = \square CP$. viz. $\square CF - \square PF = \square CP$ |
| 17, Hence | 18 | $CP = \sqrt{aa - \frac{1}{4} bb} = 1,86632012$. Q. E. D. |

Con-

*Consectary.*

Hence if the Side of any *regular Dodecagon* be given, the *Radius* of its *inscrib'd Circle* may be easily obtain'd, and thence the *Area* found; as in the last Problem.

The Work of the 'foregoing *Polygons*, being well consider'd, will help the young *Geometer* to raise the like Proportions for others, if his Curiosity requires them: And not only so, but they will also help to form a true Idea of a Circle's *Periphery* and *Area*, according to the Method which I shall lay down in the next Chapter for finding them both.

## CHAP. VI.

### *A new and easy Method of finding the* Circle's Periphery *and* Area *to any assign'd Exactness (or Number of Figures) by one Æquation only. Also a new and facile Way of making Natural* Sines *and* Tangents.

LET us suppose *(what is very easy to conceive)* the Circle's *Area* to be compos'd or made up of a vast Number of *plain Isosceles Triangles*, having their *acutest Angles* all meeting in the Circle's Center. And let us imagine the *Bases* of those *Triangles* so very small, that their Sides and their Perpendicular Heights, *viz.* the *Radius*'s of their circumscrib'd and inscrib'd Circles *(vide* Problem 6.) may become so very near in *Length* to each other, as that they may be taken one for another without any sensible Error: Then will the *Peripheries* of their circumscribing and inscrib'd Circles become *(altho' not co-incident, yet)* so very near to each other, as that either of them may be indifferently taken for one and the same Circle.

But how to find out the Sides of a *Polygon* (viz. *the Bases of those Isosceles Triangles)* to such a convenient Smallness as may be necessary to determine and settle the Proportion betwixt a Circle's *Diameter* and its *Periphery (to any assign'd Exactness)* hath hitherto been a Work which requir'd great *Care* and much *Time* in its Performance; as may easily be conceived from the *Nature* of the *Method* us'd by all those who have made any considerable Progress in it, viz. *Archimedes*, *Snellius*, *Hugenius*, *Mœtius*, *Van Culen*, &c. These proceeded with the bisecting of an *Arch*, and found the *Value* of its *Chord* to a convenient Number of Figures at every single Bisection, repeating their *Operations* until they had approach'd to the *Chord* design'd.

And this *Method* is made Choice of by the learned Dr. *Wallis* in his Treatise of *Algebra* ; wherein, after he hath given us a large Account of the different Enquiries made by several *(very eminent in Mathematical Sciences)* in order to find out some easier and more expeditious Way of approaching to the Circle's *Periphery*, as in *Chap.* 82, 84, 85, 86, and several other Places, he comes to this Result, (*Page* 321.)

" 'Tis true, *saith he*, we might in like Manner proceed by con-
" tinual Trisection, Quinquisection, or other Section, if we had
" for these as convenient Methods of Operation as we have for
" Bisection: But because *Euclid* shews how to bisect an Arch
" Geometrically, but not to trisect, *&c.* and the one may be done
" *(Algebraically)* by resolving a Quadratick Æquation, but not
" those other, without Æquations of a higher Composition, I
" therefore make Choice of a continual Bisection, *&c.*"

And then he lays down these following Canons:

| | | |
|---|---|---|
| The Subtense of $\frac{1}{6}$ | 1 | into 6 |
| of $\frac{1}{2}$ | $\surd:2-\surd 3$ | into 12 |
| of $\frac{1}{24}$ | $\surd:2+\surd:2+\surd 3$ | &c. 24 |
| of $\frac{1}{48}$ | $\surd:2-\surd:2+\surd:2+\surd 3$ | 48 |
| of $\frac{1}{96}$ | $\surd:2-\surd:2+\surd:2+\surd:2+\surd 3$ | 96 |
| &c. | $\surd:2-\surd:2+\surd:2+\surd:2+\surd:2+\surd 3$ | 192 |
| | $\surd:2-\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd 3$ | 384 |
| | $\surd:2-\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd 3$ | 768 |
| | &c. | &c. |

*How tedious and troublesome the Work of these complicated* Extractions *is*, *I leave to the Consideration of those, who either have had Experience therein, or out of Curiosity will give themselves the Trouble of making Trial.*

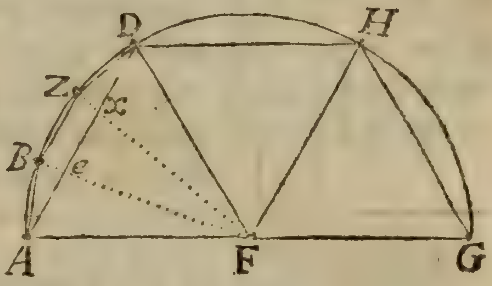
Again, in *Page* 347, the *Doctor* inserts a particular Method proposed by *Libnitius*, publish'd in the *Acta Eruditorum* at *Leipsick*, for the Month of *February* 1682, in order to find the Circle's *Area*, and consequently its *Periphery*, which is this:

As $1 : \text{to}\ \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}$, &c. infinitely : : so is the Square of the *Diameter* to the Circle's *Area*. But this convergeth so very slowly, that it is not worth the Time to pursue it.

I shall here propose a new Method of my own, whereby the Circle's *Periphery*, and consequently its *Area*, may be obtain'd infinite-

infinitely near the Truth, with much greater Eaſe and Expedition than either that of *Biſection*, or that of *Libnitius*, as above, or any other Method that I have yet ſeen ; it being perform'd by *reſolving* only one *Æquation*, deduced by an eaſy Proceſs from the Property of a Circle, *(known to every Cooper)* which is this :

*The* Radius *of every Circle is equal to the* Chord *of one ſixth Part of its* Periphery. That is, $AD = DH = HG$, the *Chords* of one third Part of the *Semicircle*, are each equal to $AF$ its *Radius*. Then if the Arch $AD$ be triſected, it will be $AB = BZ = ZD$.




Let $\begin{cases} R = AF = 1 \\ c = AD = 1 \\ a = AB. \end{cases}$ Quære $a$.

| | | |
|---|---|---|
| Then | 1 | $R : a :: a : \frac{aa}{R} = Be$ |
| And | 2 | $R : a :: R - \frac{aa}{R} : c - 2a$ |
| That is, For | 3 | $FB : BZ :: Fe : ex = AD - 2a$ <br> $\triangle AFB$, and $\triangle BAe$, are alike. <br> And $AB = Ae = Dx$, &c. |
| 2 ∵ | 4 | $Rc - 2Ra = Ra - \frac{aaa}{R}$ |
| 4 × &c. | 5 | $3R^2a - aaa = RRc$. That is, $3a - aaa = 1$ <br> Here $a =$ the Chord of $\frac{1}{18}$ Part of the *Circle*. <br> For $\frac{1}{3}$ of $\frac{1}{6} = \frac{1}{18}$ |

Next, *To triſect the* Arch $AB$.

| | | |
|---|---|---|
| Let | 1 | $3y - y^3 = a$ the laſt Chord. |
| 1 ⊚5 | 2 | $27y^3 - 27y^5 + 9y^7 - y^9 = a^3$ |
| 1 × 3 | 3 | $9y - 3y^3 = 3a$ |
| 3 − 2 | 4 | $9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = 1$ <br> Here $y =$ the Chord of $\frac{1}{54}$ Part of the Circle. |

Again, *To triſect the* Arch *whereof* y *is the* Chord.

| | | |
|---|---|---|
| Let | 1 | $3a - a^3 = y$ |
| 1 ⊚3 | 2 | $27a^3 - 27a^5 + 9a^7 - a^9 = y^3$ |
| 1 ⊚5 | 3 | $243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} = y^5$ |

1 ⊚7

| | | |
|---|---|---|
| 1 $\circledcirc^{7}$ | 4 | $2187a^7 - 5103a^9 + 5103a^{11} - 2835a^{13} + 945a^{15} = y^7$ |
| 1 $\circledcirc^{9}$ | 5 | $19683a^9 - 59049a^{11} + 78732a^{13} - 61236a^{15} = y^9$ |
| 1 × $\overline{9}$ | 6 | $27a - 9a^3 = 9y$ |
| 2 × 30 | 7 | $810a^3 - 810a^5 + 270a^7 - 30a^9 = 30y^3$ |
| 3 × 27 | 8 | $6561a^5 - 10935a^7 + 7290a^9 - 2430a^{11} + 405a^{13} + 27a^{15} = 27y^5$ |
| 4 × $\overline{9}$ | 9 | $19683a^7 - 45927a^9 + 45927a^{11} - 25515a^{13} + 8505a^{15} = 9y^7$ |
| 6 − 7 + 8 − 9 + 5 | 10 | $\left\{ 27a - 819a^3 + 7371a^5 - 30888a^7 + 72930a^9 - 107406a^{11} + 104652a^{13} - 69768a^{15} \right\} = 1$ <br> Here $a$ = the *Chord* of $\frac{1}{162}$ Part of the Circle. |

Proceeding on in this Method of continually trifecting the Arch of every new Chord, and ftill connecting the produced *Æquations* into one, as in the *two laft Trifections*, 'twill not be difficult to obtain the *Chord* of any affign'd Arch, how fmall foever it be.

Now, in order to facilitate the Work of raifing thefe *Æquations* to any confiderable Height, 'twill be convenient to add a few ufeful Obfervations concerning their Nature, and of fuch Contractions as may be fafely made in them; which, being well underftood, will render the Work very *eafy*.

1. *I have obferv'd, that every* Trifection *will gain or advance one Figure in the Circle's* Periphery, *but no more. Therefore fo many Places of Figures as are at firft defign'd to be perfect in the* Periphery, *fo many* Trifections *muft be repeated to raife an Æquation that will produce a* Chord *anfwerable to that Defign.*

2. *I have alfo found, that all the fuperior Powers (of* a) *whofe Indices are greater than the Number of* Trifections, *(viz. whofe Indices are greater than the Number of defign'd Figures) may be wholly rejected as infignificant.*

3. *When once the Number of* Trifections, *and thence the higheft Power (of* a) *is determined, the third* Procefs *(viz. the third* Trifection) *may be made a fix'd or conftant Canon; for by it, and Multiplication only, all the fucceeding* Trifections *(how many foever they are) may be compleated without repeating the feveral Involutions.*

4. *In*

4. *In raiſing and collecting the Co-efficients of the ſeveral Powers (of* a) *'twill be ſufficient to retain only ſo many ſignificant Figures (at* $a^3$*) as there is deſigned to be Places of Figures in the* Periphery *(or at moſt but two more) and every ſucceeding ſuperior Power may be allow'd to decreaſe two Places of ſignificant Figures: But herein great Care muſt be taken to ſupply the Places, of thoſe Figures that are omitted, with Cyphers, that ſo the whole and exact Number of Places may be truly adjuſted; otherwiſe all the Work will be erroneous.*

*Now the Number of thoſe ſupplying Cyphers may be very conveniently denoted by Figures placed within a* Parentheſis, *thus:* 576 (8) $a^3$, *may ſignify* 57600000000$a^3$, *as in the following Æquations. The like may be done with* Decimal Parts, thus: (,7)658 *may ſignify* ,0000000658, &c. *which will be found very uſeful in the Solution of theſe and the like Æquations.*

The aforeſaid Contractions may be ſafely made, becauſe both the ſuperior Powers of *a*, which are rejected; as alſo thoſe Numbers that are omitted in the *Co-efficients* (and ſupply'd with Cyphers) would produce Figures ſo very remote from Unity, as that they would not affect the *Chord* deſign'd; that is, they would not affect the *Chord* in that Place wherein the deſign'd *Periphery* is concerned; as will in Part appear in the following Example.

If theſe Directions be carefully minded, 'twill be eaſy to raiſe an *Æquation* that will produce the Side of a *regular Polygon*, whoſe Number of Sides ſhall be vaſtly numerous, conſequently infinitely ſmall: But, I preſume, 'twill be ſufficient for an *Example* to find the Side of a *Polygon* conſiſting of 258280326 equal Sides; that is, if I find the *Chord* of $\frac{1}{258280326}$ Part of the *Circle*'s Periphery, and that requires but *ſixteen Triſections*, which being order'd, as before directed, will produce this *Æquation*.

$$\left\{\begin{array}{l} 43046721a - 3323601794869686\,12(4)a^3 \\ +769837653199714(20)a^5 - 849121853284\,1(35a^7 \\ +5463333\,1143(50)a^9 - 230083348(66)a^{11} \\ +6830988(79)a^{13} - 15072(94)a^{15} \end{array}\right\} = 1$$

Here the Value of *a* will have 23 Places of *Figures* true; that is, the Sides of the *inſcrib'd* and *circumſcrib'd Polygons* will be exactly the ſame to 23 Places of *Decimal Parts*, but no farther; all which may be eaſily obtain'd at *two Operations*. And for the firſt, 'twill be ſufficient to take only *three Terms* of the *Æquation*, which will admit of being yet farther contracted, thus:

Let

$$\text{Let} \left\{ \begin{array}{l} 43046721a - 3323601794(12)a^3 \\ +76983765(27)a^5 \end{array} \right\} = 1$$

And let $r + e = a$; then rejecting all the *Powers* of $e$, that arise by Involution above $eee$,

It will be $r^3 + 3rre + 3ree + eee = aaa$

And $r^5 + 5r^4e + 10r^3ee + 10rreee = a^5$

Then the first single Value of $r$ may be thus found:

$43046721)\ 1,00000000\ (,00000002 = r$

This $,00000002 = r$ being duly involv'd, and its Powers multiply'd into their respective *Co-efficients*, will produce

$$\left. \begin{array}{l} +,86093441 + 43046721e \\ -,02658881 - 3988322e - 199416(9)ee - 3324(18)eee \\ +,00024635 + 61587e + 6159(9)ee + 308(18)eee \end{array} \right\} = 1$$

*viz.* $,83459196 + 39119986e - 193257(9)ee - 3016(18)eee = 1$

Hence $39119986e - 193257(9)ee - 3016(18)ee = 0,16540804$

All the *Terms* of this last *Æquation* being divided by 193257(9) the *Co-efficient* of $ee$, it will then become

$,0000002024e - ee - ,156(5)eee = ,0000000000000008558968 = D$

*Consequently*, $\left\{ \dfrac{D + 156(5)eee}{,0000002024 - e} = e \right.$

*Operation.*

$,0000002024)\ \ ,0000000000000008558968\ \ (,000000004 = e$

$-\ e\ \ ,0000000043\ :+ ,000000000000000000009984\ \ = 156\ (5)\ eee$

1 *Di.* $,000000198)\ \ ,0000000000000008568952\ \ (,000000004327$

2 *Di.* $,0000001981$

792

6489
5943

5465
3962

&c.

First $r = ,00000002$

$+\ e = ,000000004327$

$r + e = ,000000024327 = a.$

Or rather new $r$ for a second Operation.

*Now, if this first Value of a* $= ,000000024327$ *were not continued to more Places of Figures by a second Operation, but only multiply'd into the Number of* Chords, *viz.* $,000000024327 \times 258280326 = 6,28318539$, &c. *the Periphery of that* Circle *whose Diameter is* 2, *nearer than either* Archimedes, *or* Mœtius's *Proportion: For* Ar-

Archimedes *makes it* 6,285714, &c. *viz. As* 7 *to* 22. *And* Mœtius *makes it* 6,28318584, &c. *viz. As* 113 *to* 355.

But if the whole *Æquation* before propos'd be now taken, and we proceed to a ſecond *Operation*, the Value of $a$ may be increas'd with twelve Places of *Figures* more, and thoſe may be obtain'd by plain *Diviſion* only.

Thus, let $r + e = a$, as before, and let all the Powers of $e$ be now rejected as inſignificant;

$$\text{Then will} \left\{ \begin{array}{l} r + e = a \\ r^3 + 3r^2e = a^3 \\ r^5 + 5r^4e = a^5 \\ r^7 + 7r^6e = a^7 \end{array} \right\} \text{and} \left\{ \begin{array}{l} r^9 + 9r^8e = a^9 \\ r^{11} + 11r^{10}e = a^{11} \\ r^{13} + 13r^{12}e = a^{13} \\ r^{15} + 15r^{14}e = a^{15} \end{array} \right.$$

The ſeveral Powers of $r = ,000000024327$ being rais'd, and multiply'd into their reſpective *Co-efficients*, will produce theſe following Numbers:

$$\left. \begin{array}{lrl} + & 1,047197581767 & + \; 43046721e \\ - & ,047849196598394865 & - \; 5900751e \\ + & ,000655906484595355 & + \; 134810e \\ - & ,000004281440413375 & - \; 1232e \\ + & ,000000016302517803 & + \; 6e \\ - & ,000000000040631167 & - \; 0e \\ + & ,000000000000071388 & + \; 0e \\ - & ,000000000000000093 & - \; 0e \end{array} \right\} = 1$$

*Viz.* $1,000000026474745106 + 37279554e = 1$

Hence $37279554e = -,000000026474745106 = D$: Or rather $-37279554e = ,000000026474745106 = D$

*Conſequently*, $\left\{ \dfrac{D}{37279554} = -e \right.$

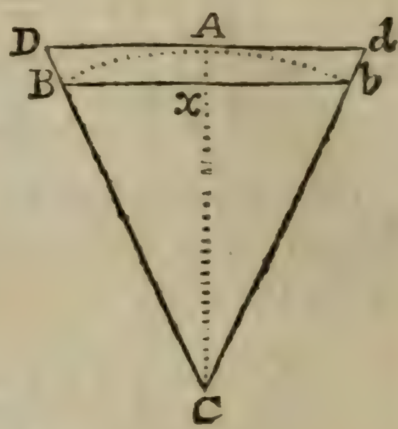
*Operation.*

```
37279554) ,000000026474745106( (,15)710167967 = — e
                 260956878
                 ---------
                  37905730
                  37279554
                  --------
                   62617660
                   37279554
                   --------
                      &c.
```

Laſt $r$ = ,000000024327
— $e$ = ,00000000000000007101679 67

$r - e$ = ,000000024326999289832033 = $a$ the *Chord* or *Side* of the *Polygon* required.

The next Work will be to examine how many Places of theſe *Figures* will hold true to the *Circle's Periphery :* In order to that, let $a$ be repreſented by the *Chord* B $b$, in the *annexed Scheme*; and let $Bx = xb$. Then will $Bx = \frac{1}{2}a =$ (,7)1216349964491 60165 and $\square BC - \square Bx = \square Cx$. Let the *Radius* $BC = 1$ as before. Then will the $\sqrt{\square BC - \square Bx} = Cx$ = ,9999999999999999, &c. But $Cx : xB :: CA : AD$ } per *Fig.* or $Cx : Bb :: CA : Dd$ } *Ergo* $Dd$ = (,7)243269992898320354 the Side of the *Circumſcribing Polygon*. Then will $a \times 25828 0326$ be the *Perimeter* of the *Inſcrib'd Polygon.* And $Dd \times 258280326$ will be the *Perimeter* of the *Circumſcribing Polygon.* That is, 6,2831853071795859 = the *Perimeter* of the *Inſcrib'd Polygon.* And, 6,2831853071795865 = the *Perimeter* of the *Circumſcrib'd Polygon.*




Hence 'tis evident, that the *Circle's Periphery*, whoſe *Diameter* is 2, may be concluded 6,2831853071795864 true, becauſe the *Perimeters* of the inſcrib'd and circumſcrib'd *Polygons* are ſo far very near being *Co-incident*, or the ſame.

'Tis poſſible there may be ſome who will think this is tedious and troubleſome Work; but if thoſe pleaſe to conſider, that, if this *Periphery* were to be found by the aforeſaid *Method* of *Biſection*, it would require theſe following *Extractions :*

*Viz.* √ : 2 — √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2
+ √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2
+ √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2
+ √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 2 + √ : 3 *multi-
ply'd* into 402809984.

Here the firſt Root (*viz.* √ 3) muſt be *extracted* at leaſt to one hundred and two Places of *Figures.* The ſecond Root (*viz.*

*(viz.* $\surd : 2 + \surd\ 3$) muſt have 99 Places of *Figures* in it. The third Root *(viz.* $\surd : 2 + \surd : 2 + \surd\ 3$) muſt have 96 Places in it, &c. every *Extraction* being allow'd to *decreaſe three Places*, that ſo the laſt *Root* (viz. *the Chord ſought)* may conſiſt of 24 Places of *Figures*, as above.

I ſay, whoever duly conſiders the Trouble of theſe ſo often repeated *Extractions* will, I preſume, be pleas'd with what I have done. For truly, when I conſider the great Time and Care required in them, I cannot but admire at the Patience of the laborious *Van Culen*, who proceeded that Way until he had found the *Circle's Periphery* to *Thirty-ſix Places of Figures*, to wit, 6,28318530717958647692528676655900576. *Theſe Numbers are ſaid to be engraven upon his Tomb-Stone in St.* Peter*'s Church in* Leyden, *for a Memorial of ſo great a Work.*

Having thus obtain'd the *Circle's Periphery*, its *Arch* may eaſily be found *(to the ſame Number of Figures)* by *Problem* 6. That is, if *Half the Periphery* of any *Circle* be *multiply'd* into *Half* its *Diameter*, the *Product* will be that *Circle's Area*, as will appear farther on. Therefore 3,14159265358979 3 will be the *Area* of the *Circle* whoſe *Diameter* is 2.

Thus I have ſhew'd the young *Geometer* how to find the *Circle's Periphery* and *Area* to what Exactneſs he pleaſes to approach; for preciſely true it cannot be found, notwithſtanding the late *Pretenſions* of a certain *Frenchman* who hath publiſhed to the World (in the *Works of the Learned)* that after twenty-five Years *Study* he had found the *Quadrature* of the *Circle:* But if he had perus'd the 83d *Chapter* of Dr. *Wallis*'s *Algebra*, he might there have ſeen his *Error*, *viz.* the *Impoſſibility* of what he pretended to; for it is as impoſſible to ſquare the *Circle (that is, to find its true Area)* as it is to find the *Root* of a *Surd Number*.

*Note*, What I have here propos'd and done by the Triſection of an Arch, may as eaſily and much more ſpeedily be perform'd by Quinqueſection or Septiſection, *&c.* But becauſe the Scheme for Triſection is more ſimple, and may be eaſier underſtood by a Learner than thoſe of the other Sections (of which ſee my *Compendium of Algebra*, Pages 76 and 79) I have for that Reaſon made Choice of Triſection.

As to the *Proportion* of one *Circle* to another, and of the *Circle* to the *Ellipſis*, &c. thoſe ſhall be fully ſhew'd when we come to the Fifth Part.

Before I conclude this Part, I ſhall make ſome *Uſe* or *Application* of the above-found *Periphery*, in finding the *Quantity* of *Angles*, which is done by the Help of *Right-lines*, call'd *Sines* and *Tangents*, the Length whereof are calculated to every Degree and Minute of a *Quadrant*, by much Labour. But I ſhall here ſhew how to find the natural *Sine* (and conſequently the natural *Tangent)* of any propos'd *Arch* or *Angle*, by two *Æquations*, without the Help of any *precedent Sine*, as uſual; which I did ſome Years ago communicate to the ingenious Mr. *Joſeph Ralphſon*, and he ſo well approv'd of them as to make them the 20th and 21ſt *Problems* in the *ſecond Edition* of his *Analyſis Æquationum Univerſalis.*

And becauſe, in finding the Quantity of *Angles*, every *Circle* is ſuppos'd to be *divided* into 360 *equal Parts*, call'd *Degrees*; every *Degree* is *ſubdivided* into 60 *Parts*, call'd *Minutes*; and every *Minute* into 60 *Seconds*, &c. (See *Page* 294.)

Therefore 360) 6,2831853, &c. (0,0174532925, &c. is an *Arch* of the above-found *Periphery*, equal to the *Arch* of one *Degree.*

And 60) 0,0174532925, &c. (0,0002908882, &c. = the *Arch* of one *Minute.*

Then if the given *Arch (or Angle)* be leſs than 45 *Degrees*, reduce it into *Minutes*, and *multiply* thoſe *Minutes* into this conſtant *Multiplicator*, *viz.* 0,0002908882 calling the *Product* $p$. And for the *Sine* ſought put $a$. Then it will be $-aaaa + 12paaa - 195aa - 36ppaa + 240pa = 45pp$.

*Example.*

Let it be required to find the Sine of 19°. 13′. = 1153′. Here $0,0002908882 \times 1153 = 0,3353940946 = p$. And $-a^4 + 4,024729a^3 - 199,049611aa + 80,494583a = 5,06201394$.

Let $r + e = a$

$$\text{Then}\begin{cases} rr + 2re + ee = aa \\ rrr + 3rre + ee = aaa \\ rrrr + 4\,rrre + 6rree = aaaa \end{cases}$$

Note, *In this Caſe the firſt* r *may always be taken equal to the firſt Figure in the Product* = p. Viz. *here* r = 0,3 *which being involved as its Powers direct, and thoſe Powers multiply'd into the reſpective Co-efficients of the Æquation*; *it will be*

$$\left.\begin{cases} +\ 24,1483 + 80,49e \\ -\ 17,9144 - 119,43e - 199,05ee \\ +\ 0,1086 + 1,08e + 3,62ee \\ -\ 0,0081 - 0,11e - 0,54ee \end{cases}\right\} = 5,06201394$$

---

$$Viz.\ 6,3344 - 37,97e - 135,97ee = 5,06201$$

Hence

Hence $37,97ee + 195,97ee = 1,27239$
And $0,123e + ee = 0,006492 = D$

THEOREM $\left\{ \frac{D}{,193 + e} = e \right.$

*Operation.* 0,193) 0,006492 (0,029 = $e$
$+ e =$ ,029 42

1. *Divisor* ,21 2292
1998

2. *Divisor* ,222

First $r = 0,3$
$+ e = 0,029$

$r + e = 0,329 = r$ for a second *Operation.*

Which being *involv'd* and *multiply'd*, &c. as before, will produce these *Numbers:*

$+ 26,48271781 + 80,49458e$
$- 21,54532894 - 130,97464e - 199,0496ee$
$+ 0,14332578 + 1,30692e + 3,9724ee$
$- 0,01171611 - 0,14244e - 0,6494ee$

*Viz.* $5,06899854 - 49,31558e - 195,7266ee = 5,06201394$

Hence $49,31558e + 195,7266ee = ,0069846$; which being divided by 195,7266 the *Co-efficient* of $ee$, will become $,25196e + ee = ,0000356854 = D$

Then $\left\{ \frac{D}{,25196 + e} = e \right.$

*Operation.* 0,25196) ,0000356854 (0,0001415 = $e$
$+ e =$ 0,00014 2520

1. *Divisor* 0,2520 104854

2. *Divisor* 0,25210 100840

40140
25210

Last $r = 0,329$ &c.
$+ e = 0,0001415$

$r + e = a = 0,3291415$ being the natural *Sine* of 90°. 13'. As was required.

Thus you may find the *Right Sine* of any *Arch* or *Angle* less than 45 *Degrees.* But,

But, if the given *Arch* be greater than 45 *Degrees*, you muſt take its Complement to 90°. *viz.* ſubtract it from 90 *Degrees*, and reduce the Remainder into *Minutes*, as before. Then *multiply* the *Square* of theſe *Minutes* into this conſtant *Multiplicator*, 0,000000084616 calling their *Product* $p$, and putting $a =$ the *Sine* ſought, as before. Then will $a^4 + 28a^3 + 195aa + 36paa + 108pa - 28a = 196 - 81p$.

*Example.*

Suppoſe it were required to find the *Sine* of 75°. 32′. or (which is the ſame Thing) to find the *Co-ſine* of 14°. 28′. = 868′, whoſe *Square* 753424 × 0,000000084616 = 0,06375172518 = $p$. Hence the *Æquation* in *Numbers* will be $aaaa + 28aaa + 197{,}295062aa - 21{,}114814a = 190{,}8361102588$.

Let $r - e = a$ And $r = 1$

$$\text{Then}\begin{cases} rr - 2re + ee = aa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$$

Note, *I here take* $r = 1$ *becauſe the Arch is ſo near to* 90°. *and therefore I make it* $r - e = a$.

$$\text{Then}\left\{\begin{array}{rrr} -\ 21{,}1148 & +\ 21{,}11e & + \\ +\ 197{,}2956 & -\ 394{,}59e & +\ 197{,}29ee \\ +\ 28{,}0000 & -\ 84{,}00e & +\ 84{,}00ee \\ +\ 1{,}0000 & -\ 4{,}00e & +\ 6{,}00ee \end{array}\right\} = 190{,}8361$$

*Viz.* $205{,}1808 - 461{,}48e + 287{,}29ee = 190{,}8361$

Hence $461{,}48e - 287{,}29ee = 14{,}3447$

And $1{,}606e - ee = {,}049930 = D$

THEOREM $\left\{ \dfrac{D}{1{,}606 - e} = e \right.$

*Operation.* 1,606) ,049930 (0,031 = $e$
$-e =$ 0,031 471

1. *Diviſor* 1,57 2830
1575
2. *Diviſor* 1,575
&c.

Firſt $r = 1{,}000$
$-e = 0{,}031$
$r - e = 0{,}969 = r$

for a ſecond *Operation*; which, being *involv'd* as before, will produce theſe following Numbers:

— 29,

$$\begin{array}{rrr}
-\ 20,460254766 & +\ 21,11481e & \\
+\ 185,252368710 & -\ 382,35783e & +\ 197,2951ee \\
+\ 25,475889852 & -\ 78,87272e & +\ 81,5960ee \\
+\ 0,881647759 & -\ 3,63941e & +\ 5,6337ee \\
\hline
\end{array}$$

*Viz.* $191,149651515 - 443,75515e + 284,5248ee$
$= 190,836110259$

Hence it will be $443,75515e - 284,5248ee = 0,313541256$
And $1,55963e - ee = ,0011019821 = D$

Then $\left\{ \dfrac{D}{1,55963 - e} = e \right.$

*Operation.* $1,55963)\ 0,0011019821\ (0,0007068 = e$
$-e = 0,00070$ 109123

1. *Divisor* 1,5589 1075210
935358

2. *Divisor* 1,55893
1398520
1247144, &c.

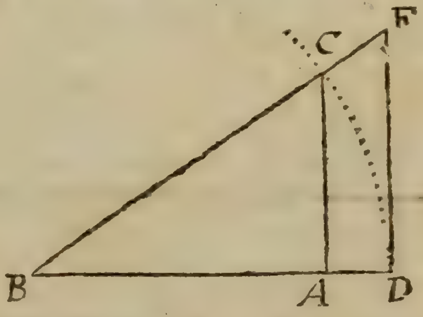
Last $r = 0,969$
$-e = 0,0007068$

$r - e = a = 0,9682932$ the *Sine* of 75°. 32′. as was required.

Having found the *Sine* and *Co-sine* of any *Arch*, the *Tangent* is usually found by this *Proportion:*

Viz. $\left\{\right.$ *As the Co-sine of any Arch : is to the Sine* of that *Arch* : : *so is the Radius : to the Tangent of the same Arch.*

For supposing $BC = BD$ *Radius*, $AC$ the *Sine* of the *Arch* $CD$. Then $BA$ is the *Co-sine*, and $FD$ the *Tangent* of the same *Arch*. But $BA : CA :: BD : FD$, &c. Now by this *Proportion* there is *required* to be given both the *Sine* and *Co-sine* of the *Arch*, to find the *Tangent*. 'Tis true, if the *Radius*, and either the *Sine* or the *Co-sine* be given,




the other may be found, thus, $\sqrt{\square BC - \square CA} = BA$. Or $\sqrt{\square BC - \square BA} = CA$. But, if either the *Sine* or *Co-sine* be given, the *Tangent* may (I presume) be more easily found by the following *Theorems:*

 Let

Let $BC = 1$. $CA = S$. $BA = x$ and $FD = T$. Then, if $S$ be given, $T$ may be found by this

$$\text{Theorem}\left\{ \sqrt{\frac{SS}{1 - SS}} = T \right.$$

Or if $x$ be given, $T$ may be found by this

$$\text{Theorem}\left\{ \sqrt{\frac{xx}{1 - xx}} = T \right.$$

Let the *Sine* of 90°. 13′. *(before found)* be given, *viz.* 0,3291415 $= S$, to find $T$ the *Tangent* of the ſame *Arch.* Firſt 0,3291415 × ,3291415 = 0,108334127 = $SS$. Again 1 — 0,108334127 = 0,891665873 = 1 — $SS$. Then 0,891665873) 0,108334127 (0,1214963253 and √ 0,1214963253 = 0,3485632 = $T$, the *Tangent* of 19°. 13′. As was required. And ſo you may proceed to find $T$ = the *Tangent*, when $x$ = the *Co-ſine* is given.

*Perhaps* it may here be *expected*, that I ſhould have ſhew'd and *demonſtrated (or at leaſt have inſerted)* the *Proportions* from whence the foregoing *Æquations* for making *Sines* were *produced*; but I have omitted that, as alſo their *Uſe* in computing the *Sides* and *Angles* of *plain Triangles* by the *Pen* only *(viz. without the Help of Tables)* for the *Subject* of my *Diſcourſe* hereafter, if *Health* and *Time* permit.

In the mean *Time*, what is here done may ſuffice to ſhew, that the making of *Sines* by ſuch a *laborious* and *operoſe* Way, as was formerly uſed, is in a great *Meaſure* overcome; which, I think, I may juſtly claim as my own.

# AN INTRODUCTION TO THE Mathematicks.

## PART IV.

### CHAP. I.

### *Definitions of a* Cone, *and its* Sections.

THERE are ſeveral Definitions given of a *Cone:* The Learned Dr. *Barrow*, upon *Euclid*, hath it thus:

" A Cone (*ſaith he*) is a Figure made when one Side of " a Rectangle Triangle, (*viz.* one of thoſe Sides that contain the " Right Angle) remaining fix'd, the Triangle is turn'd round " about, 'till it return to the Place from whence it firſt moved: " And if the fix'd Right Line be equal to the other which con- " taineth the Right Angle, then the Cone is a Rectangled Cone: " but if it be leſs, 'tis an Obtuſe-angled Cone; if greater, an Acute- " angled Cone. The *Axis* of a Cone is that fix'd Line about which " the Triangle is mov'd: The *Baſe* of a Cone is the Circle, which " is deſcrib'd by the Right Line mov'd about."

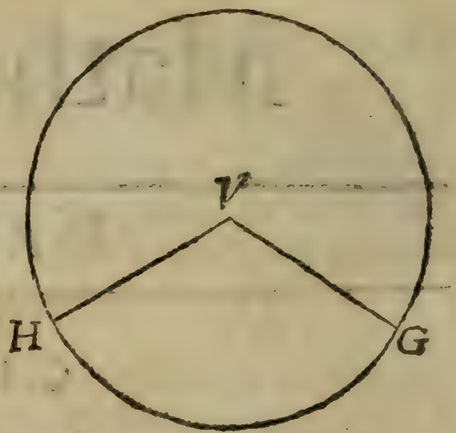
" (*Defin.* 18, 19, 20. *Euclid.* 11.)

Sir *Jonas Moor*, in his Treatiſe of *Conical Sections* (taken out of the Works of *Mydorgius*) defines it thus:

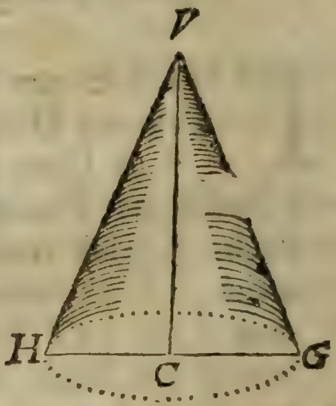
" If a Line of ſuch a Length as ſhall be needful ſhall, upon a " Point fix'd above the Plain of a Circle, ſo move about the Cir- " cle, until it return to the Point from whence the Motion began, " the Superficies that is made by ſuch a Line is call'd a *Conical* " *Superficies*; and the ſolid Figure contain'd within that Superficies " and the Circle is call'd a *Cone.* The Point remaining ſtill is the " Vertex of the Cone, &c."

Altho' both thefe *Definitions* are equally true, and, with a little Confideration, may be pretty eafily underftood ; yet I fhall here propofe one very different from either of them ; and, as I prefume, more plain and intelligible, efpecially to a *Learner*.

If a Circle defcrib'd upon ftiff Paper *(or any other pliable Matter)* of what Bignefs you pleafe, be cut into *two*, *three*, or more *Sectors*, either equal or unequal, and one of thofe *Sectors* be fo roll'd up, as that the *Radii* may exactly meet each other, it will form a *Conical Superficies*.

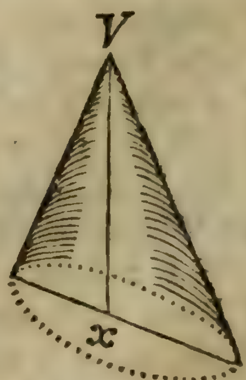



That is, if the *Sector HVG* be cut out of the Circle, and fo roll'd up as that the *Radii VH* and *VG* may juft meet each other in all their Parts, it will form a *Cone*, and the Center *V* will become a *Solid Point*, call'd the *VERTEX* of the *Cone*; the *Radius VH*, being every-where equal, will be the Side of the Cone, and the Arch *HG* will become a Circle, whofe *Area* is call'd the *Cone's Bafe*.




A *Right Line* being fuppos'd to pafs from the *Vertex*, or Point *V*, to the *Center* of the Cone's *Bafe*, as at *C*, that Line (viz. *VC*) will be the *AXIS*, or *perpendicular Height* of the Cone.

If a *Solid* be actually made in fuch a Form, it will be a compleat or perfect Cone; which I fhall all along call a Right Cone, becaufe its *Axis VC* ftands at *Right Angles* with the Plain of its *Bafe HG*, and its Sides are every-where equal.




Any Cone, whofe *Axis* is not at *Right Angles* with the Plain of its *Bafe*, may be properly call'd an *imperfect Cone*, becaufe its Sides are not every-where equal *(as in the annexed Figure.)* Now, fuch an imperfect Cone is ufually call'd a *Scalene*, or *Oblique Cone*.

Any folid Cone may be cut by Plains *(which I fhall all along hereafter call Right Lines)* into *five Sections*.

*Sect.*

*Sect.* 1.

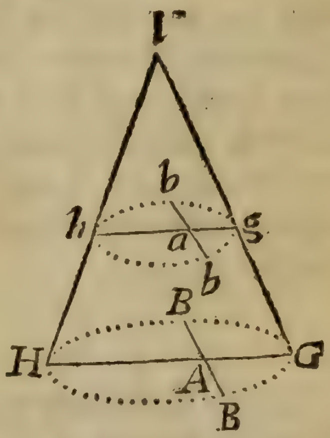
If a Right Cone be cut directly thro' its *Axis*, the *Plain* or *Superficies* of that *Section* will be a plain *Isosceles Triangle*, as *HVG* *Fig.* 2, viz. the Sides *(HV* and *VG)* of the Cone will be the Sides of the Triangle, the Diameter *(HG)* of the Cone's *Base* will be the *Base* of the Triangle, and *(VC)* its *Axis* will be the *perpendicular Height* of the Triangle.

*Sect.* 2.

If a Right Cone be cut *(any where)* off by a Right-line parallel to its *Base*, as *h g (it will be easy to conceive, that)* the Plain of that *Section* will be a Circle, because the Cone's *Base* is such: wherein one Thing ought to be clearly understood, which may be laid down as a *Lemma*, to demonstrate the Properties of the following *Sections*.

LEMMA. { *If any two Right Lines, inscrib'd within a Circle, do cut or cross each other* (as *h g* doth *b b* in the annexed Figure) *the Rectangle made of the Segments of one of the Lines will be equal to the Rectangle made of the Segments of the other Line.* (See *Theorem* 15. *Page* 315.)

That is, $ha \times ga = ba \times ab$ }
And $HA \times GA = BA \times AB$ } &c.
consequently if $ba = ab$, and if $BA = AB$, then it will be $ha \times ga = \square\, ba$, and in the *Cone*'s Base $HA \times GA = \square\, BA$.



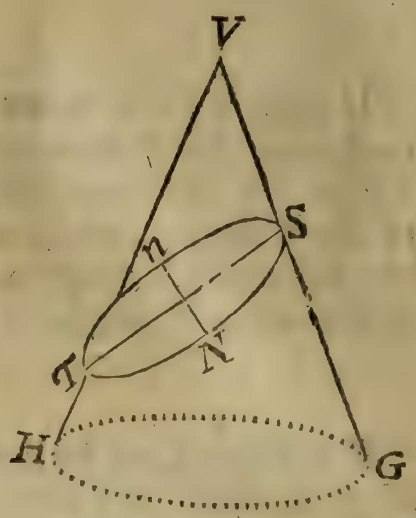

*Sect.* 3.

If a *Right Cone* be *(any where)* cut off by a *Right Line* that cuts both its *Sides*, but not *parallel* to its *Base (as TS in the following Figure)* the Plain of that Section will be an *Ellipsis (vulgarly called an* Oval) *viz.* an oblong or imperfect Circle, which hath several *Diameters*, and *two particular Centers*. That is,

1. Any *Right Line* that divides an *Ellipsis* into *two* equal Parts is call'd a *Diameter*; amongst which the *longest* and the *shortest* are particularly distinguish'd from the rest, as being of most general Use; the other are only applicable to particular Cases,

 2. The

2. The longeſt *Diameter* (as *T S*) is called the *Tranſverſe Diameter*, or *Tranſverſe Axis*, being that *Right Line* which is drawn thro' the Middle of the *Ellipſis*, and doth ſhew or limit its Length.




3. The ſhorteſt *Diameter*, call'd the *Conjugate Diameter*, is a *Right Line* that doth interſect or croſs the *Tranſverſe Diameter* at *Right Angles*, in the Middle or common Center of the *Ellipſis* (as *N n*) and doth limit the *Ellipſis*'s *Breadth*.

4. The *two Points*, which I call particular *Centers* of an *Ellipſis* (*for a Reaſon which ſhall be ſhew'd farther on*) are two Points in the *Tranſverſe Diameter*, at an equal Diſtance each Way from the *Conjugate Diameter*, and are uſually call'd NODES, FOCI, or *burning Points*.

5. All *Right Lines* within the *Ellipſis* that are parallel to one another, and can be divided into two equal Parts, are called ORDINATES with Reſpect to that *Diameter* which divides them: And if they are parallel to the *Conjugate*, viz. at *Right Angles* with the *Tranſverſe Diameter*, then they are call'd *Ordinates rightly apply'd*. And thoſe two that paſs through the *Foci* are remarkable above the reſt, which, being equal and ſituated alike, are call'd both by one Name, *viz.* LATUS RECTUM, or *Right Parameter*, by which all the other *Ordinates* are regulated and valued; as will appear farther on.

*Sect.* 4.

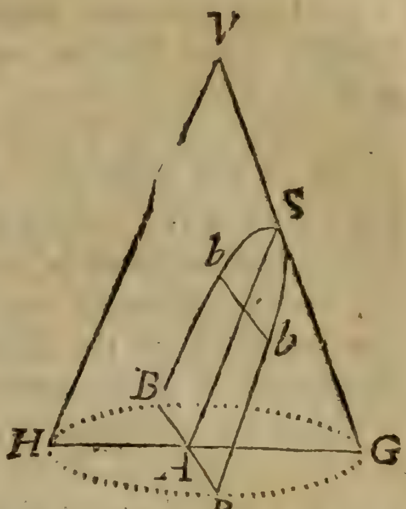
If any Cone be cut into *two Parts* by a *Right-line* parallel to one of its Sides (*as S A in the following Scheme*) the Plain of that Section (*viz. S b B A B b S*) is call'd a PARABOLA.

1. A *Right Line* being drawn thro' the Middle of any *Parabola* (as *S A*) is call'd its *Axis*, or intercepted Diameter.

2. All *Right Lines* that interſect or cut the *Axis* at *Right Angles* (*as B B and b b are ſuppos'd to cut or croſs S A*) are call'd *Ordinates rightly apply'd* (*as in the Ellipſis*) and the greateſt *Ordinate*, as *B B*, which limits the *Length* of the *Parabola's Axis* (*S A*) is uſually call'd the Baſe of the *Parabola*.

3. That

3. That *Ordinate* which paſſes thro' the *Focus*, or burning Point of the *Parabola*, is called the *Latus Rectum*, or *Right Parameter (as in the Ellipſis)* becauſe by it all the other *Ordinates* are *proportion'd*, and may be found.

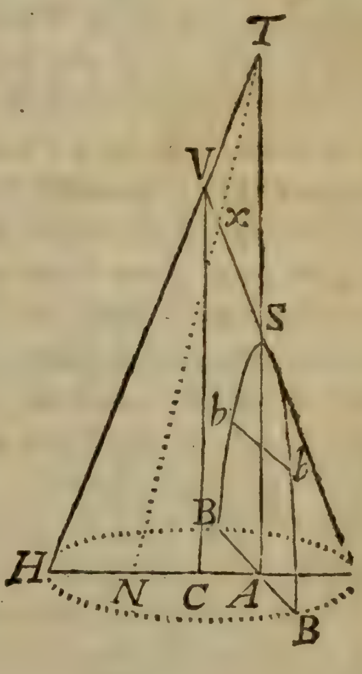



4. The *Node*, *Focus*, or *burning Point* of the *Parabola*, is a Point in its Axis *(but not a Center, as in the Ellipſis)* diſtant from the *Vertex*, or Top of the Section, (viz, *from S)* juſt $\frac{1}{4}$ Part of the *Latus Rectum*; as ſhall be ſhewn farther on.

5. All *Right Lines* drawn within a *Parabola* parallel to its *Axis* are call'd *Diameters*; and every *Right Line*, that any of thoſe *Diameters* doth biſect or cut into two equal Parts, is ſaid to be an *Ordinate* to the *Diameter* which biſects it.

*Sect.* 5.

If a Cone be any where cut by a *Right Line*, either parallel to its *Axis (as S A, or otherwiſe as x N)* ſo as the cutting Line being continued thro' one Side of the Cone (as at *S* or *x)* will meet with the other Side of the Cone if it be continued or produced beyond the *Vertex V*, as at *T*; then the Plain of that Section (viz. *the Figure S b B B b S)* is call'd an HYPERBOLA.




1. A *Right Line* being drawn thro' the Middle of any *Hyperbola*, viz. within the Section *(as S A, or x N)* is call'd the *Axis* or intercepted *Diameter (as in the Parabola)* and that Part of it which is continued or produced out of the Section, until it meet with the other Side of the Cone continued, viz. *T S* or *T x*, &c. is call'd the *Tranſverſe Diameter*, or *Tranſverſe Axis* of the *Hyperbola*.

2. All *Right Lines* that are drawn within an *Hyperbola*, at *Right Angles* to its *Axis*, are call'd *Ordinates rightly apply'd*; as in the *Ellipſis* and *Parabola*.

3. That

3. That *Ordinate* which paſſes thro' the *Focus* of the *Hyperbola* is call'd *Latus Rectum*, or *Right Parameter*, for the ſame Reaſon as in the other Sections.

4. The middle Point of the *Tranſverſe Diameter* is call'd the *Center* of the *Hyperbola*: from whence may be drawn *two Right Lines* (out of the Section) call'd ASYMPTOTES, becauſe they will always incline *(that is, come nearer and nearer)* to both Sides of the *Hyperbola*, but never meet with (or touch) them, altho' both they and the Sides of the *Hyperbola* were infinitely extended; as will plainly appear in its proper Place.

Theſe five Sections, *viz.* the *Triangle*, *Circle*, *Ellipſis*, *Parabola*, and *Hyperbola*, are all the Plains that can poſſibly be produced from a Cone; but of them, the *three laſt* are only called *Conick Sections*, both by the ancient and modern *Geometers*.

*Scholium.*

Beſides the 'foregoing *Definitions*, it may not be amiſs to add, by Way of Obſervation, how one Section may *(or rather doth)* change or degenerate into another.

An *Ellipſis* being that Plain of any Section of the Cone which is between the *Circle* and *Parabola*, 'twill be eaſy to conceive that there may be great Variety of *Ellipſes* produced from the ſame Cone; and when the Section comes to be exactly parallel to one Side of the Cone, then doth the *Ellipſis* change or degenerate into a *Parabola*. Now a *Parabola*, being that Section whoſe Plain is always exactly parallel to the Side of the Cone, cannot vary, as the *Ellipſis* may; for ſo ſoon as ever it begins to move out of that *Poſition* (viz. *from being parallel to the Cone's Side)* it degenerates either into an *Ellipſis*, or into an *Hyperbola:* That is, if the Section incline towards the Plain of the Cone's Baſe, it becomes an *Ellipſis*; but if it incline towards the Cone's Vertex, it becomes an *Hyperbola*, which is the Plain of any Section that falls between the *Parabola* and the *Triangle*. And therefore there may be as many *Varieties* of *Hyperbola's* produced from one and the ſame Cone, as there may be *Ellipſes*.

To be brief, a *Circle* may change into an *Ellipſis*, the *Ellipſis* into a *Parabola*, the *Parabola* into an *Hyperbola*, and the *Hyperbola* into a plain *Iſoſceles Triangle:* And the Center of the Circle, which is its *Focus* or burning Point, doth, as it were, part or divide itſelf into two *Foci* ſo ſoon as ever the Circle begins to degenerate into an *Ellipſis*; but when the *Ellipſis* changes into a *Parabola*, one End of it flies open, and one of its *Foci* vaniſhes, and the

the remaining *Focus* goes along with the *Parabola* when it degenerates into an *Hyperbola:* And when the *Hyperbola* degenerates into a plain *Isosceles Triangle*, this *Focus* becomes the *vertical* Point of the *Triangle* (*viz.* the *Vertex* of the Cone); so that the Center of the Cone's Base may be truly said to pass gradually through all the Sections, until it arrives at the *Vertex* of the Cone, still carrying its *Latus Rectum* along with it: For the *Diameter* of a Circle being that *Right Line* which passes through its *Center* or *Focus*, and by which all other *Right Lines* drawn within the Circle are regulated and valued, may *(I presume)* be properly called the Circle's *Latus Rectum:* and although it loses the Name of *Diameter* when the Circle degenerates into an *Ellipsis*, yet it retains the Name of *Latus Rectum*, with its first Properties, in all the Sections, gradually shortening as the *Focus* carries it along from one Section to another, until at last *it* and the *Focus* become co-incident, and terminate in the *Vertex* of the Cone.

I have been more particular and fuller in these *Definitions* than is usual in Books of this Subject, which I hope is no Fault, but will prove of Use, especially to a *Learner:* And altho' they may perhaps seem a little strange, and at first hard to be understood, yet, when they are well considered, and compar'd with a Cone cut into such Sections as have been defined, they will not only be found true, but will also help to form a true and clear *Idea* of each Section.

## CHAP. II.

### *Concerning the Chief Properties of an* Ellipsis.

NOTE, *If the transverse Diameter of an* Ellipsis, *as TS in the following* Figure, *be intersected or divided into any two Parts by an* Ordinate rightly apply'd, *as at the Points A, C, a,* &c. *then are those Parts TA, TC, Ta, and SA, SC, Sa,* &c. *usually called* Abscissæ (*which signifies Lines or Parts cut off) and by the Rectangle of any two* Abscissæ *is meant the Rectangle of such two Parts as, being added together, will be equal to the Transverse Diameter.*

$$\text{As } TA + SA = TS. \quad \text{And } TC + SC = TS.$$
$$\text{Or } Ta + Sa = TS, \text{ \&c.}$$

*Sect.*

*Section* 1.

Every *Ellipsis* is proportion'd, and all such Lines as relate to it are regulated, by the Help of one general *Theorem.*

THEOREM. { *As the Rectangle of any two* Abscissæ : *is to the Square of Half the* Ordinate *which divides them* : : *so is the Rectangle of any other two* Abscissæ : *to the Square of Half that* Ordinate *which divides them.*

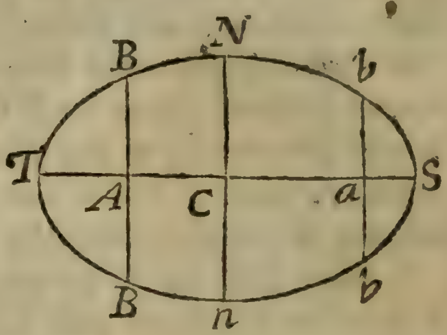
That is,

$TA \times SA : \square BA :: Ta \times Sa : \square ba$

$TA \times SA : \square BA :: TC \times SC : \square NC$

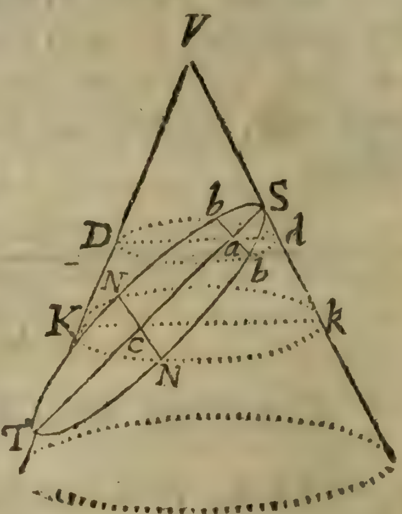
$TC \times SC : \square NC :: Ta \times Sa : \square ba$

&c.




## Demonstration.

Let the annexed Figure represent a Right Cone, cut thro' both Sides by the Right Line *TS*; then will the Plain of that Section be an *Ellipsis* (by *Sect.* 3. *Chap.* 1.) *TS* will be the *Transverse Diameter*, *NCN* and *bab* will be *Ordinates rightly apply'd*; as before. Again, if the Lines *Dd* and *Kk* be parallel to the Cone's Base, they will be Diameters of Circles (by *Sect.* 2. *Chap.* 1.) Then will △ *TCK* and *TaD* be alike. Also, △ *Sad* and △ *SCk* will be alike.



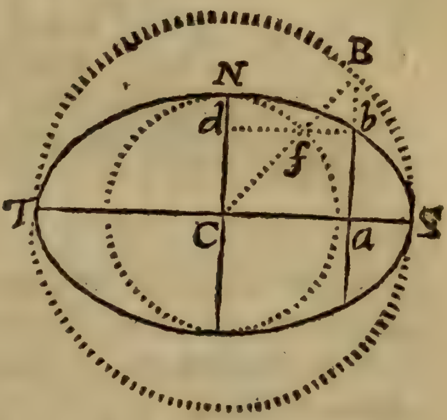

| | | | |
|---|---|---|---|
| *Ergo* | 1 | $Sa : ad :: SC : Ck$ | } per *Theorem* 13. |
| And | 2 | $TC : CK :: Ta : aD$ | |
| 1 ∵ | 3 | $Sa \times Ck = ad \times SC$ | |
| 2 ∴ | 4 | $Ta \times CK = TC \times aD$ | |
| 2 × 3 | 5 | $Sa \times Ck \times Ta \times CK = ad \times SC \times TC \times aD$. | Per *Axiom* 3. |
| But | 6 | $CK \times Ck = \square NC$ | } per *Lemma Sect.* 2. |
| And | 7 | $aD \times ad = \square ba$ | |
| Then | | for $CK \times Ck$, and $aD \times ad$, take $\square NC$ and $\square ba$ | |
| 5, 6, 7 | 8 | $Sa \times Ta \times \square NC = TC \times SC \times \square ba$. | Per *Axiom* 5. |
| Hence | 9 | $Sa \times Ta : \square ba :: TC \times SC : \square NC$. | See *Page* 194. |

Q. E. D.

Or

Or, the Truth of these Proportions may be otherwise prov'd by a Circle, without the Help of the Cone; thus: Let any *Ellipsis* be circumscrib'd and inscrib'd with Circles, as in the following Figure; then from any Point in the circumscrib'd Circle's Periphery, as at $B$, draw the Right Line $B\,a$, parallel to the semi-conjugate Diameter $N\,c$, then will $b\,a$ be a *Semi-ordinate* rightly apply'd to the transverse Diameter $TS$, as before. Again, from the Point $b$ (in the *Ellipsis's Periphery)* draw the Right Line $b\,d$ parallel to the Transverse $TS$; and draw the *Radius* $BC$. Then will $\triangle\,BCa$ and $\triangle\,Cfd$ be alike.

| | | |
|---|---|---|
| Therefore | 1 | $BC : Ba :: Cf : dC$ per *Theorem* 13. |
| But | 2 | $TC = BC$, $NC = Cf$ and $ba = dC$ |
| Conseq. | 3 | $TC : Ba :: NC : ba$ |
| Or | 4 | $TC : NC :: Ba : ba$ |
| 4 in □'s | 5 | $\square TC : \square NC :: \square Ba : \square ba$ |
| But | 6 | $Ta \times Sa = \square\, Ba$ per Lem. *Sect.* 2. *Ch.* 1. |
| Therefore | 7 | $Ta \times Sa : \square\, ba :: TC \times SC = \square\, TC : \square\, NC$, as before. |




And so for any other *Abscissæ* and their *Semi-ordinates.*

These Proportions being found to be the true and common Properties of every *Ellipsis*, all that is farther requir'd in (or about) that *Section* may be easily deduced from them.

## Sect. 2. *To find the* Latus Rectum, *or* Right Parameter *of any Ellipsis.*

There are several Ways of finding the *Latus Rectum*, but I think none so easy, and shews it so plainly to be the Third Principal Line in the *Ellipsis*, as the following.

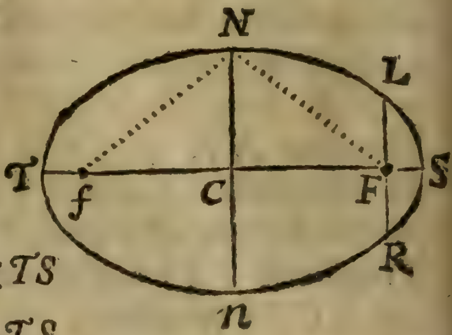
THEOREM. { *As the* Transverse Diameter : *is in Proportion to the* Conjugate : : *so is the* Conjugate : *to the* Latus Rectum.

Viz. *(in the following Fig.)* $TS : Nn :: Nn : LR$ the *Latus Rectum.*

### Demonstration.

From the last *Proportions* take either of the Antecedents, and its Consequent, *viz.* either $TC \times SC : \square\, NC$, or $Ta \times Sa : \square\, ba$, and

and make $TS$ the third Term, to which find a fourth Proportional, and it will be $= LR$:




| | | |
|---|---|---|
| Thus | 1 | $TC \times SC : \square NC :: TS : LR$ |
| But | 2 | $\begin{cases} TC = SC \\ \text{and } NC = Cn \end{cases}$ |
| Therefore | 3 | $TC \times SC = \frac{1}{4} \square TS$ |
| And | 4 | $\square NC = \frac{1}{4} \square Nn$ |
| 1, 3, 4 | 5 | $\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$ |
| 5 ∵ | 6 | $\frac{1}{4} \square TS \times LR = \frac{1}{4} \square Nn \times TS$ |
| 6 × 4 | 7 | $\square TS \times LR = \square Nn \times TS$ |
| 7 ÷ TS | 8 | $TS \times LR = \square Nn$ which gives the following *Analogy*. |
| *viz* | 9 | $TS : Nn :: Nn : LR$ |
| Again | 10 | $TC \times SC : \square NC :: Ta \times Sa : \square ba$ by common *Properties*. |
| 1, 10 | 11 | $TS : LR :: Ta \times Sa : \square ba$. |

From hence 'tis evident that $LR$, thus found, is that *Ordinate* by which the other *Ordinates* may be regulated and found. Therefore (according to its Definition *Sect.* 3, *Chap.* 1.) it is the true *Latus Rectum*. Q. E. D.

*Consectary.*

Hence it follows, that if the tranſverſe and conjugate Diameters of any *Ellipſis* are given *(either in Lines or Numbers)* the *Latus Rectum* may be eaſily found; and then any *Ordinate*, whoſe Diſtance from the Conjugate is given, may be found, as above.

## Sect. 3. *To find the* Focus *of any* Ellipſis.

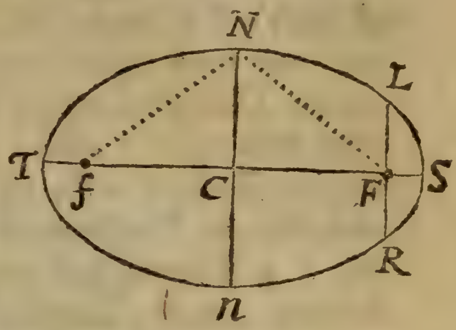
The *Focus* is the Diſtance of the *Latus Rectum* from the Conjugate or Middle of the *Ellipſis (vide* Definition 4, Page 364.) and that Diſtance is always a *Mean Proportional* between the half Sum and half Difference of the tranſverſe and conjugate Diameters, which gives this *Theorem*.

THEOREM. { *From the* Square *of half the* Tranſverſe *ſubtract the* Square *of half the* Conjugate, *the ſquare Root of their Difference will be the Diſtance of each* Focus *from the Middle or common Center of the Ellipſis.*

That is, ſuppoſing the Points $f$ and $F$ to be the two *Foci*, viz. $fC = Cf$, and $TC = \frac{1}{2} TS$ . $NC = \frac{1}{2} Nn$. Then, $TC + NC : fC :: FC : TC - NC$. Ergo $\square FC = TC - \square NC$. Conſequently, $FC = \sqrt{\square TC - \square NC}$.

## Demonstration.

| | | |
|---|---|---|
| First, | 1 | $TS \times LR = \square Nn$, by 8th Step of the last Process. |
| And | 2 | $TS : LR :: TF \times SF : \square LF$, common Properties. |
| That is, | 3 | $TS : LR :: \overline{TC + CF} \times \overline{TC - CF} : \frac{1}{4} \square LR = \square LF$ |
| 3 ∵ | 4 | $\frac{1}{4} \square LR \times TS = \overline{\square TC - \square CF} \times LR$ |
| $4 \div LR$ | 5 | $\frac{1}{4} LR \times TS = \square TC - \square CF$ |
| $\overline{1 \div 4}$ | 6 | $\frac{1}{4} TS \times LR = \frac{1}{4} \square Nn \square = NC$ |
| 5, 6 | 7 | $\square NC = \square TC - \square CF$ |
| 7, $\overline{+}$ | 8 | $\square CF = \square TC - \square NC$ |
| 8 $\omega^2$ | 9 | $CF = \sqrt{\overline{\square TC - \square NC}}$ |

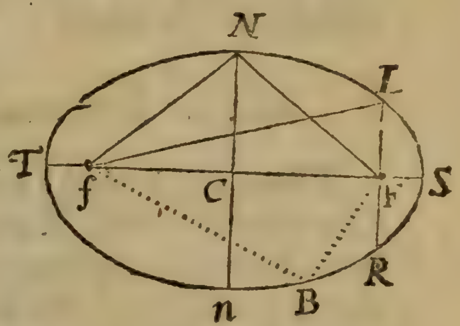



Now from hence is deduced that notable Proposition, upon which is grounded the usual Method of describing an *Ellipsis*, and drawing of *Tangents*, &c.

PROPOSITION. *If from the two* Foci *of any* Ellipsis *there be drawn two* Right Lines, *so as to meet each other in any Point of the* Ellipsis's Periphery, *the* Sum *of those* Lines *will be equal to the* Transverse.

Viz. $fN \times NF = TS$. $fL \times LF = TS$. Or $fB + BF = TS$, &c.

## Demonstration.




| | | |
|---|---|---|
| First, | 1 | $\square CF \times \square NC = \square TC$ *by 8th of the last.* |
| But | 2 | $\square CF + \square NC = \square NF$ *by* Theorem 11. |
| 1, 2 | 3 | $\square NF = \square TC$ *by* Axiom 5. |
| 3 $\omega^2$ | 4 | $NF = TC$ |
| Hence | | $2 NF = 2 TC = TS$ |
| Again, | 5 | $TS : LR :: TF \times FS : \square LF$, by common Properties. |
| Conseq. | 6 | $\frac{1}{2} TS : \frac{1}{2} LR :: TF \times FS : \square LF$ |
| But, | | $\frac{1}{2} TS = TC$. And $\frac{1}{2} LR = LF$ |
| *Ergo* | 7 | $TC : LF :: \overline{TC + CF} \times \overline{TC - CF} : \square LF$ |
| 7 ∵ | 8 | $TC \times LF = \square TC - \square CF$ |
| But, | 9 | $\square fF + \square LF = \square fL$, *by* Theorem 11. |
| That is, | 10 | $4 \square CF + \square LF = \square fL$, for $2 CF = fF$ |
| $\overline{8 \times 4}$ | 11 | $4 \square TC - 4 \square CF = 4 TC \times LF$ |
| $10 + 11$ | 12 | $4 \square TC + \square LF = 4 TC \times LF + \square fL$ |
| $12 -$ | 13 | $4 \square TC - 4 TC \times LF + \square LF = \square fL$ |

| | | |
|---|---|---|
| 13 ꟺ² | 14 | $2\,TC - LF = fL$ |
| 14 + $LF$ | 15 | $2\,TC = fL + LF$. But $2\,TC = TS$ |
| *Ergo* | | $TS = fL + LF$. Q. E. D. |

And this *Proposition* must needs hold true to every Point in the *Ellipsis*'s *Periphery*, *viz.* at *B*, &c. As will evidently appear to any one who rightly considers, That, as a Thread just the Length of the Diameter of any Circle having its two Ends ty'd together, and then mov'd about a Point in the Center (viz. *by making it a double Radius)* will, by drawing another Point in its Extremity, describe the *Periphery* of a Circle; [*vide* Definition *Page* 280] even so, if a Thread just the Length of the transverse Diameter *(TS)* having its two Ends so fix'd upon the two *Foci (f* and *F)* that it may be mov'd about them, by drawing a Point in its Extremity (viz. *at its full Stretch)* it will describe the true *Periphery* of an *Ellipsis*.

Now, altho' this easy Way of describing, or, as usually phras'd, drawing an *Ellipsis*, be mechanical, and known even to most *Joiners*, *Carpenters*, &c. yet it gives as compleat and clear an Idea of that Figure as any other Way whatsoever; and by describing it thus about its two *Foci*, as a Circle is about its Center, doth plainly shew that those two Points are not improperly call'd particular Centers in *Definition* 4, *Sect.* 3, *Chap.* 1. for each of them bears much the same Respect to the *Ellipsis*'s Periphery, as the Circle's Center doth to its Periphery.

### Sect. 4. *To describe or delineate an* Ellipsis *several Ways.*

There are several *(other)* Ways of describing an *Ellipsis*, both Geometrically and Numerically, according to peculiar Occasions, but I shall only mention two or three of them, leaving the rest to the *Learner's Genius*. Now, in order to that Work, it will be convenient to consider what *Lines* are requisite to limit or bound its Form, which I take to be chiefly these following.

1. If the Transverse and Conjugate are given, the *Ellipsis* is perfectly limited; (vide *Consectary Page* 363.) for if $TS$ and $Nn$ be set at *Right Angles* in their Middle at $C$, and $TC$ or $CS$ be set *off* from $N$, or $n$, both Ways upon the Transverse to $f$ and $F$, (viz. *make* $fN = TC = NF$) then will those Points $f$ and $F$ be the two *Foci* (by *4th Step of the last Process)* and then the *Ellipsis* may be describ'd as above.

2. If

2. If the *Transverse Diameter* and *Latus Rectum* are given, the *Ellipsis* is truly limited, because by them the Conjugate may be found, by Sect. 2.

3. Or if only the *Transverse*, and the Proportion it hath either to the *Conjugate* or *Latus Rectum*, be given, the *Ellipsis* is thereby limited. As for Instance; suppose the given *Ratio* between the Transverse and Conjugate to be, as $a$ : to $d$:

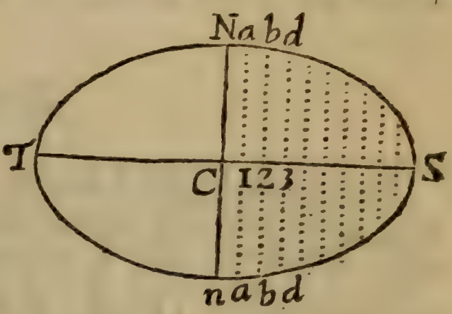
Viz. $a : d :: TS : Nn$, then $\frac{TS \times d}{a} = Nn$, &c.

4. If either the Transverse or Conjugate, and the Distance of the *Focus* from the *Conjugate* be given, the *Ellipsis* is limited, because by them the Conjugate or Transverse may be found.

These being premis'd, and the *precedent Work* a little consider'd, it must be easy to describe or delineate any *Ellipsis* in *Plano*, either Geometrically or Numerically.

1. *To describe an* Ellipsis *Numerically* by *Points.*

Suppose the *Transverse Diameter* $TS = 20$, and the *Conjugate* $Nn = 12$, *(either Inches, or any other equal Parts)* and let them cross each other at *Right Angles* in their Middles, as in the Point $C$; then will $TC = CS = 10$, and $NC = Cn = 6$, and it will be $20 : 12 :: 12 : 7, 2 =$ the *Latus Rectum.*




Again $20 : 7, 2$. Or rather take their *Ratio*.

Thus $\begin{cases} 1 : 0,36 :: \overline{10+1} \times \overline{10-1} : \square a. \parallel 1. \\ 1 : 0,36 :: \overline{10+2} \times \overline{10-2} : \square b. \parallel 2. \\ 1 : 0,36 :: \overline{10+3} \times 10-3 : \square d. \parallel 3. \text{ \&c.} \end{cases}$

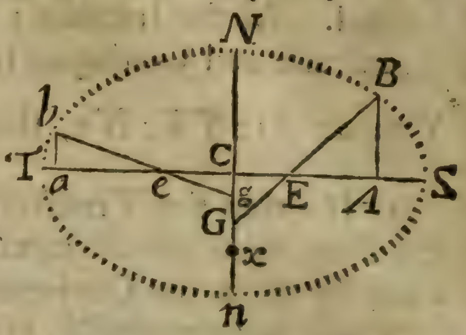
*Viz.* $\begin{cases} \overline{100-1} \times 0,36 = \square a.1. \text{ Hence } \sqrt{\overline{90 \times 0}} = 5,97 \text{ \&c.} = a.\ 1 \\ \overline{100-4} \times 0,36 = \square b.2. \quad \sqrt{\overline{90 \times}} = 5,88 \text{ \&c.} = b.\ 2 \\ \overline{100-9} \times 0,36 = \square d.3. \quad \sqrt{\overline{91 \times 0,36}} = 5,72 \text{ \&c.} = d.\ 3 \end{cases}$

If so many *Semi-ordinates* as may be thought convenient *(the more the better)* be found in this Manner, and every one of them be set off at *Right Angles* from its respective Point in the *Transverse Diameter* each Way, *viz.* from 1 to $a$, from 2 to $b$, from 3 to $d$, &c. Then if a Curve Line be carefully drawn with an even Hand thro' those extreme Points $a$, $b$, $d$, &c. it will be the *Ellipsis's Periphery* requir'd.

2. *To*

### 2. *To describe an* Ellipsis *Geometrically by Points.*

Having the Transverse and Conjugate Diameters given, *viz.* $TS$ and $Nn$, placed at *Right Angles* in their *Middles*, as before: Then from either End of the Conjugate, *viz.* $N$ (or $n$) set off half the Transverse Diameter to $x$. That is, make $Nx = TC$ (*continuing the* Conjugate $Nn$ *when it is shorter than* $TC$) Or, which is all one, make $Cx = TC - NC$. Then take any Point in the Line $Cx$ at Pleasure; suppose it at $G$, and from that Point at $G$ set off the Distance $Cx$ to the Transverse (as at $E$) *viz.* make $GE = Cx$, and join the Points $GE$ with a Right Line, produced so far beyond $E$ as to make $EB = NC$. Consequently $GB = TC$.




Then, I say, where-ever the Point $G$ was taken between $C$ and $x$, the Point $B$ will just touch (*or fall in*) the *Ellipsis's Periphery.*

## Demonstration.

Draw the Right Line $BA$ perpendicular to $TS$, *viz.* let $BA$ be a *Semi-ordinate* rightly apply'd to the transverse Diameter $TS$; then $\triangle GCE$ and $\triangle BAE$ will be alike.

| | | |
|---|---|---|
| Consequently | 1 | $CE : AE :: EG : EB$, by Theorem 13. |
| 1, and | 2 | $CE + AE : AE :: EG + EB : EB$. See p. 192. |
| But | 3 | $CE + AE = CA . EG \times EB = TC$. And $EB = NC$ |
| Therefore | 4 | $CA : AE :: TC : NC$ |
| 6, in $\square$'s | 5 | $\square CA : \square AE :: \square TC : \square NC$ |
| 5, ∵ | 6 | $\frac{\square CA \times \square NC}{\square TC} = \square AE$ |
| But | 7 | $\square NC - \square AB = \square AE$ |
| That is, | | $\square EB - \square AB = \square AE$ |
| 6, 7 | 8 | $\frac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$ |
| 8 × $\square TS$ | 9 | $\square CA \times \square NC = \square NC \times \square TC - \square AB \times \square TC$ |
| 9 ± | 10 | $\square NC \times \square TC - \square CA \times \square NC = \square AB \times \square TC$ |
| 10, *Analogy* | 11 | $\square TC : \square NC :: \square TC - \square CA : \square AB$ |
| That is, | 12 | $TC \times CS : \square NC :: \overline{TC + CA} \times \overline{TC - CA} : \square AB$ |

which is according to the common *Properties* of the *Ellipsis*: Therefore the Point $B$ is truly found. Q. E. D.

Hence

Hence it follows, that if a convenient Number of ſuch Lines as *G E B* be ſo drawn (as above directed) from the like Number of Points taken between *C* and *x*, &c. their extream Points (as at *B)* will be thoſe Points by which *(with an even Hand)* the *Ellipſis* may be truly deſcrib'd, as before.

But, if this be well underſtood, it will be very eaſy to conceive how to deſcribe an *Ellipſis* very readily, without drawing thoſe Lines, by having a thin, ſtreight, narrow *Ruler* juſt the Length of *T C*, made ſomewhat ſharp at both Ends, upon which, from one of its Ends, ſet off the Length of *N C*. Then, if the Point upon the Ruler which repreſents *E* be gradually or eaſily moved along the Tranſverſe *T S*, and at the ſame Time the Point or End repreſenting *G* be kept ſliding cloſe along the Conjugate *N n*, 'tis evident from the Work above, that the End of the Ruler repreſenting *B* will, by that Motion, aſſign the true *Periphery* of the *Ellipſis* required; for by that Motion the ſtreight Edge of the Ruler doth ſupply an infinite Number of the aforeſaid Lines; as will appear very plain and eaſy in Practice.

*Scholium.*

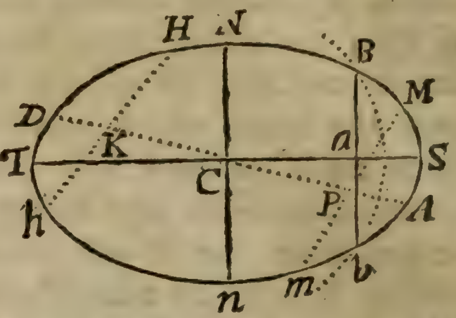
Now from hence was deduced the firſt Invention of that well-contrived *Inſtrument* for drawing an *Ellipſis* by one Motion, commonly called the *Elliptical Compaſſes*, being uſually made of Braſs, and compos'd of three Parts, two of which repreſent *(or rather ſupply)* the tranſverſe and conjugate Diameters ſet together at *Right Angles*; and the third Part is a moveable Ruler, which performs the Office of the laſt-mentioned thin Ruler. But becauſe the making of it is ſo well known to moſt Mathematical Inſtrument-makers, *eſpecially to that accurate and ingenious Artiſt Mr* JOHN ROWLEY, *Mathematical Inſtrument-maker, under St.* Dunſtan'*s Church in* Fleet-ſtreet, London; *who, for his great* Skill *in contriving, framing, and graduating all kind of Mathematical Inſtruments, may, I believe, be juſtly called one of the beſt Workmen of his Trade in* Europe; I think it needleſs therefore to give a *particular* Deſcription of that *Inſtrument.*

Alſo from hence came that *ingenious Invention* of making *Engines* for turning all Sorts of elliptical or oval Work, as oval Boxes, Picture-Frames, &c.

Sect.

## Sect. 5. *Any Ellipsis being given, to find its* Transverse *and* Conjugate *Diameters.*

Suppose the given *Ellipsis* to be *T N S n* (in the annexed Scheme) in which let it be required to find the transverse Diameter *T S* and its Conjugate *N n*. Draw within the *Ellipsis* any two *Right Lines* parallel to each other as *H h* and *M m*, and bisect those Lines, *viz.* find the Middle Point of each, as at *K* and *P*; then thro' those Points *K* and *P* draw a Right Line, as *D A*, and it will be a Diameter; for it will divide the Ellipsis into two equal Parts, [*See* Defin. 1, *Page* 363.] consequently the Middle of *D A* will be the true Middle or common Center of the Ellipsis, as at *C*.




*For 'tis the Nature and Property of all* Diameters, *howsoever they are drawn in any* Ellipsis *(as 'tis in a* Circle) *to cut or cross one another in the common Center or Middle of the Figure, as at C.*

Upon the Point *C* describe an *Arch* of any Circle that will cut the Ellipsis's Periphery in two Points, as at *B* and *b*; then join those Points *B b* with a *Right Line*, and it will be an *Ordinate*, thro' whose Middle (as at *a*) and the common Center *C*, the transverse Diameter *T S* must pass. For *B S* = *S b*, and *B a* is at Right Angles with *T S*; therefore the Line *B b* is an *Ordinate* rightly apply'd to *T S* the transverse Diameter. And if thro' the Point *C* there be drawn the Right Line *N n* parallel to *B b*, it will become the Conjugate; as was requir'd.

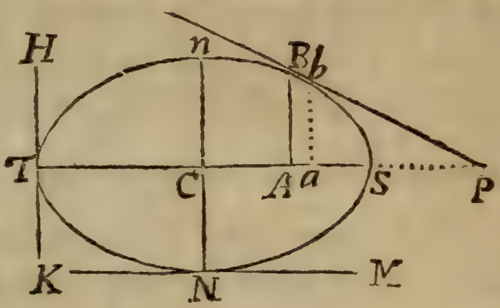
## Sect. 6. *To draw a* Tangent, *or Right Line that may touch the Ellipsis's Periphery in any assigned Point.*

The Drawing of *Tangents* to or from any *assigned Point* in the Ellipsis's Periphery, admits of three Cases.

*Case* 1. If it be requir'd to draw a *Tangent* that may touch the Ellipsis in either of the extream Points of its transverse Diameter, as at *T* or *S*, it is plain the *Tangent* must be drawn parallel to the conjugate Diameter *N n*; as *H K* in the following *Figure* is suppos'd to be.

*Case*

*Case* 2. Or, if the *Tangent* must be drawn to touch the *Ellipsis* in either of the extream Points of its *Conjugate Diameter*, as at *N* or *n*, 'tis as evident that it must be drawn parallel to the *Transverse Diameter* *T S*, as *K M*. Consequently if that *Tangent* and the *Transverse* were both infinitely continu'd, they would never meet.




*Case* 3. But if it be requir'd to draw a *Tangent* that may touch the *Ellipsis* in any other *Point*, as at *B*, &c. Then, if the *Tangent* and the *Transverse Diameter* be both continu'd, they will meet in some *Point*, as at *P*; and those two Points (viz. *B* and *P*) do so mutually depend upon each other, that one of them must be assigned in order to find the other, that so the *Tangent* may by them be truly drawn. Let $D = TS$, $y = AS$, and $z = AP$. Then, if $y$ be given, $z$ may be found by this

*Theorem* $\left\{ \frac{Dy - yy}{\frac{1}{2}D - y} = z. \right.$ Or, if $z$ be given, $y$ may be found by this *Theorem* $\left\{ \frac{D+z}{2} \pm \sqrt{\left\{ \frac{DD + zz}{4} \right.} = y. \right.$

## Demonstration.

Draw the *Semi-ordinate* *b a*, as in the *Figure*; then will $\triangle\, BAP$ and $\triangle\, baP$ be alike. Put $x = Aa$ the *Distance* between the two *Semi-ordinates* (viz. *between* *BA* and *ba*) which we suppose infinitely small.

| | | |
|---|---|---|
| Then | 1 | $z : z - x :: BA : ba$, by *Theorem* 13. |
| But | 2 | $\overline{D-y}\times y : \overline{D-y+x}\times\overline{y-x} :: \square BA : \square ba$ |
| That is, | 3 | $Dy - yy : Dy - yy + 2bx - Dx - xx :: \square BA : \square ba$ |
| 1 in □'s | 4 | $zz : zz - 2zx + xx :: \square BA : \square ba$ |
| Suppose | 5 | $x = 0$, that so $x$ may be every where rejected. |
| 3, Then | 6 | $Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$ |
| 4, And | 7 | $zz : zz - 2z :: \square BA : \square ba$ |
| 6, 7 | 8 | $Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$ |
| 8 ∵ | 9 | $2yzz - Dzz = 2yyz - 2Dyz$ |
| 9 ÷ 2z | 10 | $yx - \frac{1}{2}Dz = yy - Dy$ |
| 10 ± | 11 | $\frac{1}{2}Dz - yz = Dy - yy$ |

| | | |
|---|---|---|
| 11 ÷ | 12 | $z = \frac{Dy - yy}{\frac{1}{2}D - y}$ which is the 1ſt *Theorem*, and gives the following *Analogy*. |
| *Analogy* | 13 | $\frac{1}{2}D - y : y :: D - y : z$. Viz. $CA : SA : TA : AP$ |
| 10—$yz$ | 14 | $yy - Dy - yz = -\frac{1}{2}Dz$ |
| 14 $C\,\square$ | 15 | $yy - Dy - yz + \frac{1}{4}DD - \frac{1}{2}Dz + \frac{1}{4}zz = \frac{1}{4}DD + \frac{1}{4}zz$ |
| 15 $\omega^2$ | 16 | $y - \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{1}{4}DD - \frac{1}{4}zz}$ |
| That is, | 17 | $y = \frac{1}{2}D + \frac{1}{2}z \pm \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$ which is the 2d *Theor.* Q. E. D. |

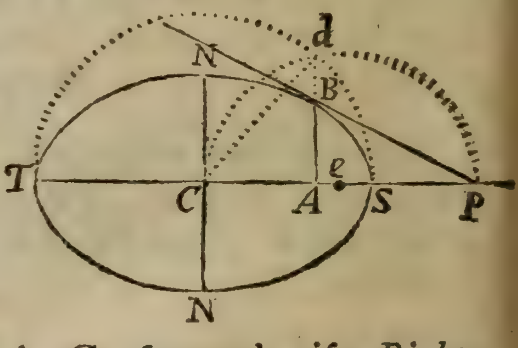
The *Geometrical* Performance of theſe two *Theorems* is very eaſy, as by the following *Figure*.

1. Suppoſe the *Point B* in the *Ellipſis Periphery* were given, and it were requir'd to find the *Point P*, &c.

Make *TC Radius*, and upon the common Center *C* deſcribe the *Semicircle T d S*, and join the *Points C* and *d* with a *Right Line*; then biſect that *Line* (by *Prob.* 2, *Page* 287) and mark the *Point* where the biſecting *Line* would croſs the *Tranſverſe*, as at ‖ *e*. Upon that *Point* ‖ *e*, with the *Radius Ce (or Cd)* deſcribe another *Semicircle*, producing the *Tranſverſe Diameter* to its *Periphery*, and it will aſſign the *Point P*.

For if $D = TS$, $y = AS$, $z = AP$, as before.

| | | |
|---|---|---|
| Then | 1 | $\overline{D - y} \times y = \square\, dA$ |
| And | 2 | $\overline{\frac{1}{2}D - y} \times z = \square\, dA$ |
| For | 3 | $TA : dA :: dA : SA$ |
| And | 4 | $CA : dA :: dA : AP$ |
| But | 5 | $CA = \frac{1}{2}D - y$, &c. |
| 1, 2 | 6 | $\frac{1}{2}Dz - yz = Dy - yy$ as at the 11th *Step* before |



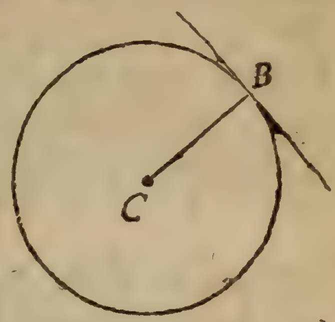

Therefore the *Point P* is truly found. Conſequently, if a *Right Line* be drawn through thoſe *Points B* and *P*, it will be the *Tangent* requir'd, according to the firſt *Theorem*.

2. The *Converſe* of this is as eaſy, to wit, if the *Point P* be given, thence to find the *Point B* in the *Ellipſis Periphery*. Thus, circumſcribe half the *Ellipſis* with the *Semicircle T d S*, as before; and biſect the Diſtance between the *Points C* and *P*, as at *e*, *viz.* Let $Ce = eP$. Then making *C e Radius*, upon the Point *c*, deſcribe the *Semicircle C d P*; and from the *Point* where the two *Semicircles* interſect or croſs each other, as at *d*, draw the *Right Line d A* perpendicular to the *Tranſverſe TS*, and it will aſſign the

the Point of Contact *B* in the *Ellipsis Periphery* through which the *Tangent* must pass.

But the *Practical Method* of drawing *Tangents* to any assign'd Point in the *Ellipsis Periphery* may *(without finding the aforesaid Point P)* be easily deduced from the following *Property* of *Tangents* drawn to a *Circle*, which is this:

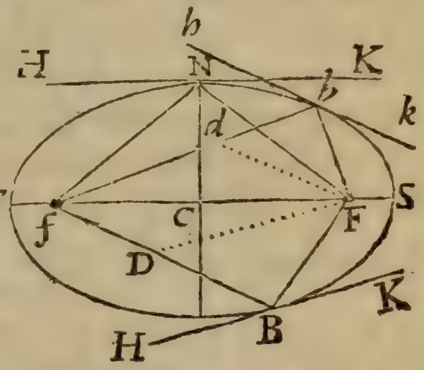
If to any *Radius* of a *Circle*, as *C B*, there be drawn a *Tangent Line (as HK)* to *touch* the *Radius* at the *Point B*; the two *Angles*, which the *Tangent* makes with the *Radius*, will always be two *Right Angles* (16, 17, 18, 19 *Euclid* 3.) that is, $\angle HBC = \angle CBK = 90°$.




In like Manner the two *Angles*, made between the *Tangent* and the two Lines drawn from the *Foci* of any *Ellipsis* to the *Point* of *Contact*, will always be equal, but not *Right Angles*, save only at the two Ends of the *Transverse Diameter*.

These being well consider'd, and compar'd with what hath been said in *Page* 366, it must needs be easy to understand the following Way of drawing *Tangents* to any *assign'd Point* in the *Ellipsis Periphery*; which is thus:

Having by the *transverse* and *conjugate Diameters* found the two *Foci f* and *F*, by *Sect.* 3. from them draw two *Right Lines* to meet each other in the *assign'd Point* of *Contact*, as *f b* and *F b* (or *f B* and *F B)* in the *annex'd Figure*. Next set off (viz. make) $b d = b F$ (or $B D = B F$) and join the *Points Fd* (or *FD*) with a *Right Line*.




Then, I say, if a *Right Line* be drawn through the *Point* of *Contact* *b* (or *B)* parallel to *d F*, or *D F*, it will be the *Tangent* requir'd. For it is plain, that as the $\angle f N H = \angle F N K$ when the *Tangent* is parallel to the *Transverse Diameter*, even so is the $\angle f b h = \angle F B k$, (and $\angle f B H = \angle F B K$) and will be every where so, as the *Point* of *Contact b* (or *B)* and its *Tangent* is carried about the *Ellipsis Periphery* with the *Lines f b F* (or *f B F)*.

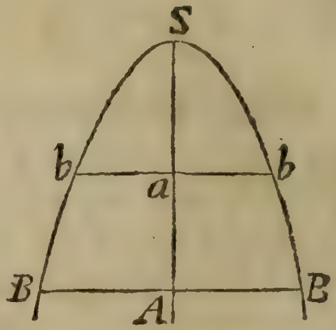
## CHAP. III.

### *Concerning the Chief Properties of every* Parabola.

NOTE, in every Parabola, the intercepted Diameter, or that Part of its Axis, which is between the Vertex and that Ordinate which limits its Length, as *Sa* or *S A*, &c. is call'd an Abſciſſa.

*Sect.* 1. The Plain or Figure of every Parabola is proportion'd by its Ordinates and Abſciſſæ, as in the following Theorem:

THEOREM { *As any one Abſciſſa : is to the Square of its Semi-ordinate : : ſo is any other Abſciſſa : to the Square of its Semi-ordinate.*




That is, if we ſuppoſe the *annex'd Figure* to be a *Parabola*, wherein *S a*, and *S A*, are *Abſciſſæ*, and *b a b*, *B A B*, *Ordinates* rightly apply'd, it will

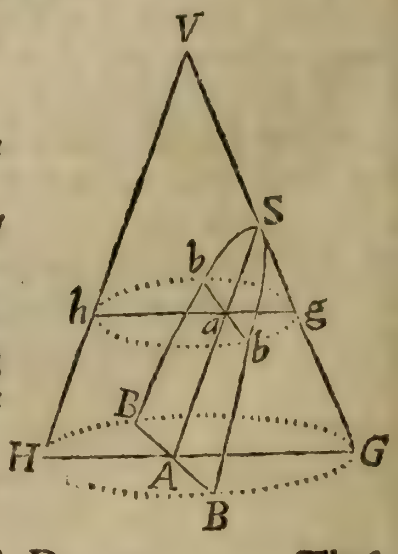
be *Sa* : □ *ba* : : *S A*: □ *B A* } wherefoever
or *Sa* : *SA* : : □ *ba* : □ *B A* } the *Points a*, *A*, are taken,

And ſo for any other *Abſciſſæ*, &c.

### Demonſtration.

Let the following Figure *HVG* repreſent a *Right Cone* cut into two Parts by the *Right Line S A*, parallel to its Side *V H*. Then the *Plain* of that *Section*, viz. *B b S b B*, will be a *Parabola*, by *Sect.* 4. *Page* 364. wherein let us ſuppoſe *S A* to be its *Axis*, and *b a b*, *B A B* to be *Ordinates* rightly apply'd to that *Axis*. Again, imagine the *Cone* to be cut by the *Right Line h g* parallel to its *Baſe HG*. Then will *h g* be the *Diameter* of a *Circle*, by *Sect.* 2. *Page* 363. and △ *S a g* like to △ *S A G*.

| | | |
|---|---|---|
| Therefore | 1 | { *Sa* : *ag* : : *S A* : *AG* By *Theorem* 13. |
| 1 ∵ | 2 | *Sa* × *AG* = *S A* × *ag* |
| 2 × *h a* | 3 | { *Sa*×*AG*×*ha*=*S A*×*ag*×*ha* By *Axiom* 3. |
| But | 4 | { *HA* = *ha*, becauſe *S A* is parallel to *VH* |
| And { | 5 | □*BA*=*AG*×*HA* } By Lem. □ *ba* = *ag* × *ha* } *P.* 363 |
| 3, 4, 5, | 6 | { *Sa* × □ *BA*=*SA*× □ *ba* By *Axiom* 5. |
| 6, *Analogy* | 7 | { *Sa* : □ *ba* : : *SA*: □ *BA* Vide *Page* 194. |




—————— Q. E. D.

Theſe

These *Proportions* being prov'd to be the common *Property* of every *Parabola*, all that is farther requir'd about that *Section*, or *Figure*, may from thence easily be deduced.

## Sect. 2. *To find the* **Latus Rectum** *or Right* **Parameter** *of any Parabola.*

The *Latus Rectum* of a *Parabola* hath the same *Ratio* or *Proportion* to any *Abscissa*, and its *Semi-Ordinate*, as the *Latus Rectum* of any *Ellipsis* hath to its *Transverse* and *Conjugate Diameters*, and may be found by this *Theorem*.

THEOREM { *As any Abscissa : is in Proportion to its Semi-ordinate : : so is that Semi-ordinate : to the* Latus Rectum.

Let $L =$ the *Latus Rectum.*

| | | | |
|---|---|---|---|
| Then | 1 | $Sa : ba :: ba : L$ | where-ever the *Points a*, and |
| And | 2 | $SA : BA :: BA : L$ | *A*, are taken in the *Axis*. |
| 1 $\because$ | 3 | $\frac{\square ba}{Sa} = L$ : Or $Sa \times L = \square ba$ | |
| 2 $\because$ | 4 | $\frac{\square Ba}{Sa} = L$ : Or $SA \times L = \square BA$ | |
| 3 = 4 | 5 | $\frac{\square BA}{SA} = \frac{\square Sa}{ba}$ Per *Axiom* 5. | |
| 5 × | 6 | $Sa \times \square BA = SA \times \square ba$, which gives this | |
| *Analogy* | 7 | $Sa : \square ba :: SA : \square BA$, the same as at the 7th | |

Step of the last *Process*; therefore $L$ (thus found) is the true *Latus Rectum*, by which all the *Ordinates* may be regulated and found, according to its *Definition* in *Section* 4, *Page* 364. For by the third Step $Sa \times L = \square ba$, and by the 4th Step $SA \times L = \square BA$. Consequently $\sqrt{Sa \times L} = ba$ and $\sqrt{SA \times L} = BA$; and so for any other *Ordinate*.

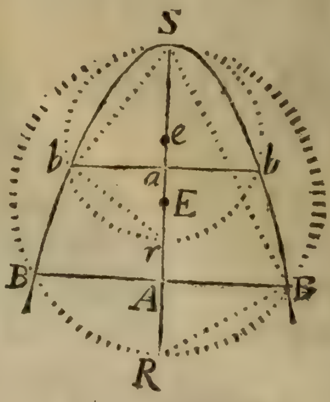
Or if the *Ordinates* are given, to find their *Abscissæ*; then it will be, $L : ba :: ba : Sa$, and $L : BA :: BA : SA$, &c.

Consequently $\frac{\square ba}{L} = Sa$, and $\frac{\square Ba}{L} = SA$, &c.

From the Consideration of these *Proportions*, it will be *easy* to conceive how to find the *Latus Rectum Geometrically*, thus:

Join

Join the *vertical Point S* of the *Axis*, and either *extream Point* of any *Ordinate* as *B* (or *b*) with a *Right Line*, viz. *S B* (or *S b*) and bisect that Line (by *Problem* 2. *Page* 287.) marking the *Point* where the bisecting *Line* doth intersect or cross the *Axis*, as at *E* (or *e*) and with the *Radius S E* (or *S e*) upon the *Point E* (or *e*) describe a *Circle*; (as in the annex'd Figure) then will the Distance between the *Ordinate* and that *Point* where the *Circle*'s *Periphery* cuts the *Axis*, viz. *A R* (or *a r*) be the true *Latus Rectum* required.




For $SA : BA :: BA : AR$, and $Sa : ba :: ba : ar$, by *Theor.* 13. therefore $AR = L$. And $ar = L$, by the 1st and 2d *Steps* above.

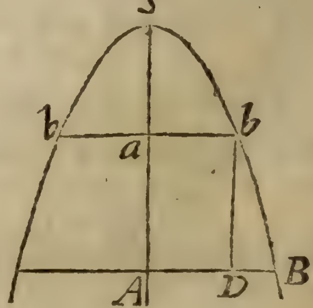
*Consectary.*

From these *Proportions* of finding the *Latus Rectum*, it will be easy to deduce and demonstrate the following *Theorem*:

THEOREM { *As the* Latus Rectum : *is to the Sum of any two Semi-ordinates* : : *so is the Difference of those two Semi-ordinates* : *to the Difference of their Abscissæ.*

Suppose any *Right Line* drawn within the *Parabola*, as *b D*, parallel to its *Axis S A*; then will that *Line* (viz. *b D*) be a *Diameter* (by *Def.* 5, *Page* 365) which will make $ED = AB + ab$, $DB = AB - ab$, and $bD = SA - Sa$. Then it will be $L : ED :: DB : bD$, according to the *Theorem*.

## Demonstration.




| | | |
|---|---|---|
| First | 1 | $SA = \frac{\square BA}{L}$, by *Step* 2. of the last *Process*. |
| And | 2 | $Sa = \frac{\square ba}{L}$ by *Step* 1. of the last *Process*. |
| 1 — 2 | 3 | $SA - Sa = \frac{\square BA - \square ba}{L}$ |
| 3 × L | 4 | $\overline{SA - Sa} \times L = \square BA - \square ba$ |
| But | 5 | $\square BA - \square ba = \overline{BA + ba} \times \overline{BA - ba}$ |
| 4, = 5, 6 | 6 | $\overline{SA - Sa} \times L = \overline{BA + ba} \times \overline{BA - ba}$ |
| 6, *Analogy* | 7 | $L : BA + ba :: BA - ba : SA - Sa$ |
| Or | 8 | $L : ED :: DB : bD$ |

{ Which gives the following *Analogy*.

This

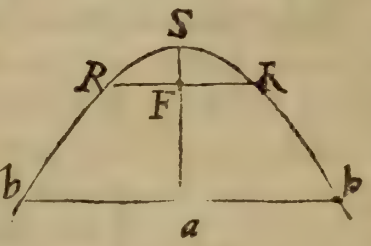
This peculiar Property of the Parabola was firft publifh'd, *Anno* 1684, by one Mr. *Thomas Baker*, Rector of *Bifhop Nympton* in *Devonfhire*, in a Treatife entitled, *The Geometrical Key : Or, the Gate of Æquations unlock'd* ; wherein he hath fhew'd the Geometrical Conftruction and Solution of all Cubick and Biquadratick Affected Æquations by one general Method, which he calls a *Central Rule*, deduced from this peculiar Property of the Parabola.

## Sect. 3. *To find the* **Focus** *of any Parabola.*

The *Focus* of every *Parabola* is that *Point* in its *Axis* through which the *Latus Rectum* doth pafs. (See *Definition* 3. *Sect.* 4. *Page* 359.) Therefore its Diftance from the *Vertex* of the *Parabola* may be eafily found, either by the *Latus Rectum* itfelf, or by any other *Ordinate*, and its *Abfciffæ*.

Thus, fuppofe the *Point* at *F* to be the *Focus*, *S* the *Vertex*, the *Ordinate* $RFR = L$ the *Latus Rectum*, and $bab$ any other *Ordinate*. Then will $SF = \frac{1}{2}L$.

Or $SF = \frac{\square ba}{4Sa}$




## Demonftration.

| | | |
|---|---|---|
| Firft | 1 | $SF \times L = \square FR$. by *Sect.* 2. *Page* 375. |
| And | 2 | $FR = \frac{1}{2}L$; for the Ordinate $RFR = L$ as above. |
| $2 \circledcirc^{2}$ | 3 | $\square FR = \frac{1}{4}\square L = \frac{1}{2}L \times \frac{1}{2}L$ |
| $1, = 3$ | 4 | $SF \times L = \frac{1}{4}\square L$ |
| $4 \div L$ | 5 | $SF = \frac{1}{4}L$, as by *Definition* 4. *Sect.* 4. *Page* 359. |
| Again | 6 | $\frac{\square ba}{Sa} = L$, by the third *Step* in Page 375. |
| Confeq. | 7 | $\frac{\square ba}{4Sa} = \frac{1}{4}L$, &c. as above. Q. E. D. |

## Sect. 4. *To* **defcribe**, *or draw a Parabola feveral Ways.*

*Note*, There are two or three Ways of drawing a Parabola inftrumentally at one Motion ; but becaufe thofe Inftruments or Machines are not only too perplex'd for a Learner to manage, but alfo a little fubject to Error, I have therefore chofen to fhew how that Figure may be (the beft) drawn by a convenient Number of Points, *viz.* Ordinates found, either Numerically or Geometrically, according to the DATA ; which if the Work of the three laft Sections be well confider'd, muft needs be very eafy.

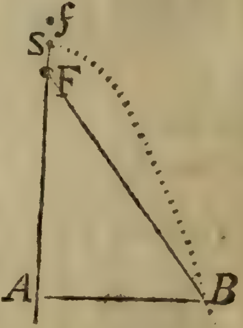
1. If any *Ordinate* and its *Abſciſſa* are given, there may by them be found as many *Ordinates* as you pleaſe to aſſign or take *Points* in the *Parabola*'s *Axis*; (by *Sect.* 4. *Page* 380) and the *Curve* of the *Parabola* may be drawn by the extream Points of thoſe *Ordinates*, as the *Ellipſis* was *Page* 373.

2. If the *Latus Rectum*, and either any *Ordinate*, or its *Abſciſſæ*, are given, then any aſſign'd Number of *Ordinates* may by them be found (by *Sect.* 2. *Page* 381) either *Numerically* or *Geometrically*, &c.

3. If only the Diſtance of the *Focus* from the *Vertex* of the *Parabola* be given, any aſſign'd Number of *Ordinates* may be found by it. For $SF = \frac{1}{4}L$ the *Latus Rectum*, and $\frac{1}{2}L = FR$ as in the laſt *Section*; and it will be, as $SF$ : is to $\square\, FR$ : : ſo is any other *Abſciſſa*, *viz.* ($Sa$ or $SA$, &c.) : to the *Square* of its *Semi-ordinate* (viz. $\square\, ba$, or $\square\, BA$) according to the common Property of the *Parabola*.

Altho' any of theſe Ways of finding the *Ordinates* are eaſy enough, yet that Way which may be deduced from the following *Propoſition* will be found much more eaſy and ready in *Practice*.

PROPOSITION. *The Sum of any Abſciſſa and focal Diſtance from the Vertex, will be equal to the Diſtance from the Focus to the extream Point of the Ordinate, which cuts off that Abſciſſa.*




For Inſtance, ſuppoſe $S$ to be the *Vertex* of any *Parabola*, the Point $F$ to be its *Focus*, and $AB$ any *Semi-ordinate* rightly apply'd to its *Axis* $SA$: Then, I ſay, where-ever the *Point* $A$ is taken in the *Axis*, it will be $SA + SF = FB$. Conſequently, if $Sf = SF$, it will be $fA = FB$.

## Demonſtration.

| | | |
|---|---|---|
| Firſt | 1 | $SF = \frac{1}{4}L$ by the 7th *Step*, *Sect.* 3. |
| *Ergo* | 2 | $fA = FA + \frac{1}{2}L$ by *Conſtruction* above. |
| 2 ⊚² | 3 | $\square\, fA = \square\, FA + \overline{FA \times L} + \frac{1}{4}LL$ |
| Again | 4 | $SA = FA + \frac{1}{4}L$ by the *Suppoſition* and *Figure*. |
| 4 × $L$ | 5 | $SA \times L = FA \times L + \frac{1}{4}LL$, but $SA \times L = \square\, AB$ |
| *Ergo* | 6 | $\square\, AB = FA \times L + \frac{1}{4}LL$ |
| 3 — 6 | 7 | $\square\, fA - \square\, AB = \square\, FA$, conſe. $\square\, fA = \square\, FA + \square\, AB$ |
| But | 8 | $\square\, FA + \square\, AB = \square\, FB$, by *Theorem* 11. |
| *Ergo* | 9 | $\square\, fa = \square\, FB$ |
| 9 ω² | 10 | $fA = FB$ Q. E. D. |

This

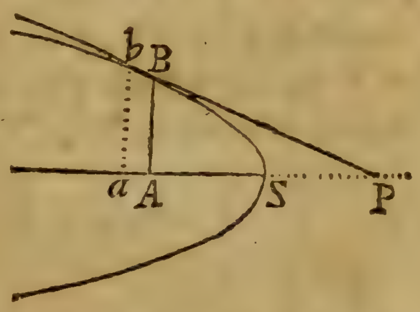
This *Propoſition* being well underſtood, 'twill be very eaſily apply'd to *Practice*, ſuppoſing the *Focal Diſtance* given, or any other *Data* by which it may be found. Thus draw any *Right Line* to repreſent the *Parabola's Axis*, and from its *vertical Point*, as at *S*, ſet off the *Focal Diſtance* both upwards and downwards, *viz.* make $Sf = SF$, the Diſtance of the given *Focus* from the *Vertex*; as in the *Scheme:* Then by the *Propoſition* 'tis evident, that, if never ſo many Lines be drawn *Ordinately* at *Right Angles* to the *Axis*, the true Diſtance between the Point *f* out of the *Parabola*, and any of thoſe Lines (or *Ordinates)* being meaſur'd or ſet off from the *Focus* *F* to the ſame Line or Ordinate, 'twill aſſign the true Point in that Line through which the *Curve* muſt paſs; that is, it will ſhew the true Limits or Length of that *Ordinate*; as at *B* in the laſt Scheme.

Proceeding on in the very ſame Manner from *Ordinate* to *Ordinate*, you may with great Expedition and Exactneſs find as many *Ordinates (or rather their Points only, like B)* as may be thought convenient, which, being all join'd together with an even Hand, will form the *Parabola* requir'd.

N. B. *The more* Ordinates *(or their* Points*) there are found, and the nearer they are to one another, the eaſier and exacter may the* Curve *of the* Parabola *be drawn. The ſame is to be obſerv'd when any other* Curve *is requir'd to be drawn by* Points.

## Sect. 5. *To draw a* Tangent *to any given* Point *in the* Curve *of a* Parabola.

*Tangents* are very eaſily drawn to the *Curve* of any *Parabola*; For, ſuppoſing *S* to be its *Vertex*, *B* the Point of *Contact (viz.* the Point where the *Tangent* muſt touch the *Curve)* and *P* the Point where the *Tangent* will interſect *(or meet with)* the *Parabola's Axis* produced: Then if from the Point of *Contact B* there be drawn the *Semi-ordinate B A* at




Right Angles to the Axis *S A*, wheresoever the Point *A* falls in the Axis, 'twill be $SP = SA$.

### Demonſtration.

Draw the *Semi-ordinate b a (as in the Figure)* then will the *B A P* and △ *b a P* be alike. Let $y = AS$ the *Abſciſſa*, and $z = SP$;

*S P*; put $x = Aa$ the Distance between the two *Semiordinates*, which we suppose to be infinitely near each other, as in the *Ellipsis*, Page 377.

| | | |
|---|---|---|
| Then | 1 | $y+z : BA :: y+z+x : ba$, per *Theorem* 13. |
| 1, Or | 2 | $y+z : y+z+x :: BA : ba$. See *Page* 192. |
| Again | 3 | $y : \square BA :: y+x : \square ba$, per *Theorem* Page 380. |
| 3, Or | 4 | $y : y+x :: \square BA : \square ba$ |
| 2 in □'s | 5 | $yy+2yz+zz : yy+2yz+2yx+zz+2zx+xx :: \square BA : \square ba$ |
| 4, 5 | 6 | $y : y+x :: yy+2yz+zz : yy+2yz+2yx+zz+2zx+xx$ |
| 6 ∵ | 7 | $yy+2yz+yx+zz+2zx+\frac{zzx}{y} = yy+2yz+2yx+zz+2zx+xx$. |
| That is, | 8 | $\frac{zzx}{y} = yx+xx$, consequently $\frac{zz}{y} = y+x$ |
| Suppose | 9 | $x = 0$ and rejected, as in the *Ellipsis*, Page 377. |
| Then | 10 | $\frac{zz}{y} = y$; consequently $zz = yy$ |
| 10 2 | 11 | $z = y$, that is, $SP = SA$ |

Q. E. D.

## CHAP. IV.

### *Concerning the chief Properties of the* Hyperbola.

*NOTE*, any Part of the *Axis* of an *Hyperbola*, which is intercepted between its Vertex and any Ordinate (*viz.* any intercepted Diameter) is call'd an *Abscissa*; as in the *Parabola*.

*Sect.* 1. *The* Plain *of every* Hyperbola *is proportion'd by this general* Theorem.

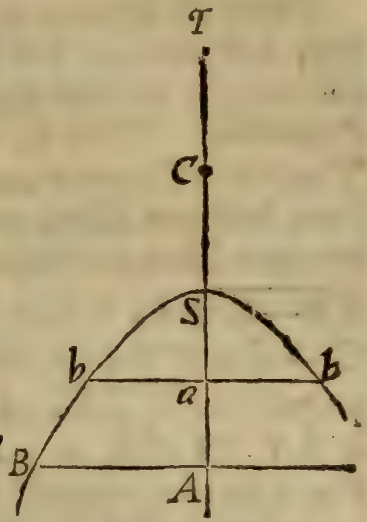
THEOREM. { As the Sum of the Transverse and any *Abscissa* multiply'd into that *Abscissa* : is to the Square of its Semi-ordinate : : so is the Sum of the Transverse and any other *Abscissa* multiply'd into that *Abscissa* : to the Square of its Semi-ordinate.

That

That is, if $TS$ be the Tranſverſe Diameter,

And $\begin{cases} Sa,\ SA \text{ Abſciſſæ.} \\ ba,\ BA \text{ Semi-ordinates.} \end{cases}$

Then is $\begin{cases} Ta = TS + Sa \\ TA = TS + SA \end{cases}$




And it will be

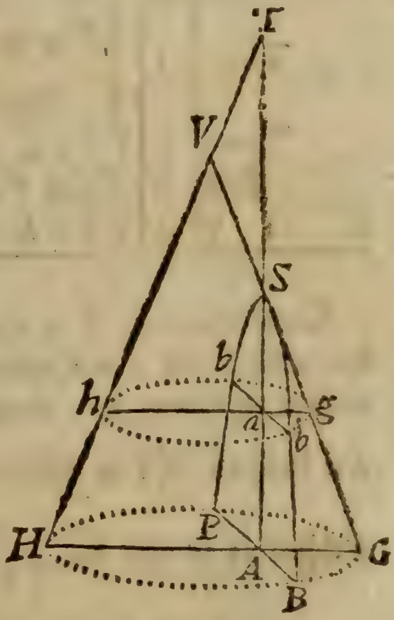
$Ta \times Sa : \square\, ba :: TA \times SA : \square\, BA.$

That is,

$\overline{TS + Sa} \times Sa : \square\, ba :: \overline{TS + SA} \times SA : \square\, BA$ &c.

## Demonſtration.

Let the following Figure *HVG* repreſent a *Right Cone* cut into two Parts by the Right Line *SA*; then will the Plain of that Section be an *Hyperbola* (by *Sect.* 5, *Chap* 1.) in which let *S A* be its Axis, or intercepted Diameter, *b a b* and *B A B Ordinates* rightly apply'd (as before in the *Parabola)* and *TS* its Tranſverſe Diameter. Again, if the Cone is ſuppos'd to be cut by *hg*, parallel to its Baſe *HG*, it will alſo be the Diameter of a Circle, &c. as in the *Ellipſis* and *Parabola*. Then will the △ *Sga* and △ *SGA* be alike; alſo the △ *Tah* and △ *TAH* will be alike; therefore it

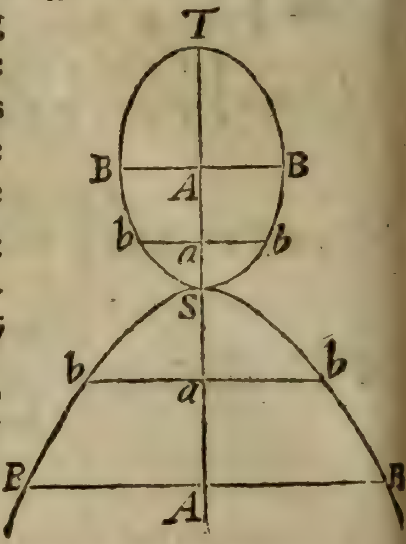



| | | |
|---|---|---|
| will be | 1 | $Sa : ag :: SA : AG$ |
| And | 2 | $Ta : ah :: TA : AH$ |
| 1 ∵ | 3 | $Sa \times AG = SA \times ag$ |
| 2 ∵ | 4 | $Ta \times AH = TA \times ab$ |
| 3 × 4 | 5 | $Sa \times Ta \times AG \times AH = SA \times TA \times ag \times ah$ |
| But | 6 | $ag \times ah = \square\, ab$ |
| And | 7 | $AG \times AH = \square\, AB$, per *Lemma* Page 363. |
| 5, 6, 7 | 8 | $Sa \times Ta \times \square\, AB = SA \times TA \times \square\, ab$ which give the following |
| 8, *Anal.* | 9 | $Sa \times Ta : \square\, ab :: SA \times TA : \square\, AB$ &c. |

Q. E. D.

 These

These *Proportions* are the common Property of every *Hyperbola*, and do only differ from those of the *Ellipsis* in the *Signs* + and —; as plainly appears in the following *Proportions*. That is, if we suppose $TS$ the Transverse Diameter common to both Sections (*viz.* both the *Ellipsis* and *Hyperbola*) as in the annexed Scheme: then in the *Ellipsis* it will be $\overline{TS - Sa} \times Sa : \square ab :: \overline{TS - SA} \times SA : \square AB$ as by *Sect.* 1, *Chap.* 2. and in the *Hyperbola* it is $\overline{TS + Sa} \times Sa : \square ab :: \overline{TS + SA} \times SA : \square AB$, as above. Therefore all, that is farther requir'd in the *Hyperbola*, may (in a manner) be found as in the *Ellipsis*, due Regard being had to changing of the *Sine*.




## Sect. 2. *To find the* Latus Rectum, *or* Right Parameter, *of any Hyperbola.*

From the last Proportion take either of the Antecedents and its Consequent, *viz.* either $Ta \times Sa : \square ab$. Or $TA \times SA : \square AB$, to them bring in the Transverse $TS$ for a third Term, and by those three find a fourth Proportional (as in the Ellipsis) and that will be the *Latus Rectum*.

| | | |
|---|---|---|
| Thus | 1 | $Ta \times Sa : \square ab :: TS : \frac{\square ab \times TS}{Ta \times Sa} =$ the *Latus Rectum*, which call $L$ (*as in the* Parabola.) |
| Then | 2 | $TS : L :: Ta \times Sa : \square ab$. |
| But | 3 | $Ta \times Sa : \square ab :: TA \times SA : \square AB$, therefore |
| 2, 3 | 4 | $TS : L :: TA \times SA : \square AB$, &c. |

Consequently $L$ is the true *Latus Rectum*, or right Parameter, by which all the *Ordinates* may be found, according to its Definition in *Chap.* 1. And because $TS + Sa = Ta$, let it be $TS + Sa$ instead of $Ta$, then it will be $\frac{\square ab \times TS}{TS \times Sa + \square Sa} = L$ and in the *Ellipsis* it would be $\frac{\square ab \times TS}{TS \times Sa - \square Sa} = LR = L$.

Sect.

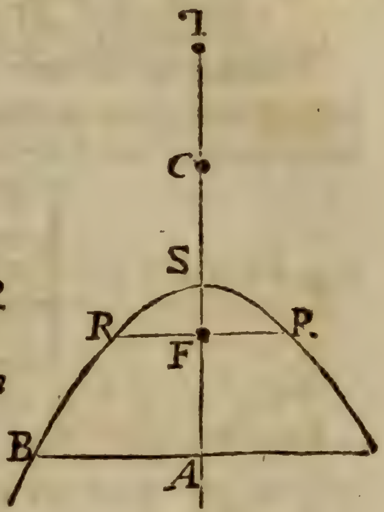
Sect. 3. *To find the* **Focus** *of any Hyperbola.*

The *Focus* being that Point in the *Hyperbola*'s Axis through which the *Latus Rectum* must pass *(as in the* Ellipsis *and* Parabola) it may be found by this *Theorem.*

THEOREM. { *To the Rectangle made of half the Transverse into half the* Latus Rectum, *add the Square of half the Transverse*; *the Square Root of that Sum will be the Distance of the* Focus *from the Center of the* Hyperbola.

## Demonstration.

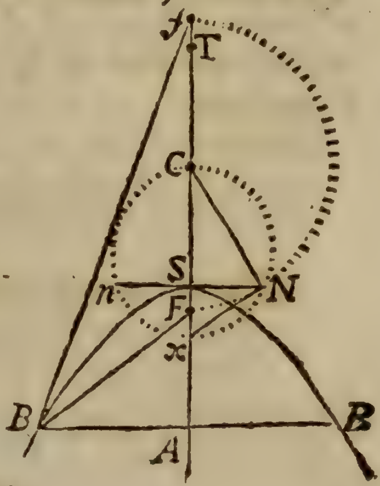
Suppose the Point at *F*, in the annex'd Scheme, to be the *Focus* sought; then will $FR = \frac{1}{2}L$. Let $TC = CS$ be half the Transverse; then is the Point *C* call'd the *Center* of the *Hyperbola* (for a Reason that shall be hereafter shew'd) Again; let $CS = d$, and $SF = a$




| | | |
|---|---|---|
| Then | 1 | $2d : L :: \overline{2d+a} \times a : \frac{1}{4}LL$ |
| That is, | 2 | $TS : L :: \overline{TS+SF} \times FS : \square FR$ |
| 1 ∵ | 3 | $\frac{1}{2}dL = 2da + aa$ |
| 3 + dd | 4 | $dd + \frac{1}{2}dL = dd + 2da + aa$ |
| 4 ω² | 5 | $\sqrt{dd + \frac{1}{2}dL} = d + a = FC$ |
| Or 5, — d | 6 | $\sqrt{dd + \frac{1}{2}dL} - d = a = SF$ |

In the *Ellipsis* 'tis, $2d : L :: \overline{2d-a} \times a : \frac{1}{4}LL$. that is, $\frac{1}{2}dL = 2da - aa$, &c.

The Geometrical Effection of the last Theorem is very easily perform'd, thus: make $Sx = \frac{1}{2}L$, viz. half the *Latus Rectum*; and let $CS = d$, as above. Upon $Cx$ (as a *Diameter)* describe a Circle, and at *S* the *Vertex* of the *Hyperbola* draw the Right Line $n\ S\ N$ at Right Angles to $Cx$; then join the Points $C\ N$ with a Right Line, and 'twill be $CN = d + a = FC$.




| | | |
|---|---|---|
| For | 1 | $CS : SN :: SN : Sx$, per *Fig.* |
| That is, | 2 | $d : SN :: SN : \frac{1}{2}L$. |
| 2 ∵ | 3 | $\frac{1}{2}dL = \square SN$ |
| But | 4 | $dd + \square SN = \square CN$ |
| 3, 4 | 5 | $dd + \frac{1}{2}dL = \square CN$ |
| 5 ω² | 6 | $\sqrt{dd + \frac{1}{2}dL} = CN = d + a$, *&c.* |

Now

Now here is not only found the Diſtance of the *Hyperbola's Focus*, either from its Center $C$, or Vertex $S$, but here is alſo found that Right Line uſually call'd its Conjugate Diameter, *viz.* the Line $n\,S\,N$, which bears the ſame Proportion to the Tranſverſe and *Latus Rectum* of the *Hyperbola*, as the Conjugate Diameter of the *Ellipſis* doth to its Tranſverſe and *Latus Rectum.* For in the *Ellipſis* $TS : Nn :: Nn : LR$. per *Sect.* 2, *Page* 363. Conſequently $\frac{1}{2}TS : \frac{1}{2}Nn :: \frac{1}{2}Nn : \frac{1}{2}LR$. But $\frac{1}{2}TS = d$, $\frac{1}{2}Nn = SN$, and $\frac{1}{2}LR = \frac{1}{2}L$. Therefore $d : SN :: SN : \frac{1}{2}L$. As at the 2d Step above.

What Uſe the aforeſaid Line $n\,S\,N$ is of, in Relation to the *Hyperbola*, will appear farther on.

## Sect. 4. *To deſcribe an* Hyperbola *in Plano.*

In order to the eaſy Deſcribing of an *Hyperbola* in *Plano*, it will be convenient to premiſe the following *Propoſition*, which differs from that of the *Ellipſis* in *Sect.* 3, *Chap.* 2, only in the *Signs.*

PROPOSITION. { *If from the* Foci *of an* Hyperbola *there be drawn two Right Lines, ſo as to meet each other in any Point of the* Hyperbola's *Curve, the Difference of thoſe Lines* (in the *Ellipſis* 'tis their Sum) *will be equal to the Tranſverſe Diameter,*

That is, if $F$ be the *Focus*, and it be made $Cf = CF$ (as in the laſt Scheme) then the Point $f$ is ſaid to be a *Focus* out of the Section (or rather of the oppoſite Section) and it will be $fB - FB = TS$.

### Demonſtration.

Suppoſe $fC$, or $Cf = z$, and $SA = x$, let $CS$, or $TC = d$, as before; then will $fA = d + x + z$, and $FA = d + x - z$. Again, let $FB = h$, and $fB = b$, then $2d = b - h$, by the *Propoſition.*

From theſe *ſubſtituted Letters* it follows,

| | | |
|---|---|---|
| That | 1 | $dd + 2dx + 2dz + xx + 2zx + zz = \square\, fA$ |
| And | 2 | $dd + 2dx - 2dz + xx - 2zx + zz = \square\, FA$ |
| But | | $\square fA + \square AB = \square fB$, and $\square fA + \square AB = \square FB$ |
| *Per* 4th of *laſt* | 3 | $dd + \frac{1}{2}dL = da + 2da + aa = \square FC = zz.$ |

3 — dd

| | | |
|---|---|---|
| 3 − *dd* | 4 | $zz - dd = \frac{1}{2} d L$ |
| 4 ÷ $\frac{1}{2}d$ | 5 | $\frac{zz - dd}{\frac{1}{2}d} = L$ |
| Again | 6 | $2d : L :: \overline{2d + x} \times x : \square AB$, by common Properties. |
| 5, 6 | 7 | $2d : \frac{zz - dd}{\frac{1}{2}d} :: 2dx + xx : \square AB$ |
| 7 ∴ | 8 | $\frac{2dzzx + zzxx - 2dddx - ddxx}{dd} = \square AB$ |
| 1 + 8 | 9 | $dd + 2dx + 2dz + xx + 2zx + zz + \frac{2dzzx + zzxx - 2d^3x - ddxx}{dd} = \square fA + \square AB = hh$ |
| 2 + 8 | 10 | $dd + 2dx - 2dz + xx - 2zx + z + \frac{2dzzx + zzxx - 2d^3x - ddxx}{dd} = \square FA + \square AB = bb$ |
| 9 + *d* | 11 | $d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zzxx = ddhh$ |
| 10 × *dd* | 12 | $d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zzxx = ddbb$ |
| 11 $\omega^2$ | 13 | $dd + dz + zx = dh$ |
| 12 $\omega^2$ | 14 | $dd - dz - zx = db$ |
| 13 ÷ *d* | 15 | $d + z + \frac{zx}{d} = h$ |
| 14 ÷ *d* | 16 | $d - z - \frac{zx}{d} = b$ |
| 16, or | 17 | $z + \frac{zx}{d} - d = b$ |
| 15—17 | 18 | $2d = h - b$ |

*Although the Æquation at the* 16*th* Step *be in itself impossible, because* z *is greater than* d *(by the* 4*th* Step) *yet from thence it will be easy to conclude, that the Difference between* d *and* $z + \frac{zx}{d}$ *will give the true Value of* b, *as in the* 17*th* Step.

But because I would leave no Room for the Learner to doubt about changing the *Æquation*, $d - z - \frac{zx}{d} = b$ into that of $z + \frac{zx}{d} - d = b$, it may be convenient to illustrate the whole Process in Numbers, whereby (I presume) 'twill plainly appear that $h - b = TS$.

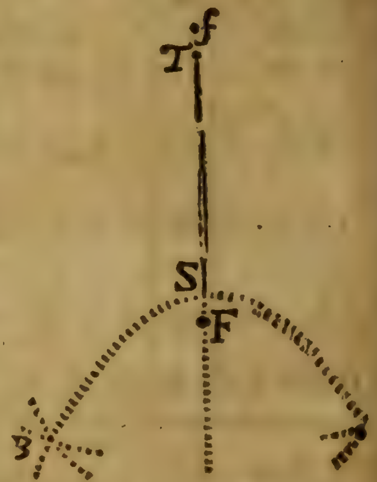
In order to that, let the Transverse $TS = 2d = 12$, then $d = 6$ suppose the Abscissa $SA = x = 4$, and the Semi-ordinate $AB = 3$

| | | |
|---|---|---|
| First | 1 | $\overline{TS + SA} \times SA : \square AB :: TS : L$, *per Sect.* 2. |
| 1, *viz.* | 2 | $12 + 4 \times 4 = 64 : 9 :: 12 : 1,6875 = L$ |
| Again | 3 | $\sqrt{dd + \frac{1}{2}dL} = d + a = CF$, *per* Sect. 3. |
| 3, *viz.* | 4 | $\sqrt{36 + 5,0625} = 6,408 = CF = z$ |
| Then | 5 | $d + x + z = 6 + 4 + 6,408 — 16,408 = fA$ |
| And | 6 | $d + x - z = 6 + 4 - 6,408 = 3,592 = FA$ |

5 S.

| | | |
|---|---|---|
| 5 $\circledast^2$ | 7 | $269{,}2224 = \square fA$ |
| 6 $\circledast^2$ | 8 | $12{,}9024 = \square FA$ |
| But | 9 | $9 = \square AB$, for $AB = 3$ by Suppofition. |
| 7 + 9 | 10 | $278{,}2224 = \square fA + \square AB = \square fB$ |
| 8 + 9 | 11 | $21{,}9024 = \square FA + \square AB = \square FB$ |
| 10 ɯ² | 12 | $16{,}68 = fB$ |
| 11 ɯ² | 13 | $4{,}68 = FB$ |
| 12—13 | 14 | $12{,}00 = fB = FB - TS$. Which was to be prov'd. |

If this *Propofition* be truly underftood, it muft needs be eafy to conceive how to defcribe the Curve of any *Hyperbola* very readily by *Points* when the *Tranfverfe Diameter* and the *Focus* are given (or any other *Data* by which they may be found, as in the precedent Rules) thus:



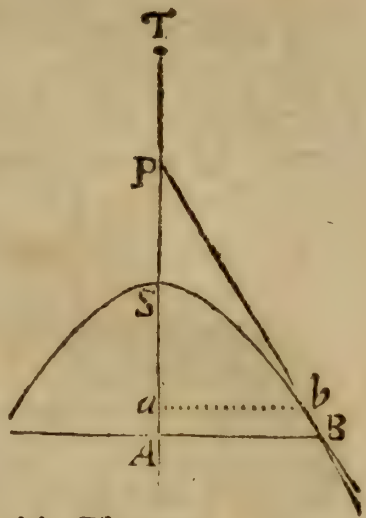

Draw any ftraight Line at Pleafure, and on it fet off the Length of the given *Tranfverfe TS*, and from its extreme Points or Limits, *viz. TS*, fet off $Tf = SF$, the Diftance of the given *Focus* (viz. *the Point f without, and F within the Section, as before)*: that done, upon the *Point f (as a Center)* with any affum'd *Radius* greater than *TS*, defcribe an Arch of a Circle; then from that *Radius* take the Tranfverfe *TS*, making their Difference a fecond *Radius*, with which, upon the Point *F* within the Section, defcribe another Arch to cut or crofs the firft Arch, as at *B*; then will that Point *B* be in the Curve of the *Hyperbola*, by the laft Propofition. And therefore 'tis plain, that, proceeding on in this Manner, you may find as many Points (like *B)* as may be thought convenient (the more there are, and nearer they are together, the better) which being all join'd together with an even Hand (as in the Parabola) will form the *Hyperbola* requir'd.

There are feveral other Ways of delineating an *Hyperbola* in *Plano*: One Way is, by finding a competent Number of *Ordinates*, as by Section 1, &c. but I think none fo eafy and expeditious as this mechanical Way: I fhall therefore, for Brevity's Sake, pafs over the reft, and leave them to the Learner's Practice, as being eafily deduced from what hath been already faid.

Sect.

## Sect. 5. *To draw a* Tangent *to any given Point in the Curve of an* Hyperbola.

The drawing of a *Tangent*, that will touch any given Point in the Curve of an *Hyperbola*, may be easily perform'd by Help of a *Theorem*; as in the *Ellipsis*, Sect. 6, Chap. 2.




Let $\begin{cases} D = TS \text{ the } \textit{Transverse Diameter.} \\ L = \text{the } \textit{Latus Rectum.} \\ y = SA \text{ the } \textit{Abscissa.} \end{cases}$

And $z = AP$ { the Distance between the *Ordinate* and that Point in the *Transverse* cut by the *Tangent*.

Then, if $y$ be given, $z$ may be found by this *Theorem*, $\left\{ \frac{Dy + yy}{\frac{1}{2}D + y} = z \right.$ [which differs from that in the *Ellipsis* only in Signs. *Vide Page* 371.]

Or, if $z$ be given, then $y$ may be found by this *Theorem*:

THEOREM. $\sqrt{\frac{DD + zz}{4}} : + \frac{1}{2}z - \frac{1}{2}D = y.$

### Demonstration.

Draw the *Semi-ordinate* $ba$, as in the Figure, and put $x = Aa$ { an infinite small Space between the two *Semi-ordinates*; as before in the *Ellipsis*, &c.

| | | |
|---|---|---|
| Then | 1 | $D : L :: Dy + yy : \square AB$ |
| That is, | 2 | $TS : L :: \overline{TS + SA} \times SA : \square AB$ |
| 1 ∵ | 3 | $\frac{DyL + yyL}{D} = \square AB$ |
| Again | 4 | $D : L :: Dy + yy - 2yx - Dx + xx : \square ab$ |
| That is, | 5 | $TS : L :: \overline{TS + Sa} \times sa : \square ab$ |
| 4 ∵ | 6 | $\frac{DyL + yyL - 2yxL - DxL + xxL}{D} = \square ab$ |
| Per Figure | 7 | $z : AB :: z - x : ab$, viz. $PA : AB :: Pa : ab$ |
| 7 in □'s | 8 | $zz : \square AB :: zz - 2zx + xx : \square ab$ |
| Suppose | 9 | $x = 0$ and every-where rejected (as in the *Ellipsis*) |
| Then 3, 9 | 10 | $zz : \frac{DyL + yyL}{D} :: zz - 2z : \square ab$ |
| 10 ∵ | 11 | $\frac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} = \square ab$ |

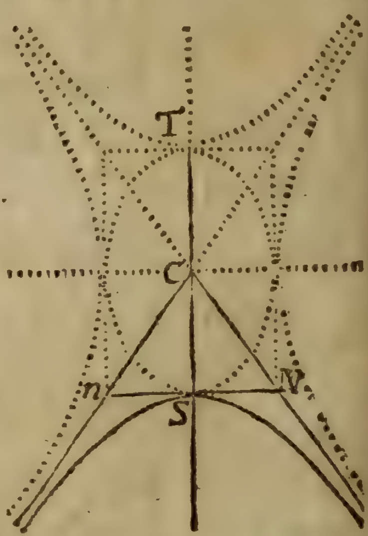
| | | |
|---|---|---|
| 6, 11 | 12 | $\frac{DyL+yyL-2yL-DL}{D}=\frac{DyLzz+yyLzz-2DyLz-2yyLz}{Dzz}$ |
| 12 reduced | 13 | $Dz+zy=Dy+yy$ |
| 13 *Analogy* | 14 | $D+y:y::D+y:z$, viz. $CA:SA::TA:AP$ |
| $13\div\frac{1}{2}D+y$ | 15 | $z=\frac{Dy+yy}{\frac{1}{2}D+y}$ which is the first *Theorem.* |
| $13-zy$ | 16 | $yy+Dy-zy=\frac{1}{2}Dz$ |
| 16 □ C | 17 | $yy+Dy-zy+\frac{DD-2Dz+zz}{4}=\frac{DD+zz}{4}$ |
| 17 $\omega^2$ | 18 | $y+\frac{1}{2}D-\frac{1}{2}z=\sqrt{\frac{DD+zz}{4}}$ |
| 18 $\pm$ | 19 | $y=\sqrt{\frac{D+zz}{4}}:+\frac{1}{2}z-\frac{1}{2}D$ which is the second *Theorem.* |

Q. E. D.

The Geometrical Effection of the first of these *Theorems* is very easy; for, by the 14th Step, 'tis evident that there are three Lines given to find a fourth proportional Line. By *Problem* 3, *Page* 308.}

*Scholium.*

From the Comparisons, which have been all-along made in this Chapter, between the *Hyperbola* and the *Ellipsis*, 'twill be easy (even for a Learner) to perceive the Coherence that is in (or between) those two Figures; but, for the better understanding of what is meant by the *Center* and *Asymptotes* of an *Hyperbola*, consider the annex'd *Scheme*, wherein it is evident (even by Inspection) that the opposite *Hyperbola*'s will always be alike, because they will always have the same Transverse Diameter common to both, &c. (see *Sect.* 1, of this *Chap.*) Also, that the middle Point, or common Center of the *Ellipsis*, is the common Center to all the four *conjugal Hyperbola*'s.




And the two Diagonals of the Right-angled *Parallelogram*, which circumscribes the *Ellipsis* (or is inscrib'd to the four *Hyperbola*'s) being continued, will be such *Asymptotes* to those *Hyperbola*'s as are defined *Chap.* 1, *Sect.* 5, *Defin.* 4.

Sect.

3

## Sect. 6. *To draw the* Asymptotes *of any* Hyperbola, &c.

Having found the *Latus Rectum* (by Sect. 2.) and the Conjugate Diameter in *n S N* in its true *Position*, by Sect. 3. Then through the Center *C* of the *Hyperbola*, and the extream Points *n N* of its Conjugate Diameter, draw two *Right Lines*, as *C N* and *C n*, infinitely continued (as in the following Figure) and they will be the *Asymptotes* required. That is, they are two such *Right Lines* as, being infinitely extended, will continually incline to the Sides of the *Hyperbola*, but never touch them.

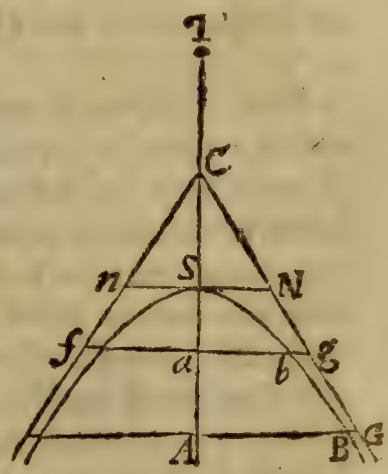
### Demonstration.

Suppose the *Semi-ordinates* *a b* and *A B* to be rightly apply'd to the *Axis* *T A*; and produced both Ways to the *Asymptotes*, as at *f g* and *FG*; then will the △ *CS N*, △ *C a g*, and △ *C A G* be alike.

Let $d = CS = TC$. And $L =$ the *Latus Rectum*; as before,

Put $\left\{ \begin{array}{l} e = Sa \\ y = SA \end{array} \right\}$ the *Abscissæ*. Then $\left\{ \begin{array}{l} d + e = Ca. \\ d + y = CA. \end{array} \right.$

| | | |
|---|---|---|
| Then | 1 | $d : SN :: d + e : ag.$ viz. $CS : SN :: Ca : ag$ |
| 1 in □'s | 2 | $dd : \square SN :: dd + 2de + ee : \square ag$ |
| But | 3 | $d = \square SN.$ per Sect. 3. |
| 2, 3 ∵ | 4 | $\frac{ddL + 2deL + eeL}{2d} = \square ag$ |
| Again | 5 | $2d : L :: 2de + ee : \square ab$, per Sect. 2. |
| 5 ∵ | 6 | $\frac{2deL + eeL}{2d} = \square ab$ |
| 4 — 6 | 7 | $\frac{dL}{2} = \square ag - \square ab$ |
| But { | 8 | $ag + ab = bf$ } per *Fig.* |
| | 9 | $ag - ab = bg$ |
| 8 × 9 | 10 | $\square ag - \square ab = bf \times bg$ |
| 7, 10 | 11 | $bf \times bg = dL$ |
| Again | 12 | $dd : \square SN :: dd + 2dy + yy : \square AG$ |
| That is, | | $\square CS : \square SN :: \square CA : \square AG$ |
| 3, 12 ∵ | 13 | $\frac{ddL + 2dyL + yyL}{2d} = \square AG$ |
| But | 14 | $2d : L :: 2dy + yy : \square AB$, per Sect. 2. |
| 14 ∵ | 15 | $\frac{2dyL + yyL}{2d} = \square AB$ |




 13∸15

| | | |
|---|---|---|
| 13 — 15 | 16 | $\frac{d\,L}{2} = \square AG - \square AB$ |
| Also | 17 | $AG + AB = BF$ per *Fig.* |
| | 18 | $AG - AB = BG$ per *Fig.* |
| 17 × 18 | 19 | $\square AG - \square AB = BF \times BG$ |
| 16  19 | 20 | $BF \times BG = \frac{1}{2} d\,L$ |
| 11, & 20 ∴ | 21 | $bg = \frac{\frac{1}{2} d\,L}{bf}$. And $BG = \frac{\frac{1}{2} d\,L}{BF}$ |

From the laſt Step 'tis evident, that the *Aſymptotes* are nearer the *Hyperbola* at $G$ than at $g$, and conſequently will continually approach to its Curve: For $BF)\ \frac{1}{2}\,dL\ (= BG$ is leſs than $bf)\ \frac{1}{2}\,dL\ (= bg$, becauſe the *Diviſor* $BF$ is greater than the *Diviſor* $bf$; and it muſt needs be ſo where-ever the *Ordinates* are produç'd to the *Aſymptotes*, from the Nature of the *Triangles*.

Again; From the 7th and 16th Steps 'tis evident, that the *Aſymptotes* can never really meet and be co-incident with the Curve of the *Hyperbola*, although both were infinitely extended, becauſe $\frac{1}{2}\,dL$ will always be the Difference between the Square of any *Semi-ordinate* and the Square of that *Semi-ordinate*, when 'tis produc'd to the *Aſymptote*.

*Confectary.*

From hence it follows, that every *Right Line* which paſſes thro' the *Center* and falls within the *Aſymptotes*, will cut the *Hyperbola*; and all ſuch Lines are call'd *Diameters* (as in the *Ellipſis)* becauſe the Properties of the *Hyperbola* and *Ellipſis* are the ſame.

*Note.* Every Diameter, both in the *Ellipſis*, *Parabola*, and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which (ſhould they be diſtinctly handled, and the Effection of all ſuch Lines as relate to them, as alſo the Nature and Properties of ſuch Figures as may be inſcribed and circumſcribed to all the Sections, with the various Habitudes or Proportions of one *Hyperbola* to another, &c.) would afford Matter ſufficient to fill a large Volume. But thus much may ſuffice by way of *Introduction*; I ſhall therefore deſiſt purſuing them any farther, being fully ſatisfied, that, if what I have already done be well underſtood, the reſt muſt needs be very eaſy to any one that intends to proceed farther on that ſubject.

AN

# AN INTRODUCTION TO THE Mathematicks.

## PART V.

THE Method of finding out any particular Quantity (*viz.* either any LINE, SUPERFICIES, or SOLID) by a regular Progression, or Series of Quantities continually approaching to it, which, being infinitely continued, would then become perfectly equal to it; is what is commonly call'd *Arithmetick of Infinites*; which I shall briefly deliver in the following *Lemma*'s, and apply them to Practice in finding the superficial and solid Contents of Geometrical Figures farther on.

### LEMMA I.

If any Series of equal Numbers (representing Lines or other Quantities) as, 1. 1. 1. 1. *&c.* or 2 2. 2. 2. *&c.* or 3. 3. 3. 3. *&c.* if one of the Terms be multiply'd into the Number of Terms, the Product will be the Sum of all the Terms in the Series.

This is so very plain, and easy to be understood, that it needs no *Example*.

### LEMMA II.

If the Series of Numbers in Arithmetick Progression begin with a Cypher, and the common Difference be 1; as, 0. 1. 2. 3. 4. *&c.* (representing a Series of Lines or Roots beginning with a Point) if the last Term be multiply'd into the Number of Terms, the Product will be double the Sum of all the Series.

That is, putting $L =$ the last Term, $N =$ the Number of Terms, and $S =$ the Sum of all the Series:

Then

Then will $NL = 2S$. Consequently, $\frac{1}{2} NL = S$. *viz.* one Half of so many times the greatest Term as there are Numbers of Terms in the Series.

Thus $\frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10}{20}$ $10 =$ the Sum of the Series $= \frac{1}{2} NL$. $20 = NL$.

And this will always be so, how many Terms soever there are, by *Consect.* 1, *Page* 185.

## LEMMA III.

If a Series of Squares whose Sides or Roots are in Arithmetick Progression, beginning with a Cypher, *&c.* (as in the last *Lemma*) be infinitely continued; the last Term, being multiply'd into the Numbers of Terms, will be Triple to the Sum of all the Series, *viz.* $NLL = 3S$, or $\frac{1}{3} NLL = S$.

That is, the Sum of such a Series will be one Third of the last or greatest Term, so many times repeated as is the Number of Terms in the Series.

*Instances in the Square Numbers.*

1. $\left\{\frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}\right.$

2. $\left\{\frac{0+1+4+9}{9+9+9+9} = \frac{14}{30} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18}\right.$

3. $\left\{\frac{0+1+4+9+16}{16+16+16+16} = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24}\right.$ *&c.*

From these Instances 'tis evident, that as the Number of Terms in the Series does increase, the Fraction or Excess above does decrease, the said Excess always being $\frac{1}{6N-6}$; which, if we suppose the Series to be infinitely continued, will then become infinitely small, *viz.* in Effect nothing at all. Consequently, *NLL* may be taken for the true or perfect Sum of such an infinite Series of Squares.

## LEMMA IV.

If a Series of Cubes, whose Roots are in Arithmetick Progression, beginning with a Cypher, *&c.* (as above) be infinitely continu'd, the Sum of all the Series will be ' $NLLL = S$.

That is, one Fourth of the last or greatest Term so many times repeated as is the Number of Terms.

*Instances*

*Instances in Cube Numbers.*

If 0 . 1 . 2 . 3. *&c.* be the *Roots* of the *Cubes*.

Then 1. $\left\{ \frac{0+1+8+27}{27+27+27+27} = \frac{36}{108} = \frac{4}{12} = \frac{1}{4} + \frac{1}{12} \right.$

2. $\left\{ \frac{0+1+8+27+64}{64+64+64+64+64} = \frac{100}{320} = \frac{10}{32} = \frac{5}{16} = \frac{1}{4} + \frac{1}{16} \right.$

3. $\left\{ \frac{0+1+8+27+64+125}{125+125+125+125+125+125} = \frac{225}{750} = \frac{45}{150} = \frac{3}{10} = \frac{6}{20} = \frac{1}{4} + \frac{1}{20} \right.$

From these *Examples* it plainly appears, that, as the Number of Terms in the Series increases, the Fraction or Excess above $\frac{1}{4}$ decreases, the Excess being always $\frac{1}{4N-4}$; which, if we suppose the Series to be infinitely continued, will become infinitely small, or rather nothing; as in the last *Lemma*. Consequently, $\frac{1}{4}$ $NLLL$ may be taken for the true and perfect Sum of all the Terms in such an infinite Series of Cubes.

## LEMMA V.

If a Series of Biquadrates, whose Roots are in Arithmetick Progression, beginning with a Cypher, *&c.* (as before) be infinitely continued, the Sum of all the Terms in such a Series will be $\frac{1}{5} NL^4$.

The Truth of this may be manifested by the like Process, as in the foregoing *Lemma*'s, and so on for higher Powers. But if any one desires a farther Demonstration of these Series, he may (I presume) meet with ample Satisfaction in Dr. *Wallis*'s History of *Algebra*, Chap. 78 and 79, wherein the Doctor concludes with these Words:

"Thus having shew'd, that in the Progression of Laterals (or "Arithmetical Proportionals) beginning at 0, the Sum of 2, 3, 4, "5, 6 Terms, is always equal to half of so many times the greatest; and there being no Pretence of Reason why we should "then doubt it in a Progression of 7, 8, 9, 10, *&c.* we conclude "it so to be, tho' such Number of Terms be suppos'd infinite.

"Again; in a Progression of their Squares having shew'd, that "in 2, 3, 4, 5, 6 Terms the Aggregate is always more than one "Third of so many times the greatest, and the Excess always such

"aliquot

" aliquot Part of the greateſt, as is denominated by ſix times the " Number of Terms wanting 1. (As, if the Terms be 2, " it is $\frac{1}{3} + \frac{1}{6}$; if three, it is $\frac{1}{3} + \frac{1}{12}$; if 4, it is $\frac{1}{3} + \frac{1}{18}$; if 5, " it is $\frac{1}{3} + \frac{1}{24}$ of ſo many times the greateſt Term, and ſo onward) " we may well conclude (there being no Pretence of Reaſon " why to doubt it in the reſt) that it will be ſo, how many ſoever " be ſuch Number of Terms. And becauſe ſuch Exceſs, as the " Number of Terms do increaſe, will become infinitely ſmall (or " leſs than any aſſignable) we conclude (from the Method of Ex- " hauſtions) that, if the Number of Terms be ſuppos'd infinite, " ſuch Exceſs muſt be ſuppos'd to vaniſh, and the Aggregate of " ſuch infinite Progreſſion ſuppos'd equal to $\frac{1}{3}$ of ſo many times " the greateſt.

" In like manner having prov'd that ſuch Progreſſion of Cubes " doth (as the Number of Terms increaſe) approach infinitely near " to $\frac{1}{4}$ of ſo many times the greateſt, and of Biquadrates to $\frac{1}{5}$, and " ſo of Surſolids to $\frac{1}{6}$ of ſo many times the greateſt, and ſo on- " wards as we pleaſe to try; and there being no Pretence of Rea- " ſon why to doubt it as to the reſt, we may take it as a ſufficient " Diſcovery, that (univerſally) the Aggregate of ſuch infinite " Progreſſion is equal (or doth approach infinitely near) to ſuch " a Part of ſo many times the greateſt, as is denominated by the " Exponent (or Number of Dimenſions) of ſuch Power (as is " that according to which the Progreſſion is made) increas'd by " 1, namely, of Laterals $\frac{1}{2}$; of Squares $\frac{1}{3}$; of Cubes $\frac{1}{4}$; of Bi- " quadrates $\frac{1}{5}$; of ſo many times the greateſt) and ſo onwards " infinitely."

This Diſcourſe of the Doctor's I thought convenient to inſert, ſuppoſing it may give ſome Satisfaction to the Learner, to hear ſo Great a Man as Dr. *Wallis*'s Argument about the Truth of theſe Series, which I have briefly deliver'd in the 'foregoing *Lemma*'s.

## LEMMA VI.

If any two Series or Ranks of Proportionals have the ſame Number of Terms (whether Finite or Infinite) it will always be { As the firſt Term of one Series : is to the firſt Term of the other Series : : ſo is the Sum of all the Terms in the one Series : to the Sum of all the Terms in the other Series.

(12. *e.* 5)

As

| | | | | |
|---|---|---|---|---|
| As in these Numbers, | 3 | 6 Or these Numbers, | 4 | 5 |
| | 2 | 6 | 12 | 15 |
| | 3 | 9 | 36 | 45 |
| | 4 | 12 | 108 | 135 |
| | 5 | 15 | 324 | 405 |
| | 6 | 18 | 972 | 1215 |

That is, 1 : 3 : : 21 : 63 And 4 : 5 : : 1456 : 1820 &c.

The Application of these *Lemma*'s to Geometrical Quantities, *viz.* to Lines, Superficies, and Solids, wholly depends upon granting the following *Hypotheses.*

## *The* Hypotheses.

1. That every Line is suppos'd to consist (or be compos'd) of an infinite Series of equidistant Points.

2. A Surface (*viz.* the Area of any Figure) to consist of an infinite Series of Lines, either streight or crooked, according as the Figure requires.

3. A Solid to consist of an infinite Series of Plains, or Superficies, according as its Figure requires.

Not that we suppose Lines, which have really no Breadth, can fill a Space or Superficies; or that Plains, which have not any Thickness, can constitute a Solid: But by what we here call Lines are to be understood small Parallelograms (or other Superficies) infinitely narrow, yet so as that their Breadths, being all taken and put together, must be equal to the Figure they are suppos'd to fill up. And those Plains or Superficies, which are here said to constitute a Solid, are to be understood infinitely thin; yet so as that their Depths or Thicknesses (which are hereafter also called Lines) being all taken together, must be equal to the Height of the propos'd Solid. Now, in order to render this Hypothesis as easy for a Learner to understand as I can, I shall here propose a very plain and familiar *Example*; *Viz.* Let us suppose any Book to be compos'd (or made up) of 100, 200, 300 (more or less) Leaves of fine Paper; such a Book, being close put together, will have Length, Breadth, and Depth or Thickness, and therefore may (not improperly) be called a Solid; and each of its Edges (being evenly cut) will be a Superficies compos'd of a Series of small Parallelograms, every one of their Breadths being only the Edge of a single Leaf of Paper; and if we conceive the Thickness of every one of those Leaves to be divided into 10, or

100, or 1000, *&c.* they will then become ſuch a Series of infinitely ſmall Lines as are (by the Hypotheſis) ſaid to compoſe or fill up a Superficies. And all the Superficies of thoſe infinitely thin or divided Leaves of Paper will become ſuch a Series of Plains, or Superficies, as are ſaid to conſtitute a Solid, *viz.* ſuch a Solid as the Bigneſs and Figure of that Book.

Now according to this Idea of Lines, Superficies, and Solids, one may, without the leaſt Prejudice to any *Demonſtration*, admit of the following *Definitions* and *Theorems.*

## Definitions.

I. The Area's of Squares, and all other Parallelograms, are compos'd or fill'd up with an infinite Series of equal Right Lines.

II. The Area of every *plain Triangle* is compos'd of an infinite Series of Right Lines parallel to its Baſe, and equally decreaſing until they terminate in a Point at the vertical Angle.

III. The Area of a *Circle* may be compoſed either of an infinite Series of concentrick or parallel Circles, or of an infinite Series of Chord Lines parallel to its Diameter, or of an innumerable Multitude of Sectors.

IV. The Area of an *Ellipſis* may be compos'd either of an infinite Series of Ordinates rightly apply'd, or of an infinite Series of Right Lines parallel to its Tranſverſe Diameter.

V. The Area's of the *Parabola* and *Hyperbola* are compos'd of an infinite Series of Ordinates; or may alſo be compos'd of Right Lines parallel to its Axis, *&c.*

VI. A *Priſm* is a ſolid Body contain'd or included within ſeveral equal Parallelograms, having its Baſes or Ends equal and alike; and it is generally nam'd according to the Figure of its Baſe: That is,

VII. A *Cube* (or Solid like a Dye) is a Priſm bounded or included with ſix equal ſquare Plains.

VIII. A *Parallelopipedon* is a Priſm that hath its Sides bounded or included within four equal Parallelograms and two ſquare Baſes or Ends.

IX. A *Cylinder* (or Solid, like a Rolling-ſtone in a Garden) is only a round Priſm, having its Baſes or Ends a perfect Circle.

X. The

X. The Solidity of every *Prism* is compos'd of an infinite Series of equal Plains, parallel and alike to that of its *Base*.

XI. A *Pyramid* is a Solid bounded or included within several plain Triangles set upon any Polygonous *Base*, having their vertical Angles all meeting together in a Point, called the Vertex, and takes its Name from the Figure of its *Base*, *viz.* if it has a square Base, 'tis call'd a square Pyramid ; if a triangular Base, 'tis call'd a triangular Pyramid, *&c.*

XII. A *Cone* is only a round Pyramid, which hath been already defined in *Page* 355, *&c.*

XIII. The Solidity of every Pyramid is compos'd or constituted of an infinite Series of Plains, parallel and alike to that of its Base, equally decreasing until they terminate in a Point at the Vertex.

XIV. A *Sphere* or *Globe*, (*viz.* a Ball) is a Solid bounded or included within one regular Superficies, being form'd or generated by the Rotation of a Semi-circle about its Diameter (call'd the Axis of a Sphere) and its Solidity is compos'd or constituted of an infinite Series of concentrick Circles, whose Diameters are the Chords of that Circle by which it was form'd.

XV. A *Spheroid* (or Egg-like Figure) is a Solid bounded with one regular Superficies, form'd by the Rotation of a Semi-ellipsis about its Transverse Diameter (call'd the Axis of the Spheroid) and its Solidity is constituted of an infinite Series of concentrick Circles, whose Diameters are the Ordinates of that Ellipsis by which it was form'd.

XVI. There is another Sort of Solid call'd an *Oblate Spheroid*, being formed by the Rotation of an Ellipsis about its Conjugate Diameter, and it is like a flat Turnep.

XVII. If a Semi-parabola be turn'd about its Axis, 'twill form a Solid call'd a *Parabolick Conoid*, being compos'd or constituted of an infinite Series of Circles, whose Diameters are the Ordinates of a Parabola.

XVIII. If a Parabola be turn'd about its Base, or greatest Ordinate, 'twill form a Solid call'd a *Pyramidoid*, but most commonly a *Parabolick Spindle*, which will be constituted of an infinite Series of Circles, whose Diameters are Right Lines parallel to the Parabola's Axis.

XIX. If an Hyperbola be turn'd about its Axis, 'twill form a Solid call'd an *Hyperbolick Conoid*, being constituted of an infinite Series of Circles, whose Diameters are the Ordinates of the Hyperbola.

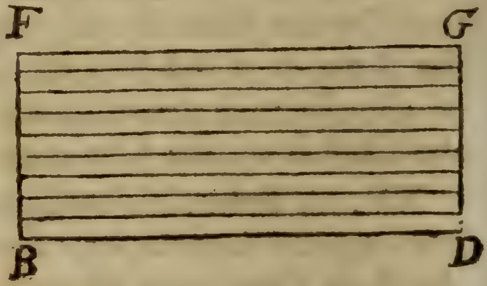
XX. The curve Superficies of all circular Solids *(viz.* Cylinders, Cones, Spheres, *&c.)* are compos'd of an infinite Series of the Peripheries of those Circles which constitute their Solidities.

Upon these Definitions are grounded all the following *Theorems*; and therefore, if they were diligently compar'd with their respective Figures, it must needs be of great Help to the Learner, and would render all that follows very easy; wherein I shall begin with what hath been already demonstrated, by way of introducing the rest.

## THEOREM I.

*The* Area *of every Right-angled* Parallelogram *is obtain'd by multiplying the Length into its Breadth.*

That is, $BD \times FB =$ the Area of the Parallelogram $BDFG$, by *Lemma* 1, compar'd with Definition 1.



*Example.*

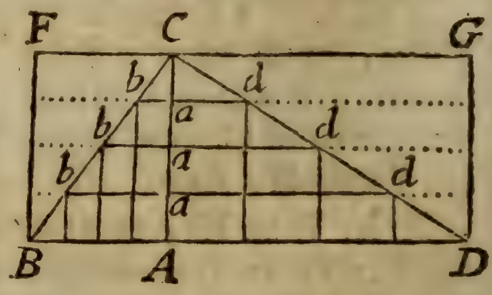
Suppose $BD = 26$, and $FB = 9$, then $26 \times 9 = 234$ the Area. See *Prob.* 1, *Page* 339.

## THEOREM II.

*The* Area *of every plain Triangle is equal to half the* Area *of its circumscribing* Parallelogram. *That is,* $\frac{BD \times CA}{2} =$ *the* Area *of* $\triangle BCD$, *in the following Figure.*

### Demonstration.

Suppose the Perpendicular $CA$ to be divided into an infinite Number of equal Parts, as at the Points $a$, $a$, $a$, &c. and through those Points there were drawn Right Lines parallel to the Base $BD$; *(viz.* $bad$, $bad$, $bad$, &c.) then will those Lines be a Series of Terms in Arithmetick Progression beginning at the Point $C$ *(viz.* 0, $bd$, $2bd$, $3bd$, &c. as is evident by the Figure, wherein $BD$ the greatest Term $= L$, and $CA$ the Number of Terms $= N$.



But $\frac{1}{2}$ $NL = S$, by Lemma 2. And $S =$ the Triangle's Area by Definition 2. Q. E. D.

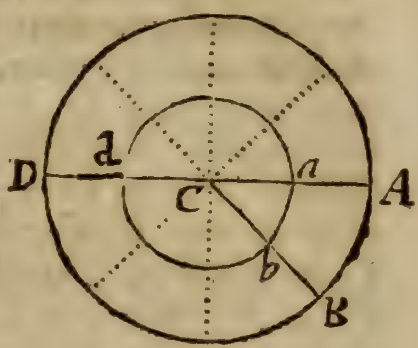
*Example.* Let $BD = 26$, and $CA = 9$, as above; then $\frac{26 \times 9}{2} = 117$, or $\frac{26}{2} \times 9 = 117$. Or thus, $26 \times \frac{9}{2} = 117$, the Area requir'd. [See *Problem* 3, *Page* 330.]

## THEOREM III.

*The Peripheries of Circles are in Proportion one to another as their Diameters are.*

### Demonstration.

Let the Periphery of a Circle be divided into any Number of equal Arches by Right Lines drawn from the Center *(viz. Radii)* suppose 'em 8, as in the annexed Figure, wherein $AB$ is one of them; then, if thro' any Point in the Radius there be drawn a concentrick or parallel Circle, its Periphery will also be divided into 8 equal Arches by those *Radii*, one whereof will be $ab$, and the $\triangle Cab$ will be like to $\triangle CAB$,




Therefore $Ca : ab :: CA : AB$, or $Ca : CA :: ab : AB$, consequently $2\,Ca : 2\,CA :: 8\,ab : 8\,AB$. But $2\,Ca = da$ the Diameter of the Circle, whose Periphery is $8\,ab$; and $2\,CA = DA$, the Diameter of the Circle, whose Periphery is $8\,AB$. Therefore, &c. as by the *Theorem*. Q. E. D.

*Example.*

In Chapter 6, Part III, it was found, that, if the Diameter of a Circle be 2, its Periphery will be 6,2831853, &c. Therefore, 2 : 6,2831853, &c. :: 1 : 3,14159265, &c. the Periphery of the Circle whose Diameter is 1.

*Corollary.*

Hence it follows, that because Unity, or 1, may be made the first Term in the Proportion, therefore 3,14159265, &c. may be made a constant or settled Factor; which, being multiply'd into any propos'd Diameter, will produce the Periphery of that Circle.

*Note,* Instead of 3,14159265, &c. it may be sufficient to take only 3,1416.

Or,

Or, in whole Numbers, the Proportion may be,

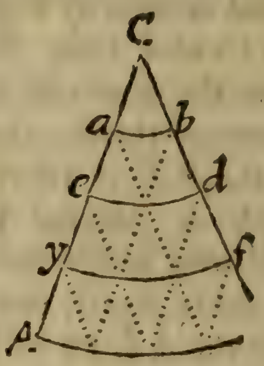
As 7 : 22 : : Diam. : Periphery }
Or 113 : 355 : : Diam. : Periphery } these Numbers may ſerve, and are often uſed in common Practice.

### THEOREM IV.

*The* Area *of any* Sector *of a* Circle *is equal to half the Rectangle of the* Radius *into its Arch.* That is, $\frac{CA \times AB}{2}$ = the Area of *ACP*.

### Demonſtration.

Suppoſe the Radius *CA* to be divided into an infinite Series of equidiſtant Points, as *a*, *e*, *y*, &c. and thro' thoſe Points there were drawn concentrick or parallel Arches, as *a b*, *e d*, *y f*, &c. then they will be a Series of Arches in Arithmetick Progreſſion, beginning at the Point *C* (*viz.* 0, 1, 2, 3, *&c.*) as plainly appears by the Figure, wherein the greateſt Term is $AB = L$, and Number of Terms is $CA = N$. But $\frac{1}{2} NL = S$, the Sum of all the Series, by Lemma 2, and $S$ = the Sector's Area, by Definition 3. Q. E. D.




Let the Radius $CA = 12$, and the Arch $AB = 8$, then $\frac{12 \times 8}{2} = 48$. Or $\frac{12}{2} \times 8 = 48$. Or $\frac{8}{2} \times 12 = 48$, the Area of the Sector *ACB*.

### THEOREM V.

*The* Area *of every Circle is equal to half the Rectangle of the* Radius *into its* Periphery. *That is, according to* Archimedes, *a Circle is equal to a Right-angled Triangle, whoſe Sides containing the Right-angle are equal, one to the* Radius, *and the other to the* Perimeter *of that Circle.* Pro. 1. de Dimenſione Circuli.

The Truth of this *Theorem* may be eaſily deduced from the laſt, thus: If we ſuppoſe the laſt Sector to be one Eighth-part of a Circle, then it follows, that $\frac{8\, AB \times CA}{2} = 4\, AB \times CA$ will be the Area of the whole Circle. But $4\, AB$ = half the Circle's Periphery, and $CA$ = half its Diameter; therefore, *&c.* as per Theorem. Q. E. D.

*Example.*

*Example.*

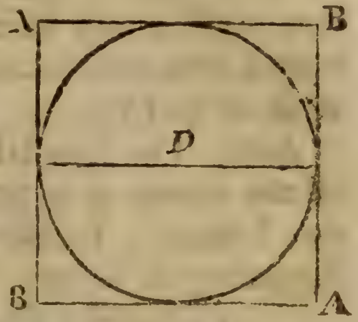
If the Diameter be Unity, or 1, the Periphery will be 3,14159265 &c. by *Theorem* 3. Then $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816$, &c. (or 0,7854 for common Uſe) will be the Area of that Circle.

*Scholium.*

From hence naturally flows the following Proportion between the Square and its inſcrib'd Circle.

PROPORTION. { As the Perimeter (*viz.* the Sum of the four Sides) of any Square : is to its Area : : ſo is the Periphery of the inſcrib'd Circle : to its Area.

That is, ſuppoſing $AB = D =$ the Side of the Square, and the Diameter of its inſcrib'd Circle; then $4\,D =$ the Perimeter, $DD =$ the Area of the Square, and $3,1416\,D =$ the Periphery of the Circle, by *Theorem* 3. But $4\,D : DD :: 3,1416\,D : 0,7854\,DD$ $=$ the Circle's Area. And if $D = 1$, then $4\,D = 4$, and $DD = 1 \times 1 = 1$, and the Periphery will be 3,1416. Then $4 : 1 :: 1 : 0,7854$ &c. as in the *Example* above. And from hence may be eaſily deduced the following *Theorems.*




## THEOREM VI.

*The* Area's *of all Circles are in Proportion one to another as the Squares of their Diameters.* (2. e. 12.)

For if $D =$ the Diameter of one Circle, and $d =$ the Diameter of another Circle, then will $0,7854\,DD$ be the Area of one Circle, and $0,7854\,dd$ will be the Area of the other Circle; as above. But $0,7854\,DD : 0,7854\,dd :: DD : dd$. Or thus, let $D$ $=$ the Diameter, and $P =$ the Periphery of one Circle; $d =$ the Diameter, and $p =$ the Periphery of another Circle;

| | | |
|---|---|---|
| Then | 1 | $\frac{1}{2}D \times \frac{1}{2}P = \frac{1}{4}DP = A$, the Area of one Circle. |
| And | 2 | $\frac{1}{2}d \times \frac{1}{2}p = dp = a$, the Area of the other Circle. |
| $1 \times \overline{4}$ | 3 | $DP = 4A$ (per laſt Theorem. |
| $2 \times \overline{4}$ | 4 | $dp = 4a$ |
| $3 \div D$ | 5 | $P = \frac{4A}{D}$ |

$4 \div d$

| | | |
|---|---|---|
| 4 ÷ *d* | 6 | $p=\frac{4a}{d}$ |
| But | 7 | $P:p::D:d$, per *Theorem* 3. |
| 5, 6, 7 | 8 | $D:d::\frac{4A}{D}:\frac{4a}{d}$ |
| 8 ∵ | 9 | $4DDa=4ddA$, that is, $DDa=ddA$ |
| 9, *Analogy* | 10 | $DD:A::dd:a$, or $A:a::DD:dd$ |

Q. E. D.

*Corollary.*

Hence it follows, that becaufe the Square of 1 is 1 (*viz.* $1\times1=1$) and 0,785398l6, *&c.* or 0,7854 is the Area of the Circle whofe Diameter is 1 (as before) therefore it will be 1 : 0,7854 : : fo is the Square of any Circle's Diameter : to its Area. And becaufe 1 is the firft Term in the Proportion, therefore 0,7854 may be made a conftant Factor ; which, being multiply'd into the Square of any propos'd Diameter, will produce the Area of that Circle.

*Note*, The four laft Theorems do plainly fhew the Reafon of all the common or practical *Problems* about a Circle, which, for the Learner's farther Satisfaction, I have here inferted together. Suppofing as before,

That { $D=$ the *Diameter*, $P=$ the *Periphery*, $A=$ the *Area* } of any propofed Circle.

| | | |
|---|---|---|
| | | *Probl.* 1. D *being given, to find* P. |
| Then | 1 | 1 : 3,1416 : : $D:P$. per Theorem 3. |
| 1 ∵ | 2 | $3,1416\,D=P$. |
| *Examp* | | Suppofe $D=32$. Then $3,1416\times32=100,5312$ the *Periphery*. |
| | | *Probl.* 2. D *being given, to find* A. |
| | 3 | 1 : 0,7854 : : $DD:A$, per Theorem 6. |
| 3 ∵ | 4 | $0,7854\,DD=A$ |
| | | Suppofe $D=32$ (as before) |
| *Examp.* | | $DD=32\times32=1024$ |
| Then | | $0,7854\times1024=804,2496$, the Area requir'd. |
| | | *Probl.* 3. P *being given, to find* D. |
| And 2 ÷ | 5 | $D=\frac{P}{3,1416}$ Or { becaufe $\frac{1}{3,1416}=0,3183$ therefore $0,3183\,P=D$. } |
| | | This, being only Converfe to the firft, needs no Exam. |

| | | |
|---|---|---|
| | | *Prob.* 4. *P* being given to find *A*. |
| 2 ⊚² | 6 | $9{,}86965\ DD = PP$ |
| 6 ÷ | 7 | $DD = \frac{PP}{9{,}86965}$, or $0{,}10132\ PP = DD$ |
| 4 ÷ | 8 | $DD = \frac{A}{0{,}7854}$, or $1{,}2732\ A = DD$ |
| For | | $\frac{1}{0{,}7854} = 1{,}2732$ |
| 7, 8 | 9 | $\frac{PP}{9{,}86965} = \frac{A}{0{,}7854}$, or $0{,}10132\ PP = 1{,}2732\ A$ |
| 9 × &c. | 10 | $\frac{PP}{12{,}5664} = A$, or $0{,}07957\ PP = A$ |
| | | *Prob.* 5. *A* being given, to find *D*. |
| 8 ⱳ² | 11 | $D = \sqrt{\frac{A}{3{,}7854}}$, or $D = \sqrt{1{,}2732\ A}$ |
| | | *Prob.* 6. *A* being given, to find *P*. |
| 10 × &c. | 12 | $PP = 12{,}5664\ A$, or $PP = \frac{A}{0{,}07957}$ |
| 12 ⱳ² | 13 | $P = \sqrt{12{,}5664\ A}$, or $P = \sqrt{\frac{A}{0{,}07957}}$ |

These six *Problems* contain all the Variety that can be proposed about finding the Periphery, Diameter, and Area of any Circle.

But if it be required to find the Area of any Segment, or Part of a Circle cut off by a Chord, that Work will require a farther Consideration.

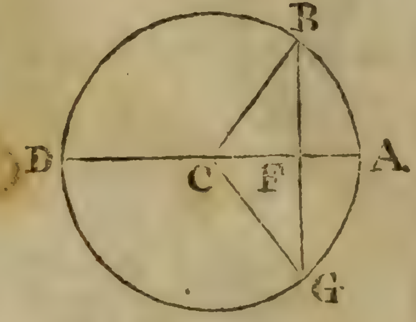
*First*, As to the *Data* there must always be given the *Diameter*; or, either the *Periphery* or *Area* of the *Circle*, in order to find the *Diameter*.

*Secondly*, There must also be given, either the *Chord* which is the *Base* of the *Segment*, or the *versed Sine*, which is the Height of the *Segment*. That is, either *B G*, or *A F*, in the following *Scheme*, must be given, that so the *Area* of the △ *BCG* may be found. Then it's evident (by the Figure) that, if the *Area* of the △ *BCG* be taken from the *Area* of the *Sector CBAG*, the Remainder will be the *Area* of the *Segment BAG*. And if the *Area* of the *Segment BAG* be taken from the whole *Area* of the *Circle*, the Remainder will be the *Area* of the other *Segment DBG*.

 *Exam-*

*Example* in *Numbers.*

Let there be given $DA = 32$. as in *Prob.* 1. and the *versed Sine* $AF = 6$; then $\frac{1}{2} DA = BC = CA = 16$, and $CA - AF = CF = 10$. But $\square BC - \square CF = \square BF$. Consequently $\sqrt{\square BC - \square CF} = BF$, *viz.* $\sqrt{156} = 12,49 = BF$.




Then, by the Doctrine of plain *Triangles*, the *Arch* $BA = \angle BCA$ may be found in Degrees and Decimal Parts. Thus $BC$ : *Radius* : : $BF$ : *Sine* $\angle BCF = 51,31$ Degrees. And then it will always hold in this Proportion;

*Viz.* { *As the Circle's Periphery in Degrees* : *is to its Periphery in equal Parts* (according to the Dimensions taken) : : *So is the Arch in Degrees* (viz. $\angle BCA$) : *to the same Arch in equal Parts.*

That is, $360^{\circ} : 100,5312 : : 51,31^{\circ} : 14,3284 = BA$. Then $14,3284 \times 16 = 229,2544$, the *Area* of the *Sector* $BCAG$; and $12,49 \times 10 = 124,9$, the *Area* of the $\triangle BCG$. Their Difference $104,3544 =$ the *Area* of the *Segm.* $BAG$.

Or the *Area* of any *Segment* may be otherwise found (as most usually it is) by a *Table* of the *Segments* of a *Circle*, whose *Area* is Unity, or 1. The *Construction* or making of such a *Table* is very well laid down in Mr. *Darie*'s Book of *Gauging*, *Chap.* 9. which he performs in this *Problem.*

## PROBLEM.

*In a Circle whose Area is Unity, and its Diameter cut by Chord Lines into* 1000 *equal Parts, to find the Segment to any versed Sine propos'd, not exceeding* 500 *of those equal Parts.*

1. Multiply the *versed Sine* propos'd by 0,002, and subtract the *Product* from an *Unit* or 1.

2. This Remainder you shall seek in the common *Table* of *Natural Sines*, (the Arch being divided into Degrees and Centesimals) which being found, let its *Co-arch* be doubled, and called $A$.

3. You must find the *correspondent Sine* to $A$; which *Sine* being found, you may call $S$, and then it holds 6,2831853) 0,0174532 925 $A - S$ (= the *Segment* required.

Now this *Segment* being thus found, if you subduct it from an *Unit*, you have the *Co-segment*, &c.

*Note*, Notwithstanding what has been said in the second Precept of this *Problem*, it very often falls out that the Remainder there spoken of cannot be truly found in the Table of Natural Sines; therefore in this Case my Advice is, that you make two Operations, one with a Sine the next greater, and one with a Sine the next less; and in so doing you will be sure to have the Segment requir'd bounded between the Results of those two Operations.

Example, *Let it be propos'd to find the correspondent Segment to the versed Sine* 263.

*First*, 263 × 0,002 = 0,526, and 1 — 0,526 = 0,474, its *Arch* is 28,29° being less than just; its Complement is 61,71°, which, being doubled, is 123,42 = *A*.
Then ,0174533 *A* = 2,154086286
= 0,8346556 = *S* the *Sine* of *A*.

6,2831853) 1,319430686 (0,209993 the *Segment*.

*Now I make a second Work.*

263 being *multiplied* with 0,002 is 526. and 1 — 526 = 0,474 its *Arch* is 28,30° being greater than just; and its Complement is 61,70°, which being doubled is 123,4 = *A*.
Then 0,0174533 *A* = 2,1537372
— 0,8348478 = *S* the *Sine* of *A*

6,2831853) 1,3188894 (0,209907 the *Segment*.

So you see by these two Operations that the *Segment* is bounded, and 'tis very probable it may be 0,20995.

But to abbreviate this large *Factor*, and this large *Divisor*, I shall here insert two Tables of them, which will be ready for Use, and exact enough too.

| *Divisor.* | | *Factor.* | |
|---|---|---|---|
| 6,2832 | 1 | ,0174533 | 1 |
| 12,5664 | 2 | ,0349066 | 2 |
| 18,8495 | 3 | ,0523599 | 3 |
| 25,1327 | 4 | ,0698132 | 4 |
| 31,4159 | 5 | ,0872665 | 5 |
| 37,6991 | 6 | ,1047197 | 6 |
| 43,9823 | 7 | ,1221730 | 7 |
| 50,2655 | 8 | ,1396263 | 8 |
| 56,5487 | 9 | ,1570796 | 9 |

Thus far Mr. *Darie*, which I have here inserted to shew the *Learner* how, by the Help of these two *Tables*, and a *Table* of *Natural Sines*, he may easily make a *Table* of *Segments*, whose Use shall be shewed farther on, *viz.* when I come to treat of practical Gauging. In the mean Time I shall here lay down another Method to find the *Area* of any *Segment* of a *Cir-*

*cle* (very near) by a new *Theorem*, without the Help either of a Table of *Sines* or *Segments*, having the same *Data* as before in Page 404.

*Viz.* Let { $R =$ the *Radius*, or $\frac{1}{2}$ *Diameter* of the given *Circle*.
$d =$ the Difference between the *versed Sine* and *Radius*.
$C =$ half the *Chord* of the *Segment*'s *Base*. }

THEOREM. { $\frac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{2}R + d} \times C = S$, the *Area* of the *Segm.* }

*Example*, Suppose $R = BC = 16$, $d = FC = 10$, and $C = BF = 12{,}49$; as before.

Then $2\frac{1}{3}RR = 597{,}3333$. $1\frac{1}{3}Rd = 213{,}3333$. $dd = 100$
$- 313{,}3333 = 1\frac{1}{3}Rd + dd$

$1\frac{1}{2}R + d = 34$ 284,0000 (8,3529. Lastly, $8{,}3529 \times 1249 = 104{,}3276$ the *Area* of the *Segment* $BAG$, as before.

## THEOREM VII.

*As Squares are to the* Area's *of their inscribed Circles, so are Parallelograms to the* Area's *of their inscribed Ellipses.*

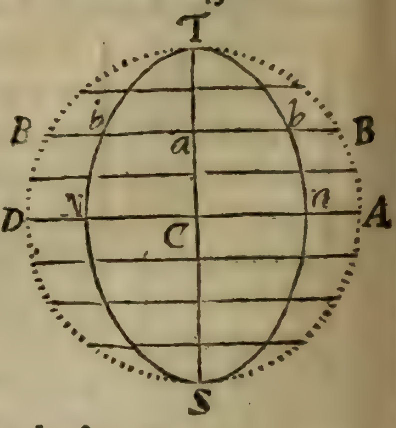
That is, { As the Square of the Diameter of any Circle : is to its *Area* : : so is the Rectangle of the Transverse and Conjugate Diameters of any Ellipsis : to its *Area*. }

### Demonstration.

Circumscribe any *Ellipsis* with a *Circle*; and suppose an infinite Number of *Chord Lines* drawn therein, all parallel to the *Conjugate Diameter*, as those in the annexed Figure; then it will be { *As (D A) the Diameter of the Circle : is to (N n) the Conjugate Diameter of the Ellipsis : : so is (B a B) any Chord in the Circle : to (b a b) its respective Ordinate in the Ellipsis.* }

For according to the Property of the *Circle*

| | | |
|---|---|---|
| it is | 1 | $\overline{TS - Ta} \times Ta = \square\, Ba$ |
| And | | by the Property of the *Ellipsis* |
| it is | 2 | $\square\, TC : \square\, NC :: \overline{TS - Ta} \times Ta : \square\, ba$ |
| 1, 2 | 3 | $\square\, TC : \square\, NC :: \square\, BA : \square\, ba$ |
| 3, Hence | 4 | $TC : NC :: Ba : ba$ |
| Conseq | 5 | $2\,TC : 2\,NC :: 2\,Ba : 2\,ba$ |
| That is | 6 | $DA : Nn :: BaB : bab$ |
| Put | 7 | $D = 2\,TC$, and $d = 2\,NC$ |
| Then | 8 | $D : d ::$ *Chord* $Bab$ : *Ordinate* $bab$, &c. |




But

But the Sum of an *infinite Series* of ſuch *Chords*, as *B a B*, do conſtitute the *Area* of the *Circle*, by *Definition* 3 : and the Sum of the like *Series* of their reſpective *Ordinates*, as *b a b*, do conſtitute the *Ellipſis's Area* by *Definition* 4. Therefore *D* : *d* : : *Circle's Area* : *Ellipſis's Area*, by *Lemma* 6. But *D* : *d* : : *D D* : *D d*. Whence it follows, that *D D* : *Circle's Area* : : *Dd* : *Ellipſis's Area*. Q. E. D. Conſequently, as 1 : is to 0,7854 : : ſo is the *Rectangle* or *Product* of the *Tranſverſe* and *Conjugate Diameters* of any *Ellipſis* : to its *Area*.

*Example*, Suppoſe *TS* = 36. and *Nn* = 16 ; then 36 × 16 = 576, and 576 × 0,7854 = 452,3904 the *Area* of the *Ellipſis*.

### *Corollaries.*

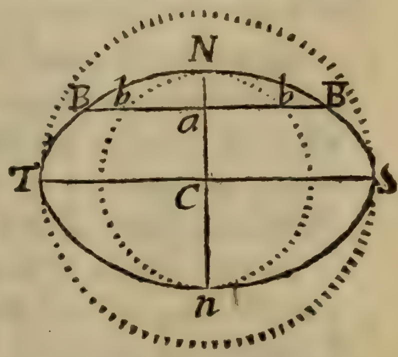
1. Hence it is eaſy to conceive, that the ſquare Root of the *Rectangle* or *Product* of the *Tranſverſe* and *Conjugate Diameters* will be the *Diameter* of a *Circle* whoſe *Area* will be equal to the *Ellipſis's Area*, viz. $\sqrt{576} = 24$ the *Diameter* of a *Circle* = to the *Ellipſis*.

2. All *Segments* of an *Ellipſis* and its *circumſcribing Circle* (whoſe Baſes are parallel to the *Conjugate Diameter*, and of the ſame Height) are in Proportion one to another, as their *Baſes* are. That is, *B a b* : *b a b* : : *Area Segment B N B* : *Area Segment b N b* ; or *T S* : *Nn* : : *Area Segment B N B* : *Area Segment b N b*.

## THEOREM VIII.

*The* Area *of every* Ellipſis *is a mean Proportional between the* Area's *of its* circumſcribing *and* inſcrib'd Circles.

The Truth of this *Theorem* may be eaſily deduced from the laſt ; for ſuppoſing *D* = *T S*, and *d* = *N n*, as before ; then it is already proved, that *D D* : *D d* : : *circumſcribing Circle's Area* : *Ellipſis's Area*. But *D D* : *D d* : : *D d* : *d d*. Therefore *Ellipſis's Area* : *inſcrib'd Circle's Area* : : *D d* : *d d*. By *Theorem* 6.




*Example*, Let *T S* = *D* = 36, and *Nn* = *d* = 16, as before ; then *D D* = 1296, and *d d* = 256.

Then

Then will $\left\{ \begin{array}{l} 1296 \times 0{,}7854 = 1017{,}8784 \text{ the great } Circle\text{'s Area} \\ 256 \times 0{,}7854 = 201{,}0624 \text{ the leſſer } Circle\text{'s Area.} \end{array} \right.$

Suppoſe $A =$ the *Ellipſis's Area*; then, according to the *Theorem*, it will be, $1017{,}8784 : A :: A : 201{,}0624$. *Ergo* $AA = 1017{,}8784 \times : 201{,}0624 = 204657{,}07401216$. Conſequently, $\surd\ 204657{,}07401216 = 452{,}3904 = A$, the *Area* of the *Ellipſis* as before in the laſt *Example*.

*Corollary.*

From hence it follows, that all *Segments* of an *Ellipſis* and its *inſcrib'd Circle*, whoſe *Baſes* are parallel to the *Tranſverſe Diameter*, and have the ſame Height, are in Proportion one to another as the *Area's* of the *Ellipſis* and *Circle* are. That is, *Area* of *Circle* : *Area* of *Ellipſis* : : *Segment b N b* : *Segment B N B*. Or, *N n* : *T S* : : *Area Segment b N b* : *Area Segment B N B*.

## THEOREM IX.

*The Solid Content of any Priſm* (what Figure ſoever its Baſe is of) *is obtained by multiplying the* Area *of its* Baſe *into its Height.*

For Inſtance, a *Parallelopipedon* (or ſquare *Priſm*) is conſtituted of an *infinite Series* of equal *Squares*; that of its *Baſe B A* being one of the *Terms*, and its Height *D B*, or *G A*, the *Number* of all the *Terms*. Conſequently, the *Area* of $BAba \times DB =$ the *Sum* of all the *Series* (by *Lemma* 1.) which is the *Solidity* of the *Parallelopipedon D B G A*, by *Definition* 10.


*Example*, Suppoſe the Side of the *Baſe B A* $= 16$ and the *Height D B* $= 42$; then will $16 \times 16 = 256$ be the *Area* of the Baſe, and $256 \times 42 = 10752$ the *Solid Content* of the *Parallelopipedon D B G A*.

In this Manner you may find the *Solidity* of all regular *Polygonous Priſms*, whoſe *Baſes* (or Ends) are parallel and alike, what Form ſoever they are of, that is, whether their *Baſes* are *Triangles*, *Pentagons*, *Hexagons*, or *Octagons*, &c.

THEO-

## THEOREM X.

*Every Pyramid is the third Part of the Prism, that hath the same Base and Height with it* (7. e. 12.)

That is, the *Solid Content* of the *Pyramid B V A* (in the last Figure) is one *Third* of its circumscribing *Prism D B G A.*

### Demonstration.

For every *Pyramid* that hath a square *Base* (as *B A b a*, in the last Figure) is constituted of an *infinite Series* of *Squares*, whose *Sides* or *Roots* are continually increasing in *Arithmetick Progression*, beginning at the *Vertex* or *Point V* (See *Theor.* 2.) its *Base B A B a*; being the greatest *Term* ($= L L$) and its *perpendicular Height V C*, or *D B*, is the Number of all the *Terms* $= N$; but $\frac{NLL}{3} = S$ the *Sum* of all the *Series*, by *Lemma* 3, and $S =$ the *Solid Content* of the *Pyramid B V A*, by *Definition* 13.

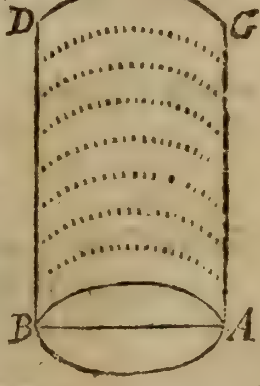
*Example*, Suppose the Side of a *Pyramid's Base* be $B A = 16$. and its *Height* be $V C = 42$. Then $16 \times 16 = 256$ the *Area* of its *Base* $B A B a = a$, and $\frac{256 \times 42}{3} = 3584$. Or $\frac{356}{3} \times 42 = 3584$ or thus, $256 \times \frac{42}{3} = 3584$, is the *Solidity* of that *Pyramid B V A.*

*Corollary.*

From hence it will be easy to conceive, that every *Pyramid* is $\frac{1}{3}$ of its *circumscribing Prism*, what Form soever its *Base* is of, *viz.* whether it be a *Square*, *Triangle*, *Pentagon*, &c.

## THEOREM XI.

*The Solid Content of every Cylinder is obtain'd by multiplying the* Area *of its Base into its Height.*




For every *Right Cylinder* is only a round *Prism*, being constituted of an *infinite Series* of equal *Circles*; that of its *Base* or End being one of the *Terms*, and its *Height B D* is the Number of all the *Terms*. Therefore the *Area* of its *Base B A*, being multiply'd into *D B*, will be its *Solidity*, by *Lemma* 1. *viz.* Let $D = B A$, and $H = G A$. Then $0{,}7854\ D D \times H =$ its *Solidity*.

*Example,*

*Example*, Let the *Diameter* of its *Base* be $D = 16$, and its *Height* $H = 42$. Then 1 : 0,7854 : : 16 × 16 = 256 : 201,0624 the *Area* of its *Base*. And 201,0624 × 42 = 8444,6208 the *Solid Content* of that *Cylinder D B G A*.

*Corollary.*

Hence it is evident that every square *Parallelopipedon* is to its *inscrib'd Cylinder*, as 1 : is to 0,7854. Or in whole *Numbers*, as 452 : to 355 very near. And that all *Prisms* are in Proportion to their *inscrib'd Cylinders*, as the *Area's* of their *Bases* are.

## THEOREM XII.

*The Curve Superficies of every Right Cylinder is equal to the Rectangle made of its Height into the Periphery of its Base.*

That is, *D B* multiply'd into the *Periphery* of the *Diameter B A*, will produce the *Curve Superficies* of the last *Cylinder D B G A*. For the *Cylinder* is constituted of an *infinite Series* of equal *Circles* (according to the last *Theorem.)* Therefore its *Curve Superficies* is compos'd of the *Peripheries* of those *Circles*, bv *Definition* 20. But the *Periphery* of its *Base B A* is one of the *Terms*, and its Height *D B* is the Number of *Terms*. Therefore, &c. as by *Lemma* 1. To which, if there be added the *Area*'s of both its Ends (or *Bases)* the *Sum* will be the *Superficies* of the whole *Cylinder*.

*Example.* Suppose the *Diameter* of its *Base* to be $B A = 16$, and its Height $D B = 42$; as before, then 1 : 3,1416 : : 16 : 50,2656 the *Periphery* of its *Base*. Again, 1 : 0,7854 : : 16 × 16 = 256 : 201,0624 the *Area* of each End or *Base*.

Then 50,2656 × 42 = 2111,1552 the *Curve Superficies*, to which add 201,0264 × 2 = 402,1248 both the End *Area's*.

The *Sum* = 2513,2800 is the *Superficies* of the whole *Cylinder*.

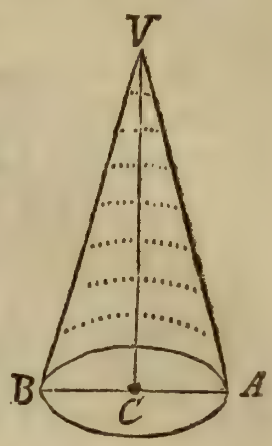
## THEOREM XIII.

*Every Cone is the third Part of a Cylinder, having the same Base with it, and their Altitudes equal.* (10. *e.* 12.)

Demon-

## Demonstration.

The Truth of this *Theorem* may be eaſily conceiv'd by only conſidering, that a *Cone* is but a round *Pyramid*, and therefore it muſt needs have the ſame *Ratio* to its circumſcribing *Cylinder* as the ſquare *Pyramid* hath to its circumſcribing *Parallelopipedon*, *viz.* as 1 : to 3. However, to make it yet clearer, let it be farther conſidered, that every *Right Cone* is conſtituted of an *infinite Series* of *Circles*, whoſe Diameters do continually encreaſe in *Arithmetick Progreſſion* beginning at the *Vertex* or *Point V*, the *Area* of its *Baſe B A* being the greateſt *Term*, and its perpendicular Height *V C* the *Number* of all the *Terms*; therefore the *Area* of the Circle $B\,A \times \frac{1}{3}\,V\,C$ will be the Sum of all the *Series*, by *Lemma* 3, which is the *Cone*'s *Solidity*.



*Example.* Let the Diameter of its *Baſe* be $B\,A = 16$, and its Height $V\,C = 42$; Then $1 : 0{,}7854 :: 16 \times 16 = 256 : 201{,}0624$ the *Area* of the *Baſe*; and $\frac{201{,}0624 \times 42}{3} = 2814{,}8736$ the *Solidity* of the *Cone B V A*. Or thus, $201{,}0624 \times \frac{42}{3} = 2814{,}8736$, &c.

*Corollary.*

Hence it follows, that every ſquare Pyramid is to its inſcrib'd Cone, as 1 : 0,7854. (Or as 452 : 355) conſequently, that all Pyramids have the ſame *Ratio* to their inſcrib'd *Cones* as the *Area*'s of their Baſes have.

## THEOREM XIV.

*The Curve Superficies of every Right Cone is equal to half the Rectangle of the Periphery of its Baſe into the Length of its Side.*

The Truth of this *Theorem* is ſelf-evident from the *Definition* of a Cone, *Chap.* 1, *Part* IV, where it appears, that the *Curve Superficies* of every Right Cone (as *B V A*) is equal to the *Area* of a *Sector* of that Circle whoſe *Radius* is the Side of the Cone (*V B*) and its *Arch* equal to the *Periphery* of the Cone's Baſe (*B A*). But the *Area* of any *Sector* is equal to half the *Rectangle* of the *Radius* into its *Arch*, by *Theorem* 4. Therefore, &c.

 *Exam-*

*Example.* Suppose the Length of the Cone's Side to be *V B*, or *V A* = 42,7551, and the Diameter of its Base, *viz.* *B A* = 16 (as before) then will 50,2656 be the *Periphery* of its Base, and $\frac{50,2656 \times 42,7551}{2} = 1074,5553$, &c. the *Curve* of the *Superficies*; to which if there be added the *Area* of its Base, the Sum will be the *Superficies* of the whole (viz. *all the*) Cone.

That is, 1074,5553
\+ 201,0624 the *Area* of the *Base.*

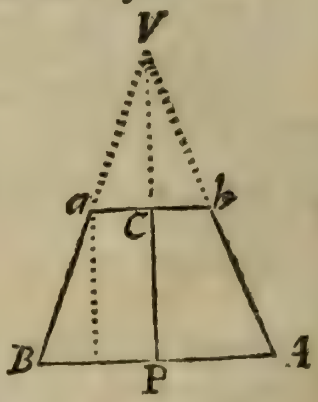
Sum 1275,6177 is the total *Superficies*, &c.

Note, *The Truth of this Theorem may be prov'd from the Consideration of the last Theorem and Definition* 20.

*Scholium.*

From the 10th and 13th *Theorems* may be easily deduced several *Theorems* for finding the *solid Content* of any *Frustum* or *Part*, either of a *Pyramid* or *Cone*, cut by a plain Parallel to its *Base.*

Suppose a square *Pyramid*, as *B V A*, to be cut by a *Plain* at *a b*, parallel to its *Base* *B A*, and it were requir'd to find the *Solidity* of the *Frustum* or *Part a b A B*; let there be given *D* = *B A* the Side of the greater Base. *d* = *b a* the Side of the lesser Base. *H* = *C P* the perpendicular Height.




| | | |
|---|---|---|
| First, | 1 | $D - d : H :: d : \frac{dH}{D-d} = VC$ by the Figure. |
| Then | 2 | $DD \times \frac{H + VC}{3} =$ the whole Pyramid *B V A.* By *Theorem* 10. |
| And | 3 | $dd \times \frac{1}{3} VC =$ the Pyramid *a V b* cut off. |
| *Viz.* 1, 2 | 4 | $\frac{DDDH}{3D - 3d} =$ the whole Pyramid *B V A.* |
| And 1, 3 | 5 | $\frac{dddH}{3D - 3d} =$ the Pyramid *a V b.* |
| 4 — 5 | 6 | $\frac{DDDH - dddH}{3D - 3d} =$ the Frustum *a b A B.* |
| 6. *Reduc.* | 7 | $\overline{DD + Dd + dd} \times \frac{1}{3} H =$ the Frustum *a b A B.* |

Which in Words gives this following *Theorem.*

THEO-

## THEOREM XV.

*To the Rectangle of the Sides of the two* Bases, *add the Sum of their* Squares; *that Sum, being multiply'd into one Third of the* Frustum's *Height, will give its Solidity.*

*Example.* Suppose the Side of the greater Base $BA = 16$, and the Side of the lesser Base (or Top) $ab = 12$ the Height $CP = 9$. Then $16 \times 12 = 192$. $16 \times 16 = 256$. and $12 \times 12 = 144$. Next $192 + 256 + 144 = 592$. and $\frac{592 \times 9}{3} = 1776$. Or $592 \times \frac{9}{3}$ $= 1776$ the Content of the *Frustum* of a square *Pyramid.*

And if it were the like *Frustum* of a *Right Cone*, it may be found by the same *Theorem.* Supposing $D =$ the Diameter of the greater Base, $d =$ the Diameter of the lesser, and $H =$ the Height of the *Frustum*, then the Sum of all the *Squares* which constitute the *Frustum* of a square *Pyramid*, are to the Sum of all the Circles which constitute the like *Frustum* of a right Cone, in the *Ratio* of 1 : to 0,7854 (or of 452 : to 355) therefore it will be $1 : 0{,}7854 :: \overline{DD + Dd + dd} \times \frac{1}{3} H : \overline{0{,}7854\,DD + 0{,}7854\,Dd + 0{,}7854\,dd} \times \frac{1}{3} H =$ the Cone's *Frustum*, that is, in the last *Example*, $1 : 0{,}7854 :: 1776 : 1394{,}8704$ the like *Frustum* of a right Cone. Or because $\frac{1}{0{,}7854} = 1{,}273236$, *&c.* Therefore it may be made $1{,}273236)\ \overline{DD + d + dd} \times \frac{1}{3} H$ (= the same *Frustum*; that is, $1{,}273236)\ 1776\ (1394{,}87$, *&c.* as before. And if you take the *Triple* of this Divisor, *viz.* $1{,}273236 \times 3$, it will be $3{,}8197)\ \overline{DD + Dd + dd} : \times H$ (= the *Frustum*, &c.

Again,

| | | |
|---|---|---|
| Suppose | 1 | $x = D - d$, and $F =$ the *Frustum.* |
| Then | 2 | $DD + Dd + dd = \frac{3F}{H}$, by the 7th Step of the last. |
| 1 ⓐ 2 | 3 | $xx = DD - 2Dd + dd$ |
| 2 — 3 | 4 | $3Dd = \frac{3F}{H} - xx$ |
| 4 ÷ 3 | 5 | $Dd = \frac{F}{H} - \frac{1}{3}xx$, or $Dd + \frac{1}{3}xx = \frac{F}{H}$ |
| 5 × H | 6 | $\overline{Dd + \frac{1}{3}xx} \times H = F$ the *Frustum* $abAB$. |

Hence we have another easy *Theorem* for finding the same *Frustum.*

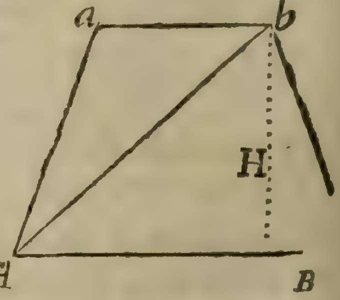
## THEOREM XVI.

*To the Rectangle of the Sides of the two Bases, add one third Part of the Square of their Difference; that Sum, being multiply'd into the Height, will produce the Solidity.*

*Example.* Let $D = 16$. $d = 12$. and $H = 9$, as before; then $D d = 192$. $D - d = 4 = x$. $\frac{1}{3} xx = \frac{4 \times 4}{3} = 5,3333$, and $192 + 5,3333 = 197,3333$. Lastly, $197,3333 \times 9 = 1775,9997$ the *Solidity* of the *Frustum* of the square Pyramid, as before. And 3,81968) 1775,9997 (1394,87, &c. the like *Frustum* of a right Cone, as before.

Either of the two last *Theorems* (being rightly apply'd) will produce the true *solid Content* of all *Frustums* of any kind of Pyramids, that are intercepted between two parallel and alike *Plains* or *Bases*: As above.

But if such *Frustums* are cut through the Extremities of both Bases by a Diagonal Plain (as *A b* in the annexed Figure) into two Parts, *A a b* and *A B b*, call'd *Hoofs*; then the *Solidity* of those *Hoofs* is usually found by dividing the middle Term *D d* of the *Æquation* $D d + D d + d d$ into two Parts, and adding one of those Parts to the *Square* of each Base. Thus, $\overline{DD + \frac{1}{2} Dd} \times \frac{1}{3} H =$ the great *Hoof A B b*, and $\overline{dd + \frac{1}{2} Dd} \times \frac{1}{3} H =$ the lesser *Hoof A a b* of the *Frustum* of any square Pyramid. Then 3,8197) $\overline{DD + \frac{1}{2} Dd \times H}$ (=) the greater *Hoof* of a *Cone*. And 3,8197) $\overline{dd + \frac{1}{2} Dd \times H}$ (=) the lesser *Hoof*, &c.




These are the *Theorems* made Use of by Mr. *Darie*, in his Book of *Gauging*, and are pretty near the Truth, but not exactly so; for they give the *Solidity* of the upper Hoof *A a b* a small Matter too big, and the lower Hoof *A B b* as much too little.

Now, in order to rectify that small Error, I shall here propose the two following *Theorems*, which come very near the Truth, and are more easily perform'd than those propos'd in the *first Impression of this Book.*

First, $\overline{D D + \frac{1}{2} D d + D - d} \times \frac{1}{3} H$ will be the *Solidity* of the greater *Hoof A B b.*

Secondly,

Secondly, $\overline{dd+\frac{1}{2}Dd+d-D}\times\frac{1}{3}H$ will give the *Solidity* of the leſſer *Hoof A a b*, of the *Fruſtum* of any ſquare *Pyramid*.

And for the like *Hoofs* of the *Fruſtum* of any *right Cone*, it will be

Thus, 3,8197) $\overline{DD+\frac{1}{2}Dd+D-d}\times$ * $H$ (= the greater *Hoof*.

And 3,8197) $\overline{dd+\frac{1}{2}Dd+d-D}\times$ $H$ (= the leſſer *Hoof*.

Note, *In order to avoid many Words in the following* Demonſtrations, *let* ⊙ *ſignify any* Circle *in general*; *and if any two Letters be join'd to it, thus,* ⊙ *B A*, &c. *it then denotes the* Area *of ſuch a* Circle *as thoſe two Letters repreſent the* Radius *of*.

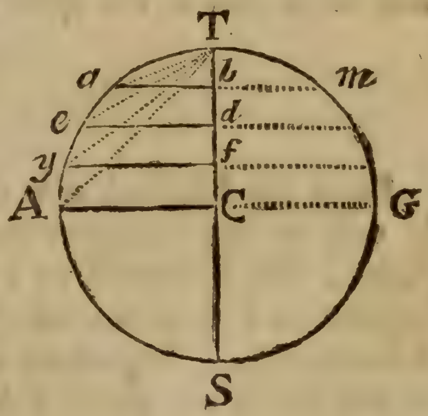
## THEOREM XVII.

*The Superficies of every* Sphere *(or* Globe) *is equal to four Times the* Area *of its greateſt* Circle.

That is, of a *Circle* whoſe Diameter is the *Axis* of the *Sphere*.

### Demonſtration.

If any *Semicircle* (as *A T G S)* be turn'd or mov'd about its Diameter *(T S)* it will deſcribe a ſolid Body call'd a *Sphere*, which will be conſtituted of an infinite Series of *concentrick* or *parallel Circles*, whoſe Diameters are Chords, *viz.* ⊙ *a b*, ⊙ *e d*, ⊙ *e f*, &c. by Definition 14. Conſequently, the *Superficies* of the *Sphere* will be compos'd of the *Peripheries* of thoſe Circles which conſtitute its *Solidity*, by Definition 20.




Let $D = TS$, the *Axis* of any *Sphere*. Then, according to the Property of a Circle, it

| | | |
|---|---|---|
| will be | 1 | $\overline{D-Tb}\times Tb = \square\, ab$ |
| That is, | 2 | $D\times Tb - \square\, Tb = \square\, ab$ |
| Therefore | 3 | $D\times Tb = \square\, aT$, for $\square\, ab + \square\, Tb = \square\, aT$. |
| And | 4 | $D\times Td = \square\, eT$ |
| | 5 | $D\times Tf = \square\, yT$, &c. |

* The Error is here corrected, which Mr. *J. Robertſon* takes Notice of in his Book, entitled, *A Compleat Treatiſe of Menſuration*, Page 160.

Hence

Hence 'tis evident, that the *Series* □ *a T*, □ *e T*, □ *y T*, &c. are in the fame *Ratio* with *T b*, *T d*, *T f*, &c. *viz.* in *Arithmetick Progreffion*; whence it follows, that the ⊙ *a T* = the Sum of all the Circle's *Peripheries* between *T* and *b*, and ⊙ *e T* = the Sum of all the Circle's *Peripheries* between *T* and *d*, &c. Confequently, that the ⊙ *A T* = the Sum of all the Circle's *Peripheries* included between *T* and *C*; that is, ⊙ *A T* = the *Superficies* of the *Hemifphere*. And becaufe □ *A C* + □ *T C* = □ *A T*, and □ *A C* = □ *T C*. Therefore ⊙ *A T* = 2 ⊙ *A C* is the *Superficies* of the *Hemifphere*. Confequently, 4 ⊙ *A C* will be the *Superficies* of the whole *Sphere*. Q. E. D.

*Example*. Suppofe the Axis *T S* = *D* = 16. Then *D D* = 256, and 1 : 0,7854 : : 256 : 201,0624 = ⊙ *A C*, for $\frac{1}{2}$ *D* = *A C*. Then 201,0624 × 4 = 804,2496, the *Superficies* of the whole *Sphere*. Or, becaufe 3,1416 is four Times 0,7854, therefore it will always be 1 : 3,1416 : : *D D* : 3,1416 *D D*, the *Superficies* of the *Sphere* (as before); and it is equal to the *curve Superficies* of the *right Cylinder*, whofe Diameter and Height are each = *D* the Axis of the *Sphere*. For 3,1416 *D* = the *Periphery* of the *Cylinder*'s Bafe, and that, multiply'd with *D* its Height, will be 3,1416 *DD* the *curve Superficies* of the *Cylinder*, by *Theorem* 12. And if to this there be added the *Area* of its two Bafes (or *Ends*) *viz.* 1,5708 *DD*, then 'tis evident, that the whole *Superficies* of the *Cylinder* will be to that of the *Sphere* in Proportion of 3 to 2.

*Scholium.*

From the Method here ufed in proving the laft *Theorem* 'twill be eafy to find the *curve Superficies* of any *Segment*, or *Part* of a *Sphere*, that is cut off by a Right Line or Plain, *viz.* fuch as the Segment *a T m* in the laft *Scheme*, whofe *curve Superficies* is ⊙ *a T* (as above). Therefore (becaufe □ *a b* + □ *T b* = □ *a T*) it will be ⊙ *a b* + ⊙ *T b* = the *curve Superficies* of that *Segment*.

But if the Axis *T S*, and Height *T b*, of the *Segment* are given, then will it be *T S* × *T b* □ *a T*; as in the third Step above: Which gives this Proportion or Theorem;

Viz.

Viz. { *As the* Axis *of the* Sphere : *is to the whole* Superficies *of the* Sphere : : *so is the* Height *of any* Segment *to its curve* Superficies.

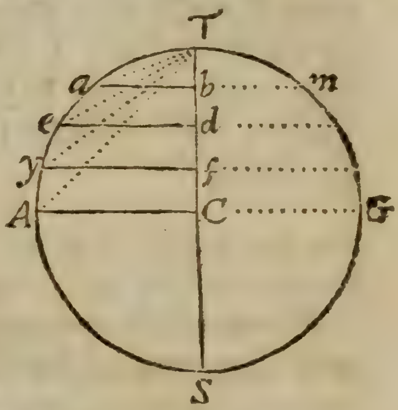
To which if there be added the *Area* of the Segment's Base, the Sum will be the Superficies of the whole Segment.

## THEOREM XVIII.

*Every Sphere is equal to two Thirds of its circumscribing Cylinder.*

That is, of a Cylinder whose Height and Diameter of its Base are each equal to the Axis of the Sphere.

### Demonstration.




According to the Work in the last *Theorem* it appears, that $\odot\, ab$, $\odot\, ed$, $\odot\, yf$, &c. do constitute the Solidity of the Sphere; and that $\square\, aT$, $\square\, eT$, $\square\, yT$, &c. are a Series of Terms in *Arithmetick* Progression, $\square\, AT$ being the greatest *Term*, and $TC$ the *Number* of *Terms*; therefore $\odot\, AT \times \frac{1}{2} TC =$ the Sum of all the Series, per *Lemma* 2. And because $\square\, aT - \square\, Tb = \square\, ab$, $\square\, eT - \square\, Td = \square\, ed$, $\square\, yT - \square\, Tf = \square\, yd$, $\square\, AT - \square\, TC = \square\, AC$, &c. wherein $\square\, Tb$, $\square\, Td$, $\square\, Tf$, &c. are a Series of Squares whose Roots $Tb$, $Td$, $Tf$, are in Arithmetick Progression, $\square\, TC$ being the greatest *Term*, and $TC$ the *Number* of *Terms*; therefore $\odot\, TC \times \frac{1}{3} TC =$ the Sum of all that Series, *per* Lemma 3, consequently, $\odot\, AT \times \frac{1}{3} TC - \odot\, TC \times \frac{1}{3} TC =$ the Sum of the *Series* $\odot\, ab$, $\odot\, ed$, $\odot\, yf$, &c. which constitute the Solidity of the half Sphere $ATG$. Put $D = 2\,TC$ the Axis of the Sphere; then $\frac{1}{4} D = \frac{1}{2} TC$, and $\frac{1}{6} D = \frac{1}{3} TC$. And because $\square\, AT = 2\, \square\, TC$; therefore $\odot\, AT = 2\, \odot\, TC = 1{,}5708\, DD$. And $1{,}5708\, DD \times \frac{1}{4} D = 0{,}3927\, DDD$.

Again, $D\, TC \times \frac{1}{3} TC = 0{,}7854\, DD \times \frac{1}{6} D = 0{,}1309\, DDD$, then $0{,}3927\, DDD - 0{,}1309\, DDD = 0{,}2618\, DDD$ the Solidity of the *Semi-sphere* $ATG$; consequently, $0{,}2618\, DDD \times 2 = 0{,}5236$, $DDD$ will be the solid Content of the whole *Sphere*, which is equal to two Thirds of the Cylinder whose Diameter of its *Base* and Height $= D$. For $0{,}7854\, DDD =$ the Solidity of the Cylinder, by *Theorem* 11. But $\frac{2}{3}$ of $0{,}7854\, DDD = 0{,}5236\, DDD$; as before. Therefore, &c. : as by *Theorem*.

*Example.* Suppofe the *Axis* $D = 16$, then $DDD = 4096$, and $1 : 0{,}5236 :: 4096 : 2144{,}6656$ the *folid Content* of that *Sphere.*

*Corollaries.*

1. Hence it appears, that the folid Content of every Sphere is equal to its Superficies multiply'd into one fixth Part of its *Axis.* For its Superficies is $3{,}1416\ DD$, by *Theorem* 17. But $3{,}1416 \times \frac{1}{6} D = 0{,}5236\ DDD$ the folid Content, as before.

2. And hence 'tis alfo evident, that there is the like *Ratio* or Habitude between the *Cube* and its *infcrib'd Sphere*, as is betwixt the *Square* and its *infcrib'd Circle*; and that is, as the Superficies of any *Cube* : is to the Superficies of its infcrib'd Sphere : : fo is the folid Content of that *Cube* : to the folid Content of the *Sphere.* [See the Circle's Proportion, *Page* 407.] For if $D =$ the Side of the *Cube*, then $6\ DD =$ its Superficies, and $DDD =$ its Solidity, and $3{,}1416\ DD =$ the *Sphere*'s Superficies. But $6\ DD : 3{,}1416\ DD :: DDD : 0{,}5236\ DDD$ the Solidity of the *Sphere*; as above.

*Scholium.*

From the Proof of this *Theorem* 'twill be eafy to deduce or raife *Theorems* for finding the folid Content of any *Fruftum* or *Segment* of a *Sphere*, as *a T m* in the laft Figure. For we there fuppofe the Segment *a T m* to be conftituted of an infinite Series of Circles, which have the fame *Ratio* with all thofe Circles that conftitute the *Semi-fphere.* Therefore it follows, that $\odot\ at \times \frac{1}{2} Tb - \odot\ bT \times \frac{1}{3} Tb$ will be the Sum of all the Circles intercepted between *T* and *b*. Confequently 'twill be the Solidity of that Segment. And becaufe $\square\ ab + \square\ Tb = \square\ aT$ : therefore $\overline{\odot\ ab + \odot\ Tb} \times \frac{1}{2} Tb - \odot\ Tb \times \frac{1}{3} b =$ the fame *Solidity.*

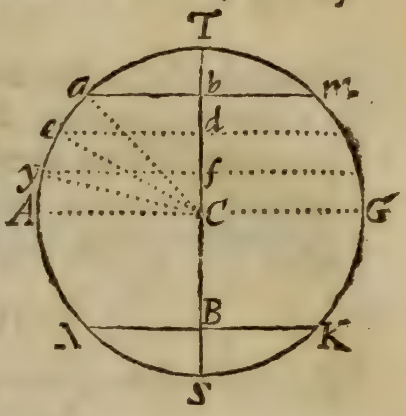
Let $c = ab$ half the *Segment*'s Bafe; $h = Tb$ its Height; and $S =$ the *Solidity* of the *Segment* or *Fruftum* : Then $\odot\ ab = 3{,}1416cc$, and $\odot\ Tb = 3{,}1416hh$. Confequently,

$\frac{3{,}1416cch + 3{,}1416hhh}{2} - \frac{3{,}1416hhh}{3} = S$, which being reduced will become $\overline{3cch + hhh} \times 0{,}5236 = S$. Or $1{,}909855)\ 3cch + hhh$ $(= S$. for $0{,}5236)\ 1{,}0000\ (1{,}909855$. Which is one *Theorem* for finding the *Fruftum*'s Solidity.

*Note,*

*Note*, Here we ſuppoſe the Height of the Segment, and the Diameter of its Baſe to be given; but if the *Axis* of the Sphere, and the Height of the Segment be given, then putting $D =$ the Sphere's Axis, $h =$ the Segment's Height, and $c$ as before, 'twill be $\overline{D-h}\times h = cc$, viz. $Dh - hh = cc$. Therefore $3Dhh - 2hhh = 3cch + hhh$. conſequently $\overline{3Dhh - 2hhh}\times 0{,}5236 = S$, the *Fruſtum*'s Solidity. Or 1,90985) $3Dhh - 2hhh$ ( $= S$, as before. Which is a ſecond *Theorem* for finding the ſame *Fruſtum a T m*.

And if it be requir'd to find the middle Part *a m N K*, uſually call'd the middle *Zone* of a Sphere, then becauſe 'tis ſuppos'd that $am = NK$, or which is all one, that $bC = CB$, therefore it is plain, that, if twice the Segment *a T m* be taken from the Solidity of the whole Sphere, there will remain the Middle *Zone a m N K*. But, becauſe that Work is a little troubleſome, I ſhall here ſhew how to raiſe a Theorem for the doing it.




Firſt, Becauſe $AC = yC = eC = aC = TC$. Therefore it will be $\square AC - \square Cf = \square yf$. $\square AC - \square Cd = \square ed$. $\square AC - \square Cb = \square ab$, &c. Here becauſe $\square AC$. $\square AC$. $\square AC$, &c. are a Series of *Equals*, and $Cb$ the Number of all the Terms, therefore $\square AC \times Cb =$ the Sum of all that Series, by *Lemma* 1. And $\square Cf$. $\square Cd$. $\square Cb$, &c. being a Series of Squares whoſe *Roots* are in *Arithmetick Progreſſion*, beginning at the Center or Point $C$, viz. 0, $Cf$, $Cd$, $Cb$, &c. wherein the greateſt Term is $\square Cb$, and Number of Terms is $Cb$. *Ergo* $\square Cb \times \frac{1}{3} Cb =$ the Sum of all the Series by *Lemma* 3. Conſequently, the $\odot AC \times Cb - \odot Cb \times \frac{1}{3} Cb =$ the Sum of all the Series $\odot yf$. $\odot ed$. $\odot ab$, &c. which do conſtitute the Solidity of the *half Zone a m A G*. And becauſe $\square AC - \square Cb = \square ab$. *Ergo* $\odot AC - \odot ab = \odot Cb$. Conſequently $\odot AC \times Cb - \frac{\overline{\odot AC - \odot ab}\times Cb}{3} = \overline{2\odot AC + \odot ab}\times \frac{1}{3} Cb$ will be the Solidity of the *half Zone*.

Put $D = AG = 2AC$. $x = am$. and $H = bB = 2Cb$. Then $\odot AC = 0{,}7854\,DD$. $\odot ab = 0{,}7854\,xx$. And if we turn the common Factor 0,7854 into the Diviſor 1,27323,

 and

and then take the Triple of that Divisor, *viz.* 3,8197 (as before in the Frustum of Pyramids) the Result of the precedent Work will produce this following Theorem.

THEOR. XIX. $\left\{\frac{2DD+xx}{3{,}8197}\right. : \times H = \left\{\right.$ the *middle Zone am NK.*

### THEOREM XX.

Spheres *are in Proportion one to another as the* Cubes *of their Diameters.* (18. *e.* 12.)

### Demonstration.

Suppose $D =$ the *Diameter* or *Axis* of any *Sphere*, and $d =$ the Diameter of another *Sphere*, either greater or lesser. Then is 0 5236 $DDD =$ the *Solidity* of one *Sphere*, and $0{,}5236ddd =$ the *Solidity* of the other *Sphere*, by *Theorem* 18. But $DDD : ddd :: 0{,}5236\,DDD : 0{,}5236ddd$. Q. E. D.

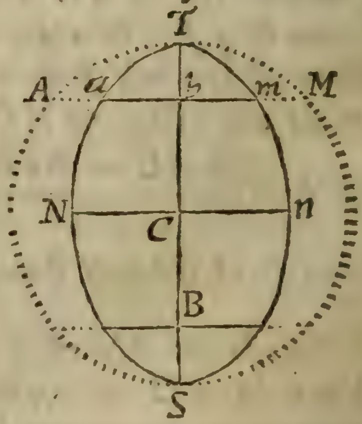
### THEOREM XXI.

*The solid Content of every* Spheroid *is equal to two Thirds of its circumscribing* Cylinder.

### Demonstration.

Suppose the Figure *NTnSN* in the annex'd *Scheme*, to represent a *Spheroid*, form'd by the Rotation of the *Semi-Ellipsis TNS*, about its *Transverse Axis TS* (as by Definition 15.)

Let $D = TS$, the Length of the *Spheroid*, and the *Axis* of its circumscribing *Sphere*; and $d = Nn$, the Diameter of the greatest Circle of the *Spheroid*. Then because □ $TC$ : □ $NC$ :: □ $Ab$ : □ $ab$, by Step 3 in *Theor.* 7, therefore it will be $DD : dd$ :: □ $Ab$ : □ $ab$ :: ⊙ $Ab$ : ⊙ $ab$, &c. But the Sum of an *infinite Series* of such Circles as ⊙ $Ab$ (whose Diameters are Chords) do constitute the Solidity of the *Sphere*, (as before at *Theorem* 18) and the Sum of an *infinite Series* of such Circles as ⊙ $ab$ (viz. whose *Diameters* are Ordinates of the Ellipsis) do constitute the Solidity of the *Spheroid*, by Definition 15. *Ergo* $DD : dd :: 0{,}5236\,DDD : 0{,}5236ddD =$ the Solidity of the *Spheroid*, by *Lemma* 6.




But

But 0,5236$dd\,D$ = $\frac{2}{3}$ of the *Cylinder* whose Diameter is = $d$, and Height = $D$, by *Theorem* 11. Q. E. D.

Now, from this *Proportion* between the *Sphere* and its inscrib'd *Spheroid*, 'twill be very easy to deduce *Theorems* for finding the *Solid Content* either of the *Segment* or middle *Zone* of any *Spheroid*, having the same Height with that of the *Sphere*.

For { *As the Solidity of the whole Sphere : is to the Solidity of the whole Spheroid : : so is any Part of the Sphere : to the like Part of the Spheroid*, by the *Converse* to *Lemma* 6.

As for Instance; suppose it were requir'd to find the middle *Zone* of any *Spheroid*: Let $D = TS$, and $d = Nn$, as above; and $H = b\,B$, $x = A\,M$, as in *Theorem* 19, and let $c = a\,m$. Then $\frac{2\,DD + xx}{3,8197} \times H$ = the middle *Zone* of the *Sphere*. And 0,5236 $DDD$ : 0,5236 $ddD$ : : $\frac{2\,DD + xx}{3,8197} \times H : \frac{2dd \times H}{3,8197} + \frac{xx\,dd \times H}{3,8197\,DD}$ = the middle *Zone* of the *Spheroid*.

Again, $DD : dd :: xx : cc$, therefore $\frac{xx\,dd}{DD} = cc$. consequently, $\frac{xx\,dd}{DD} \times \frac{H}{3,8197} = \frac{cc}{3,8197} \times H$, which being taken instead of $\frac{xx\,dd \times H}{3,8197\,DD}$, there will arise this following

THEOREM XXII. $\left\{ \frac{2\,dd + cc}{3,8197} \right. : \times H = \Big\{$ the middle *Zone* of the *Spheroid* being the very same with *Theorem* 19.

Note, *In the same Manner you may raise* Theorems *for finding the* Segment *of a* Spheroid, *cut off at either of its Ends*, &c.

## THEOREM XXIII.

*The* Area *of every* Parabola *is equal to two Thirds of its circumscribing* Parallelogram.

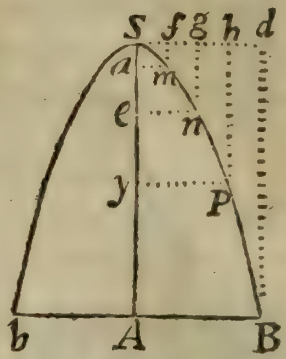
### Demonstration.

Let the Figure $S\,A\,B$ represent half a *Parabola*. Make $D\,B$ parallel to the *Axis* $S\,A$, and $S\,d$ parallel to the *Semi-Ordinate* $A\,B$, and suppose $S\,d$ to be divided into an *infinite Series* of equidistant

 *Points*,

*Points*, as *f*, *g*, *h*, &c. and from those *Points* imagine a Series of *parallel Lines*, viz. *f m*, *g n*, *h p*, &c. to touch the *Curve* of the *Parabola*, and meet the *Semiordinates m a*, *n e*, *y p*, &c. Then, according to the Property of the *Parabola*, it will

| | | |
|---|---|---|
| be | 1 | $SA : \square AB :: Sa : \square am$ |
| | 2 | $SA : \square AB :: Se : \square en$ |
| | 3 | $SA : \square AB :: Sy : \square yp$, &c. |
| But | | $Sa = fm.\ Se = gn.\ Sy = hp.\ SA = dB$ |
| | | Therefore alternately it will be |
| 3, | 4 | $\square AB : dB :: \square yp : hp$ |
| 2, | 5 | $\square AB : dB :: \square en : gn$ |
| 1, | 6 | $\square AB : dB :: \square am : fm$, &c. |




In these *Proportions* $\square\, am$, $\square\, en$, $\square\, yp$, &c. are a *Series* of *Squares* whose *Roots* $Sf$, $Sg$, $Sh$, &c. are in *Arithmetick Progression*, beginning at the Point *S*. And because the Lines *h p*, *g n*, *f m*, &c. have the same *Ratio*, therefore they are as such a *Series* of *Squares*, wherein *d B* is the greatest *Term*, and *S d* the *Number* of *Terms*. Consequently $\frac{dB \times Sd}{3}$ = the *Sum* of all those Lines, by *Lemma* 3. But $SA \times AB = dB \times Sd$. Therefore $\frac{SA \times AB}{3}$ = the *Sum* of all that *Series* of *Lines*; but all those *Lines* do constitute the *Area* of the *Semi-Parabola's Complement*, viz. the *Area* of what half the *Parabola* wants of compleating or filling up the *Parallelogram S d A B*. Wherefore $SA \times AB - \frac{1}{3} SA \times AB = \frac{2 SA \times AB}{3}$ will be the *Area* of half the *Parabola S A B*. Consequently, $\frac{2}{3} SA \times bB$ will be the *Area* of the whole *Parabola b S B*. Q. E. D.

*Example*. Suppose the *Base*, or greatest *Ordinate*, of a *Parabola* to be $bB = 24$, and its intercepted Diameter (or Axis) be $SA = 33$; then $2 SA \times bB = 66 \times 24 = 1584$. and 3) 1584 (528 the *Area* of that *Parabola*.

## THEOREM XXIV.

*Every Parabolick Conoid is equal to one Half of its circumscribing Cylinder.*

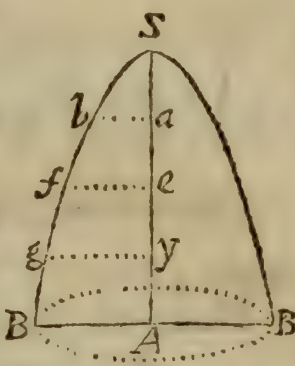
Demon-

## Demonstration.

If any *Semi-Parabola* (as $B\ S\ A$) be turn'd or mov'd about its *Axis* ($S\ A$) 'twill form a *solid Parabolical Conoid*, constituted of an *infinite Series* of *Circles*, viz. $\odot\ b\ a$, $\odot\ f\ e$, $\odot\ g\ y$, &c. by *Definition* 17.

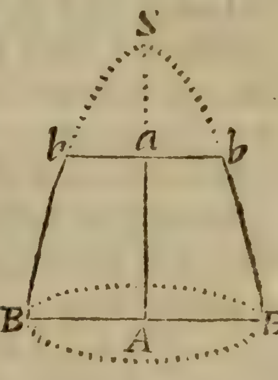
Now, according to the *Property* of every *Parabola*, it will be, $S A : A B :: A B : \frac{\square\ AB}{SA} = L$, the *Latus Rectum*.

Then $\begin{cases} S a \times L = \square\ b a \\ S e \times L = \square\ f e \\ S y \times L = \square\ g y, \text{\&c.} \end{cases}$




Here $S a \times L$, $S e \times L$, $S y \times L$, &c. are a *Series* of *Terms* in Arithmetick *Progression*: therefore $\square\ b\ a$, $\square\ f e$, $\square\ g y$, &c. are also a *Series* of *Terms* in the same *Progression*, beginning at the Point $S$; wherein $\square\ A B$ is the greatest *Term*, and $S A$ the *Number* of all the *Terms*. Therefore $\square\ A B \times \frac{1}{2}\ S A =$ the *Sum* of all the *Series* by *Lemma* 2. Consequently, $\odot\ A B \times \frac{1}{2}\ S A =$ the *Sum* of all the *Series* $\odot\ b a$, $\odot\ f e$, $\odot\ g y$, &c. which do constitute the *Solidity* of the *Conoid*. And putting $D = 2\ A B$, and $H = S A$. Then 0,7854 $D D \times \frac{1}{2} H =$ 0,3927 $D D H$ will be the solid *Content* of the *Conoid*, which is just half the *Cylinder* whose *Base* $= D$ and *Height* $= H$. [See *Theorem* 11.] Q. E. D.

This being understood, 'twill be easy to raise a *Theorem* for finding the lower *Frustum* of any *Parabolick Conoid*. For supposing $h = a\ A$ the *Height* of the *Frustum*, and $p = S\ a$ the *Height* of the Part $b\ S\ b$ cut off; then $h + p = S\ A$, the *Height* of the whole *Conoid*. Consequently, $\frac{\odot\ A B \times h + .\ A B \times p}{2} =$ *Solidity* of the whole *Conoid*. And $\frac{ba + p}{2} =$ the *Solidity* of the Part cut off.




| | | |
|---|---|---|
| *Ergo* | 1 | $\frac{\odot\ A B \times h + \odot\ A B \times p - \odot\ ba \times p}{2} =$ the *Solidity* of the *Frustum*. |
| But | 2 | $h + p : \square\ A B :: p : \square\ b a$ |
| Conseq. | 3 | $h + p : \odot\ A B :: p : \odot\ b a$ |
| 3 ∴ | 4 | $\odot\ A B \times p = \odot\ ba \times h + \odot\ b a \times p$ |

4 —

| | | |
|---|---|---|
| $4 - \odot ba \times p$ | 5 | $\odot AB \times p - \odot ba \times p = \odot ba \times b$ |
| $1 \times 2$ | 6 | $\odot AB \times b + \odot AB \times p - \odot ba \times p = 2F$ |
| $6 - 5$ | 7 | $\odot AB \times b = 2F - \odot ba \times b$ |
| $7 + \odot ba \times b$ | 8 | $\odot AB \times b + \odot ba \times b = 2F$ |
| $8 \div 2$ | 9 | $\frac{\odot AB + \odot ba}{2} \times b = F$ the *Frustum*'s Solidity. |

Let $D = AB$, as before, and $d = 2ba$ the Diameter of the *Part* cut off; then we ſhall have this following

THEOREM XXV. $\left\{ \overline{0{,}3927\,DD + 0{,}3927\,dd} \times b = \right.$ the *Solidity* of the *Fruſtum* requir'd.

Or $\left\{ \frac{DD + dd}{2{,}5464} \times b = \right.$ the *Fruſtum*; for ,3927) 1,0000 (=2,5464 and becauſe $2{,}5464 + \frac{2{,}5464}{2} = 3{,}8196$; therefore it may be made $3{,}8196)\overline{DD + dd \times \tfrac{1}{2} b}$ (= the ſame *Fruſtum*, &c.

Note, *The Reaſon why I have reduced this* Theorem *to have the ſame* Diviſor *with thoſe at the* Fruſtums *of Pyramids*, &c. *will beſt appear farther on*, viz. *when they all come to be apply'd to Practice in* Gauging.

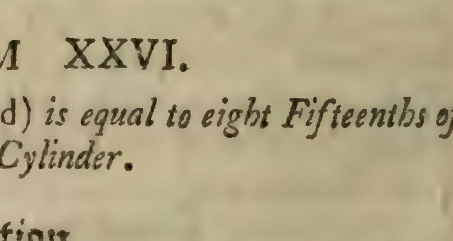
## THEOREM XXVI.

*Every Parabolick Spindle* (or Pyramidoid) *is equal to eight Fifteenths of its circumſcribing Cylinder.*

### Demonſtration.

If any *acute Parabola*, as $b\,S\,B$, be turn'd or mov'd about its greateſt Ordinate $b\,AB$, it will form a Solid call'd a *Parabolick Spindle*, conſtituted of an *infinite Series* of ⊙ $m\,a$, ⊙ $n\,e$, ⊙ $p\,y$, &c. by Definition 18.

Let us ſuppoſe the Line $S\,d$, parallel to $A\,B$, &c. (as at *Theorem* 23) then it hath already been prov'd, that the Lines $f\,m$, $g\,n$, $h\,p$, &c. are a *Series* of *Squares* whoſe *Roots* are in *Arithmetick Progreſſion:* conſequently their *Squares*, viz. □ $f\,m$, □ $g\,n$, □ $h\,p$. &c. will be a *Series* of *Biquadrates*, whoſe *Roots* will be in *Arithmetick Progreſſion:* which being premis'd, we may proceed thus.




First, $\left\{ \begin{array}{l|l} 1 & SA - fm = ma \\ 2 & SA - gn = ne \\ 3 & SA - hp = py \text{ \&c.} \end{array} \right.$

| | | |
|---|---|---|
| 1 ⊚² | 4 | $\Box SA - 2SA \times fm + \Box fm = \Box ma$ |
| 2 ⊚² | 5 | $\Box SA - 2SA \times gn + \Box gn = \Box ne$ |
| 3 ⊚² | 6 | $\Box SA - 2SA \times hp + \Box hp = \Box py$, &c. |

In these *Æquations* the $\Box SA$, $\Box SA$, $\Box SA$ being a *Series* of *Equals*, and $AB$ the *Number* of all the *Terms*; therefore it will be $\Box SA \times AB =$ the *Sum* of the *Series*, by *Lemma* 1.

2. Because $fm$, $gn$, $hp$, &c. are as a *Series* of *Squares* wherein $SA$ is the *greatest Term*, and $AB$ the *Number* of all the *Terms*; therefore $\frac{2SA \times SA \times AB}{3} = \frac{2\Box SA \times AB}{3}$ will be the *Sum* of all that *Series*, by *Lemma* 3.

3. And the $\Box fm$, $\Box gn$, $\Box hp$, &c. will be a *Series* of *Terms* in the *Ratio* of *Biquadrates*, as above; $\Box dB = \Box SA$ being the *greatest Term*, and $AB$ the *Number* of all the *Terms*; therefore it will be $\frac{\Box SA \times AB}{5} =$ the *Sum* of all that *Series*, by *Lemma* 5.

Whence it follows, that $\Box SA \times AB - \frac{2\Box SA \times AB}{3} + \frac{\Box SA \times AB}{5} =$ the *Sum* of all the *Series* of $\Box ma$, $\Box ne$, $\Box py$, &c. That is, $\frac{8\Box SA \times AB}{15} =$ the *Sum* of all the *Series* of $\Box ma$, $\Box ne$, $\Box hp$, $\Box dB$. &c. consequently, $\frac{8 \odot SA \times AB}{15} =$ the *Sum* of all the *Series* of $\odot ma$, $\odot ne$, $\odot py$, &c which do constitute the *Solidity* of half the *Spindle*, viz. of $SAB$. Therefore putting $D = 2SA$, and $H = 2AB$, (viz. $bAB$) it will be 0,41888 $DDH$ = the *Solidity* of the whole *Parabolick Spindle* $bSB$, being $\frac{8}{5}$ of 0,7854 $DDH$ the Solidity of its *circumscribing Cylinder*. Q. E. D.

From hence we may also raise a *Theorem* for finding the *Frustum* $SApy$ of the last Figure. For $\odot SA$ being the greatest Term, $\odot py$ the least Term, and $Ay$ the *Number* of all the Terms or Circles included between $A$ and $y$,

| | | |
|---|---|---|
| Therefore | 1 | $\overline{\Box SA - \frac{2SA \times hp}{3} + \frac{\Box hp}{5}} \times Ay = z$ the *Sum* of all the *Series* $\Box SA$, $\Box ma$, $\Box gn$, $\Box py$ |
| $1 \times 3$ | 2 | $\overline{3\Box SA - 2SA \times hp + \frac{3\Box hp}{5}} \times Ay = 3z$ |

| | | |
|---|---|---|
| 2 ÷ *Ay* | 3 | $3\square SA - 2SA \times hp + \frac{3\square hp}{5} = \frac{3z}{Ay}$ |
| But | 4 | $\square SA - 2SA \times hp = \square py - \square hp$, by 6th *Step.* |
| 3 — 4 | 5 | $2\square SA + \frac{3\square hp}{5} = \frac{3z}{Ay} - \square py + \square hp$ |
| 5 + &c. | 6 | $\overline{2\square SA + \square py - \frac{2}{5}\square hp} = \frac{3z}{Ay}$ |
| Conseq. | 7 | $\overline{2\odot SA + \odot py - \frac{2}{5}\odot hp} \times \frac{1}{2}Ay = z$, the *Sum* |

of all the *Series* of $\odot\, SA$, $\odot\, ma$, $\odot\, ne$, $\odot\, py$, which do conftitute the *Solidity* of the *Fruſtum* $SApy$. Therefore putting $D = 2SA$, as before, $C = 2pv$, $x = 2hp$, and $H = Ay$, it will be $\overline{1{,}5708\,DD + 0{,}7854\,CC - 0{,}31416\,xx} \times \frac{1}{3}H =$ the *Fruſtum* $SApy$. And if we make $L = 2H$. Then $\overline{1{,}5708\,DD + 0{,}7854\,CC - 0{,}31416\,xx} \times \frac{1}{3}L =$ Double of that *Fruſtum*, being the *middle Zone*. And by turning theſe *Factors* into one common Diviſor, as in the *Fruſtum* of the *Conoid* at *Theorem* 25, *Page* 430, there will ariſe this following *Theorem*.

## THEOREM XXVII.

$$3{,}8196)\ \overline{2DD + CC - 0{,}4xx} \times L\ ( =$$
the *middle Zone* of a *Parabolick Spindle.*

It may be here expected that I ſhould now proceed to ſhew how the *Area* of any *Hyperbola*, and the *Contents* of ſuch *Solids* as may be form'd by the *Rotation* of that Figure about its *Axis*, &c. may be found; but becauſe thoſe Things cannot be exactly perform'd by any certain or ſettled *Theorem*, as theſe of the *Circle*, *Ellipſis*, and *Parabola* have been, I have therefore omitted them, and refer the Reader to Dr. *Wallis*'s *Algebra*, Chap. 90, *&c.* or to the *Philoſoph. Tranſact.* Numb. 34, wherein he may find the Method of forming *infinite Series* relating to the ſquaring of an *Hyperbola*, &c. which are too tedious to be fully explain'd and demonſtrated in this ſmall Tract, it being only intended as an *Introduction*, the which I ſhall here conclude.

*F I N I S.*

AN

# APPENDIX

OF

# *Practical Gauging.*

THE Art of *Gauging* is that Branch of the Mathematicks called *Stereometry*, or the Meaſuring of Solids, becauſe the Capacities or Contents of all Sorts of Veſſels uſed for Liquors, *&c.* are computed as tho' they were really ſolid Bodies; which any one that hath made himſelf Maſter of the 'foregoing Parts of this Treatiſe may eaſily underſtand, without any farther Directions.

However, becauſe 'tis not to be ſuppos'd that every one, who deſigns to undertake the Office or Employment of a Gauger, hath made ſo great a Progreſs in Mathematical Learning, I have therefore preſented the young Gauger with this Appendix, wherein I have only inſerted ſuch Rules as are uſeful in Gauging, and have been already demonſtrated in this Treatiſe. But herein, I preſuppoſe that he hath acquir'd (or if not, 'tis very neceſſary he ſhould acquire) a competent Knowledge both in Arithmetick and Geometry: That is,

I. In Arithmetick he ſhould underſtand the principal Rules very well, eſpecially Multiplication and Diviſion, both in whole Numbers and Decimal Parts, (which may be eaſily learnt out of the 2d, 3d, and 5th Chapters of Part 1.) that ſo he may be ready at computing the Contents of any Veſſel, and caſting up his Gauges by the Pen only, *viz.* without the Help of thoſe Lines of Numbers upon Sliding Rules, ſo much applauded, and but too much practis'd, which at beſt do but help to gueſs at the Truth; I mean ſuch Pocket-Rules as are but nine Inches (or a Foot) long, whoſe Radius of the double Line of Numbers is not ſix Inches; and therefore the Graduations or Diviſions of thoſe Lines are ſo very cloſe, that they cannot be well diſtinguiſh'd. 'Tis true, when the Rules are made two or three Feet long (I had one of ſix Feet) there they may be of ſome Uſe, eſpecially in ſmall Numbers; altho' even then the Operations may be much better (and almoſt as ſoon) done by the Pen: For, indeed, the chief Uſe of Sliding-Rules is only in taking of Dimenſions, and for that Purpoſe they are very convenient.

II. In Geometry the Gauger ſhould underſtand not only how to take Dimenſions (which is beſt learnt by Practice) but alſo how to divide any irregular Figure or Superficies, as Brewers Backs or Coolers, &c. into the eaſieſt and feweſt regular Figures they will admit of, that ſo their Area's may be truly computed with the leaſt Trouble. And this may be learn'd (with a little Care and Diligence) out of the 1ſt, 2d, and 5th Chapters of Part III, which the Gauger ſhould be well acquainted with. Alſo he ought to have ſo much Skill in Solids, as to be able, even at ſight (but this muſt be acquired by Experience) to determine what ſort of Figure any Veſſel is of (*viz.* any Tun or cloſe Caſk) or what Figures it may be beſt reduced to, ſo that its Dimenſions may be truly taken, and the Content thereof computed with the leaſt Error. I ſay, with the leaſt Error, becauſe 'tis very difficult, if not impoſſible, to do it exactly; for there is not any Tun, or cloſe Caſk, &c. ſo regularly made, as by the Rules of Art 'tis requir'd to be.

III. Beſides the aforemention'd, the young Gauger muſt know, that all Dimenſions uſeful in Gauging are to be taken in Inches, and Decimal Parts of an Inch; and if they are taken in any other Meaſures, as Feet, Yards, &c. thoſe Meaſures muſt be reduced to Inches, (ſee Sect. 4. Page. 42.) becauſe the Contents of all Sorts of Veſſels (taken Notice of in Gauging) are computed by the Standard Gallon of its Kind, whoſe Content is known to be a certain Number of Cubick Inches: That is, the Beer or Ale Gallon contains 282, the Wine 231, and the Corn Gallon 268, 8 Cubick Inches. [See the five Tables, &c. in Pages 34, 35, 36, which I here ſuppoſe the Gauger to have learnt perfectly, by heart.] Conſequently, if either the ſuperficial or ſolid Content of any Veſſel, as Back, Tun, Caſk, &c. be once computed in Cubick Inches, 'twill be eaſy to know how many Gallons, either of Ale, Wine, or Corn, that Veſſel will hold.

*Note*, I have here ſaid, the Superficial Content in Cubick Inches, which may ſeem to be very improper, according to the Definition given of a Superficies in Page 279; but you muſt know, that, in the Buſineſs of Gauging, all Superficies or Area's are always underſtood to be one Inch deep, otherwiſe it could not be ſaid (as in the Gauger's Language it is) that the Area of ſuch a Back, or of ſuch a Circle, &c. is ſo many Gallons.

Theſe Things being very well underſtood, the young Gauger will be fitly prepar'd to underſtand the following *Problems*, which are ſuch as have (moſt of them) been already propos'd in the 'foregoing Parts of this Treatiſe, and only are here apply'd to Practice; and therefore I ſhall, for Brevity's Sake, often refer to thoſe Theorems and Problems.

Sect. 1. To find the *Area* of any right-lined Superficies in Gallons.

## PROBLEM I.

*To find the* Area *of any* ſquare Tun, Back, *or* Cooler, *&c. either in* Ale, Wine, *or* Corn Gallons.

RULE. { Multiply the given Length or Breadth (being here equal) into itſelf, and the Product will be the Area in Inches; then divide that Area by 282, or 231, or 268,8 and the Quotient will be the Area requir'd.

*Example.* Suppoſe the Side of a Square Tun, Back, or Cooler be 124,5 Inches, what will its Area be in Gallons?

Firſt 124,5 × 124,5 = 15500,25 the Area in Inches.

| | | | | |
|---|---|---|---|---|
| Then 282 | | 54,96 *&c.* | | *Ale Gallons.* |
| And 231 | 15500,25 | 76,10 *&c.* | the *Area* in | *Wine Gallons* |
| Or 268,8 | | 57,66 *&c.* | | *Corn Gallons.* |

But if any one would rather work by Multiplication than by Diviſion, he may turn or change any Diviſor into a Multiplicator, if he divide Unity, or 1, by that Diviſor. (Vide Probl. 3, Pag. 402.)

| | | | | |
|---|---|---|---|---|
| Thus 282 | | 0,003546 | | *AleGallons.* |
| And 231 | 1,000000 | 0,004329 | the Multipli. for | *W.Gallons.* |
| Or 268,8 | | 0,003722 | | *C. Gallons.* |

Conſequently 15500,25×0,003546 = 54,96 *&c.* the Area in Ale Gallons; as before and ſo on for the reſt.

## PROBLEM II.

*To find the* Area *of any* Tun, Back, *or* Cooler *in the Form of a Right-angled Parallelogram in* Ale Gallons, *&c.*

See the Rule for finding its Area in Inches, at Probl. 1. P. 339, then either divide (or multiply) that Area as above, and you will have the Area in Gallons.

*Example.* Suppoſe the Length of a Brewer's Tun, Back, or Cooler be 217,5 Inches, and its Breadth 85,6 Inches, what will its Area be in Ale or Beer Gallons, *&c?*

Firſt 217,5×85,6 = 18648. Then 282) 18648 (66,12, *&c.*
Or 18648×0,003546 = 66,12, *&c.* the *Area* requir'd, *&c.*

## PROBLEM III.

*To find the* Area *of any* Triangular Tun, Back, *or* Cooler, *in* Ale Gallons, *&c.*

See the Rule for finding its Area in Inches at Prob. 3, p. 340; then divide (or multiply) that Area as before, and you will have the Area requir'd.

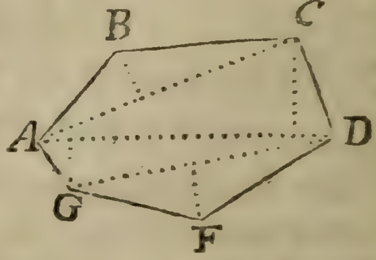
*Example.* If the Length of the Base of a Triangular Cooler be 86,4 Inches, and its Perpendicular Breadth be 57 Inches, what will its Area be in Ale Gallons?

Firft, 86, $4\times\frac{57}{2}=2462,4$. Then 282) 2462,4 (8,73 *&c.* Or $2462,4\times0,003546=8,73$ *&c.* the Area in Ale Gallons.

Proceeding thus, you may eafily find the Area of any Tun, Back, or Cooler, whether it be in the Form of a Rhombus, Rhomboides, Trapezium, or any other Polygon, either regular or irregular, in Ale or Beer Gallons, *&c.* if you firft divide it into Triangles, and then find the Area's of thofe Triangles; (as in the 2d, 4th, 5th, and 6th Problems in Chap. 5, Part III.) the Sum of thofe Area's being divided (or multiply'd) by its proper Divifor (or Multiplicator) as above, will give the Area requir'd.

Now, the Practical Way of dividing any Polygonous Tun, Back, *&c.* into Triangles, is by help of a chalk'd Line, fuch as the Carpenters ufe, and may be thus perform'd.

Suppofe any Brewer's Tun, Back, or Cooler in the Form of the annex'd Figure *A B C D F G.* Let one End of the chalk'd Line be faften'd with a Nail (or otherwife) in any Corner or Angle of the Back, as at *A*; then ftraining it to the Angle at *C*, ftrike the Diagonal Line *AC*, upon the Bottom of the Back; and ftraining it again to the Angle *D*, ftrike another Diagonal Line, as *A D*, and fo on for the Diagonal Line *G D*, &c. Then having




mark'd out all the Diagonals, the Perpendiculars may be thus found: Faften (as before) one End of the chalk'd Line in the Angle *B*, and then, by moving it *to* and *fro* upon the Stretch, find out the neareft Diftance between the Angle at *B* and the Diagonal Line *A C*; and there ftrike a Line, and it will mark out the Perpendicular from *B* to the Line *A C*, and fo on for the other Perpendiculars: Which being all mark'd out upon the Bottom of the Back, meafure them and each Diagonal by a Line of

Inches, &c. and then the Area of that Back may be computed, as directed above.

And here, by the Way, it may be obſerved, that the Number of Triangles will always be leſs by two, and the Number of the Diagonals leſs by three, than the Number of the Sides of any Right-lin'd Figure that is ſo divided.

Having found (as above) the true Area of any Brewer's Back or Cooler (which, according to the Laws of Exciſe, ought always to be fix'd or immoveable) the next Thing will be to find out the true dipping or gauging Place in that Back, that ſo the true Quantity of Worts may be computed or (caſt up) at any Depth; which may be thus done.

1. When the Bottom of the Back is covered all over (of any Depth) either with Worts or Liquor (viz. *Water*) then dip it in eight or ten ſeveral Places (more or leſs according to the Largeneſs of the Back) as remote and equally diſtant one from another as you well can, noting down the wet Inches and decimal Parts of every Dip.

2. Divide the Sum of all thoſe Dips or wet Inches by the Number of Places you dipp'd in, and the Quotient will be the mean Wet of all thoſe Dips.

3. Laſtly, find out ſuch a Place by the Side of the Back (if you can) that juſt wets the ſame with that mean Dip, and make a Notch or Mark there, for the true and conſtant Dipping-place of that Back. Then if any Quantity of Worts (which do cover the whole Back) be dipp'd or gaug'd at that Place, and the wet Inches ſo taken be multiply'd into the Area of the Back in Gallons, the Product will ſhew what Quantity (*viz.* how many Gallons) of Worts are in that Back at that Time, provided the Sides of the Back do ſtand at Right Angles with its Bottom.

Sect. 2. *To find the* Area *of any* Circular *and* Elliptical Superficies *in Gallons.*

1. I have demonſtrated in Cap. 6, Part III, and Theorem 3, 5, 6. Part V. that the Periphery of the Circle whoſe Diameter is Unity, or 1, is 3,14159265 *&c.* (or for common Uſe 3,1416) and that its Area is 0,78539816 *&c.* (or 0,7854 *fere.*)

2. Alſo, that the Peripheries of all Circles are in Proportion one to another as their Diameters are; and their Area's are in Proportion to the Squares of the Diameters. That is, as 1 : 3,1416 : : the Diameter of any Circle : to its Periphery. And 1 : 0,7854 : : the Square of the Diameter : to the Area.

Upon

Upon theſe two Proportions depend the Solutions of all the common or practical Queſtions about a Circle. [*See* Page 408, 409.]

## PROBLEM IV.

*The* Diameter *of any* Circle *being given in Inches, to find the* Periphery.

RULE. { Multiply the given Diameter with 3,1416, and the Product will be the Periphery requir'd. [*See* Prob. 1. p. 408.]

*Example.* Suppoſe the Diameter of a Circle be 54,5 Inches, and it were requir'd to find its Periphery. Then 54,5×3,1416 = 171,21, &c. Inches is the Periphery requir'd. The Converſe of this is eaſy, *viz.* by having the Periphery given, to find the Diameter. [*See* Prob. 3. Page 408.]

## PROBLEM V.

*The* Diameter *of any* Circle *being given* (in Inches) *to find its* Area *in Gallons.*

RULE. { Multiply the Square of the propos'd Diameter into 0,7854, and the Product will be the Area in Inches; [*See* Probl. 2, P. 408.] that Area being divided by 282, or 231, &c. the Quotient will be the Area required.

*Example.* Suppoſe the given Diameter be 54,5 Inches as above. Firſt, 54,5 × 54,5 = 2970,25. And 2970,25 × 0,7854 = 2332,83 the Area in Inches:

| | | | | |
|---|---|---|---|---|
| Then 282 | 2332,83 | 8,2724 | the Area in | *Ale* or *Beer* Gallons. |
| And 231 | | 10,0988 | | *Wine* Gallons. |
| Or 268,8 | | 8,6788 | | *Corn* Gallons. |

But theſe Area's in Gallons may be much eaſier found without knowing the Circle's Area in Inches, as above, by having the Square of the Diameter of that Circle whoſe Area is one Gallon; which may be thus found, by Theorem 6, Page 407.

0,785398 : 1 : : 282 : 359,05 the Square of the Diameter of the Circle whoſe Area is 282 cubick Inches, *viz.* one *Ale* Gallon.

And from this Proportion will ariſe the following Diviſors;

| | | | |
|---|---|---|---|
| *viz.* 0,785398 | 282,000000 (359,05 | will be a *Diviſor* for | *A. G.* |
| | 231,000000 (294,12 | | *W. G.* |
| | 268,800000 (342,24 | | *C. G.* |

If

If the Square of the Diameter of any Circle be divided by any one of theſe conſtant or fixed Diviſors, the Quotient will ſhew that Circle's Area in their reſpective Gallons. As for Inſtance, in the laſt Circle, whoſe Square of its Diameter is 2970,25.

| | | | | | | |
|---|---|---|---|---|---|---|
| Then 359,05 | | 8,2725 | | A. G. | |
| And 294,12 | 2970,25 | 10,0988 | the *Area* in | *W*. G. | as before. |
| Or 342,24 | | 8,6788 | | *C. G.* | |

Now theſe Diviſors may be turn'd into Multiplicators by dividing Unity or 1, as in Page 435: Or rather by dividing the Area in Inches of that Circle whoſe Diameter is 1.

That is, 0,785398 by 282. Or by 231, *&c.*

| | | | | |
|---|---|---|---|---|
| Thus 282 | | 0,002785 | | *Ale* Gal. |
| And 231 | 0,785398 | 0,003399 | the *Multiplicator* for | *Wine* Gal. |
| Or 268,8 | | 0,002922 | | *Corn* Gal. |

Theſe Multiplicators are the reſpective Area's of a Circle whoſe Diameter is 1; and therefore, if the Square of the Diameter of any Circle be multiply'd with any of theſe Numbers, the Product will be that Circle's Area in Gallons of the ſame Name:

*Viz.* 2970,25×0,002785 = 8,2725 the Area in *A. G.* as above.
And 2970,25×0,003399 = 10,0988 the Area in *W. Gal.* &c.

Thus you ſee, that if the Diameter of any Circle be given in Inches, there are three ſeveral Ways of finding its Area in Gallons, and all equally true; but that which is perform'd by the conſtant Diviſors is moſt generally practis'd.

## PROBLEM VI.

*The Tranſverſe* (or longeſt Diameter) *and the Conjugate* (or ſhorteſt Diameter) *of any* Elliptical Superficies *being given, to find its* Area *in Gallons.*

RULE. Multiply the two Diameters (*viz.* the Length and Breadth) together, and divide their Product by 359,05 for Ale Gallons, or 294, 12 for Wine Gallons, *&c.* the Quotient will be the Area requir'd. [*See* Theorem 7, Page 412.

*Example.* Suppoſe the longeſt Diameter to be 73,5 Inches and the ſhorteſt Diameter to be 51,6 Inches; what will the *Area* be in Ale Gallons?

Firſt 73,5×51,6 = 3792,6. Then 359,05) 3792,6 (10,56 the Area in Ale Gallons. Or 294,12) 3792,6 (12,89 the Area in Wine Gallons, *&c.*

*Note,*

*Note*, The two laſt Problems are of a great Uſe in Gauging of Worts amongſt Country Victuallers, who generally brew but ſhort Lengths of Ale (perhaps between 20 and 60 Gallons at a Brewing) and cool their Worts in ſeveral ſmall open Veſſels or Tubs, whoſe Baſes or Bottoms are either a Circle, or an Ellipſis, having their Sides but low, and are moſt commonly wider at the Top than at the Bottom.

Now a practical Way of computing the Quantity of Worts, that are at any Time in one of thoſe open Tubs, is briefly thus: When the Tub is dry, find the true Area of its Bottom according to its Figure (as above) and either mark that Area on the Outſide of the Tub (which was the Way I generally us'd to order, becauſe the Victuallers did often lend their cooling Tubs one to another) or elſe number the Tub, and enter its Area (and its Number) into the Stock-book; then, when any of thoſe Tubs hath Worts in it, take the Diameter of the Surface or Top of the Worts, and find that Area, adding it and the bottom Area together. If either the half Sum of thoſe two Area's be multiply'd with the Depth of the Worts (taken as near the Middle of the Tub as you well can) or, if the Sum of thoſe two Area's be multiply'd with half the Depth (ſo taken) the Product will ſhew the Quantity of thoſe Worts very near the Truth.

## PROBLEM VII.

*The* Diameter *of any* Circle, *and the verſed Sine*, viz. (the Height *of any Segment*, *being given*, *to find the* Area *of that Segment in* Gallons.

In the 410th and 412th Pages you have two Ways (and their Examples) of finding the Area of any Segment of a Circle in Inches; then if that Area in Inches be divided by 282, or 231 *&c.*) the Quotient will be its Area in Gallons. But becauſe the Area of any ſuch Segment may be readily found in Gallons (without finding its Area in Inches) by help of a Table of Segments, whoſe Conſtruction is laid down in the Problem, Page 411, *&c.* I have here inſerted a Compendium of ſuch a Table, which will ſerve very well for common Practice, not only to find the Area of any Segment of a Circle in Gallons, but alſo to find the Number of Gallons that are either drawn out, or remaining in any Cylindrick Veſſel lying along; or of any cloſe Caſk (being firſt reduced to a Cylinder) its Axis lying parallel to the Horizon, uſually call'd the *Ullage* of a Caſk; as ſhall be ſhew'd farther on.

*A Table*

A Table of the Segments of a Circle whose Area is Unity or 1, the Diameter being divided by parallel Chord-Lines into 100 equal Parts.

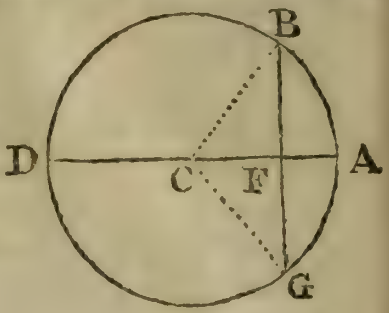
| *V.S.* | *Segment.* | *V.S.* | *Segment.* | *V.S.* | *Segment.* | *V.S.* | *Segment.* |
|---|---|---|---|---|---|---|---|
| 1 | 0,0017 | 26 | 0,2066 | 51 | 0,5127 | 76 | 0,8155 |
| 2 | 0,0048 | 27 | 0,2178 | 52 | 0,5255 | 77 | 0,8262 |
| 3 | 0,0087 | 28 | 0,2292 | 53 | 0,5382 | 78 | 0,8369 |
| 4 | 0,0134 | 29 | 0,2407 | 54 | 0,5509 | 79 | 0,8474 |
| 5 | 0,0187 | 30 | 0,2523 | 55 | 0,5635 | 80 | 0,8576 |
| 6 | 0,0245 | 31 | 0,2640 | 56 | 0,5762 | 81 | 0,8677 |
| 7 | 0,0308 | 32 | 0,2759 | 57 | 0,5888 | 82 | 0,8776 |
| 8 | 0,0375 | 33 | 0,2878 | 58 | 0,6014 | 83 | 0,8873 |
| 9 | 0,0446 | 34 | 0,2998 | 59 | 0,6140 | 84 | 0,8968 |
| 10 | 0,0520 | 35 | 0,3119 | 60 | 0,6265 | 85 | 0,9059 |
| 11 | 0,0598 | 36 | 0,3241 | 61 | 0,6389 | 86 | 0,9149 |
| 12 | 0,0680 | 37 | 0,3364 | 62 | 0,6514 | 87 | 0,9236 |
| 13 | 0,0764 | 38 | 0,3486 | 63 | 0,6636 | 88 | 0,9320 |
| 14 | 0,0851 | 39 | 0,3611 | 64 | 0,6759 | 89 | 0,9402 |
| 15 | 0,0941 | 40 | 0,3735 | 65 | 0,6881 | 90 | 0,9480 |
| 16 | 0,1032 | 41 | 0,3860 | 66 | 0,7002 | 91 | 0,9554 |
| 17 | 0,1127 | 42 | 0,3986 | 67 | 0,7122 | 92 | 0,9625 |
| 18 | 0,1224 | 43 | 0,4112 | 68 | 0,7241 | 93 | 0,9692 |
| 19 | 0,1323 | 44 | 0,4238 | 69 | 0,7360 | 94 | 0,9755 |
| 20 | 0,1424 | 45 | 0,4365 | 70 | 0,7477 | 95 | 0,9813 |
| 21 | 0,1526 | 46 | 0,4491 | 71 | 0,7593 | 96 | 0,9866 |
| 22 | 0,1631 | 47 | 0,4618 | 72 | 0,7708 | 67 | 0,9913 |
| 23 | 0,1738 | 48 | 0,4745 | 73 | 0,7822 | 98 | 0,9952 |
| 24 | 0,1845 | 49 | 0,4873 | 74 | 0,7934 | 99 | 0,9983 |
| 25 | 0,1955 | 50 | 0,5000 | 75 | 0,8045 | 100 | 1,0000 |

The Use of this Table of Segments depends upon the following Proportion,

*viz.* { As the Diameter of any propos'd Circle : is to 100 (the Diameter of the tabular Circle) : : so is the Height of any Segment of the propos'd Circle : to a versed Sine in the Table.

Then, if the tabular Segment, which stands against that versed Sine, be multiply'd into the Circle's Area (either in Inches or Gallons) the Product will be the Area of the Segment requir'd [of the same Name] *viz.* If the Circle's Area be Inches, the Segment will be Inches; if Gallons, the Segment will be Gallons.

*Example.* Let the Diameter of the given Circle be $DA = 62{,}5$ Inches, and the Height of the Segment sought be $FA = 20$ Inches; What will its Area be in Ale Gallons?




First, the Area of the whole Circle will be 10,8793 Ale Gallons (by Problem 5.) and the Proportion will stand thus, 62,5 : 100 : : 20 : 32 the versed Sine of the Table whose Segment is 0,2759. Then, $10{,}8720 \times 0{,}2759 = 3{,}0016$ Ale Gallons, being the Area of the Segment *B A G F*, as was requir'd. The like may be done for Wine Gallons, Corn Gallons, or Inches.

And, upon Occasion, the like Segments of any Ellipsis may be easily found. See the *Proportions* in the Corollaries to the 7th and 8th Theorems, Page 412, *&c.* to which I here, for Brevity's Sake, refer the Reader.

Sect. 3. To compute the Contents of such Vessels (*viz.* Tuns, *&c.*) as are in the Form of the following Solids.

*Note*, Before the young Gauger proceeds to these Computations, he should be well acquainted with such Solids as are defin'd in P. 402 and 403, and then he may easily understand what Sort of Figures are meant in the following Problems, without the Repetition of many Words.

## PROBLEM VIII.

*To find the Content of any* Prism *whose Sides are* Parallelograms *what Form soever its Base is of.*

That is, to compute the Content (in Gallons) of any Tun, &c. whose Sides are Parallelograms which stand upright, or at Right Angles with its Bottom.

First, find its solid Content in Inches, by Theorem 9, Page 414; then divide that Content by 282, or 231, or by 268,8; the Quotient will shew the Content in their respective Gallons, *viz.* in Ale, Wine, or Corn Gallons.

Or else multiply the Content in Inches with 0,003546, or 0,004329, *&c.* [See the Multiplicators, Page 435] those Products will be the Content in their respective Gallons.

Or otherwise thus:

Find the true Area of the Tun's Base or Bottom, as directed in Sect. 1, P. 435; that Area being multiply'd with the Tun's Height (*viz.* Depth within) will produce the Content in Gallons, as before.

I take

I take the Work of this Problem to be ſo very eaſy, that it needs no Example.

## PROBLEM IX.

*To find the Content of any Pyramid* (in Gallons) *whoſe Baſe is bounded with Right Lines.*

Every Pyramid is one Third-part of its circumſcribing Priſm, by Theorem 10, Page 415. Therefore, if the Area of the Baſe of any Pyramid, in Gallons, be multiply'd in one Third of its perpendicular Height; or if one Third of that Area be multiply'd with the whole Height, either of thoſe Products will be the Content of the Pyramid in Gallons, &c. But the Content of any ſquare Pyramid may be eaſily found in Gallons by this Rule:

RULE. Square the Side of its Baſe, and multiply that Square with the perpendicular Height; then divide that Product by $846 = 282 \times 3$ for Ale Gallons, or by $693 = 231 \times 3$ for Wine Gallons, or by $806,4 = 268,8 \times 3$ for Corn Gallons, the Quotient will be the Content requir'd.

Or, if you multiply the ſaid Product with 0,001182 for *A G.* or with 0,001443 for *W G.* or, laſtly, with 0,001241 for *C. G.* the Reſult will be the Content requir'd, as before.

## PROBLEM X.

*To find the Content* (in Gallons) *of the* Fruſtum *of any ſquare Pyramid, cut off by a Plain parallel to its Baſe.*

Firſt, either by Theorem 15, Page 419, or Theorem 16, P. 420, find the propos'd Fruſtum's Solidity in Cubick Inches; then divide that Content in Cubick Inches by 282 or 231, &c. and the Quotient will be the Content of the Fruſtum in their reſpective Gallons.

But, from the foreſaid Theorem 15, there may be eaſily deduced the following general Rule for finding the Content of the like Fruſtum of any Pyramid, what Form ſoever its Baſes are of (ſuppoſing them to be parallel) whether they are alike or unlike.

RULE. Firſt, find the Area of each Baſe, (*viz.* the top and bottom Area's of the propos'd Fruſtum;) then find a Geometrical Mean between thoſe two Area's (by Lemma 1, Page 83;) the Sum of thoſe two Area's and their Mean, being multiply'd into one Third of the Fruſtum's Height, will produce the Content required.

*Example.* Suppose a Tun in the Form of the lower Fruſtum of a Pyramid, whoſe Baſes are Equilateral Triangles: Let the Side of the Top be 42 Inches, the Side of the Bottom be 63,4 Inches, and its Height [*viz.* Depth] be 33 Inches; What will the Content of that Tun be in Ale Gallons?

Firſt, find the Area of that Baſe in Inches, by Probl. 7, P. 343; then find what thoſe Area's are in Ale Gallons, by Probl. 3, P. 436. Multiply thoſe two Area's together and the ſquare Root of their Product will be the mean Area, *&c.* as in this Example:

*Example.* The { Top / Bottom / Mean } Area is { 2,71 / 6,12 / 4,07 } Ale Gallons.

Their Sum 12,90

Then $12,9 \times \frac{33}{3} = 141,9$. Or $\frac{12,9}{3} \times 33 = 141,9$ the Content required.

## PROBLEM XI.

*To find the Content of any right Cylinder in Gallons.*

That is, to compute the Content of any round Tun, *&c.* whoſe Diameters at Top and Bottom are equal, and at Right Angles with its Sides.

The Content of ſuch a Tun may be found by Theorem 11, Page 415; or otherwiſe by the following Rule.

RULE. { Multiply the Square of the Diameter into the Height, and divide the Product by 359,05 (or multiply with 0,002785) *&c.* as in Page 439, that Quotient (or Product) will be the Content required.

*Exam.* Suppoſe the Diameter be 42,5, and the Height 31,5 Inches. Firſt $42,5 \times 42,5 = 1806,25$ And $1806,25 \times 31,5 = 56896,875$. Then 359,05) 56896,875 (158,46 the Content in Ale Gal. *&c.*

## PROBLEM XII.

*To find the Content of any* Cone *or* round Pyramid *in Gallons.*

Becauſe every Cone is one Third of its circumſcribing Cylinder, [See Theorem 13, Page 416] therefore its Content may be truly found by the following Rule.

RULE. { Multiply the Square of the Diameter of its Baſe into the perpendicular Height, then divide their Product by $1077,15 = 359,05 \times 3$ for Ale Gallons, or by $882,36 = 294,12 \times 3$ for Wine Gallons, *&c.* and the Quotient will be the Content required.

Or

Or if the said Product be multiply'd with 0,000928 $= \frac{0,002785,}{3}$ or with 0,001133 $= \frac{0,0034,}{3}$ those Products will be the Content in their respective Gallons.

*Example.* Suppose the Diameter of the Base be 42,5, and the perpendicular Height be 31,5 Inches, what will the Content be in Ale Gallons? (as before.

First 42,5×42,5 = 1806,25. And 1806,25 ×31,5 = 56896,875
Then 1077,15) 56896,875 (52,82. Or 56896,25×0,000928 = 52,82 the Content in Ale Gallons. And so on for Wine or Corn Gallons.

## PROBLEM XIII.

*To find the Content of the lower* Frustum *of any Cone in Gallons.*

That is, to compute the Content of any round Tun, *&c.* whose Diameters at Top and Bottom are parallel, but unequal.

The Content of such a Tun may be found by the Rule at Problem 10; but from Theorem 16, Page 420, 'twill be easy to deduce this following Rule.

RULE. To the triple Product of the Top and bottom Diameters, add the Square of their Difference; multiply that Sum into the Height (or Depth): then divide the last Product by 1077,15 for Ale Gallons, or by 882,36 for Wine Gallons; the Quotient will be the Content requir'd.

*Example.* Suppose the Diameter at the Top to be 52,4 Inches, the Diameter at the Bottom 45,6, and the Height 30 Inches.

First, 52,4×44,6 = 2337,04; and 2337,04×3 = 7011,12 } Add
Also, 52,4 − 44,6 = 7,8; and 7,8 × 7,8 = 60,84 }
The Height 30 × 7071,96 = 212158,8.

Then 1077,15) 212158,8 (196,96 } the *Content* in *Ale* Gallons.
Or 212158,8×0,000928 = 196,96 }

And so on for either Wine or Corn Gallons, as Occasion requires. But if the Tun (or Vessel) be not truly circular, that is, if either its Top or Bottom (or both of 'em) be Elliptical, whether they are alike or unlike, it matters not, the Content of such a Tun may be truly found by the general Rule at Problem 10.

## PROBLEM XIV.

*The* Axis *or* Diameter *of any Sphere or Globe being given in* Inches *to find its Content in* Gallons.

Every Sphere is two Thirds of its circumscribing Cylinder, by Theor, 18, Page 423; from whence and Theor. 20, Page 426, 'tis proved,

proved, that if the Cube of the Axis of any Sphere (taken in Inches) be multiply'd into 0,5236, the Product will be the Content of that Sphere in Inches. Consequently, if that Content be divided by 282, or by 231, *&c.* the Quotient will be the Content in Gallons.

But those two Works of multiplying with 0,5236, and then dividing by 282, or by 231, *&c.* may be contracted into one.

Thus 282 } 0,5236 { 0,001856 } will be a *Multiplicator* for { *A. G.*
And 231 } { 0,002266 } { *W. G.*

Or 0,5236 { 282 { 538,57 } will be a *Divisor* for { *Ale* Gallons.
{ 231 { 441,17 } { *Wine* Gallons.

From hence arises this following Rule.

RULE. { If the Cube of the Axis of any Sphere be divided by 538,57; or multiply'd with 0,001856: or divided by 441,17; or else multiply'd with 0,002266; the Quotient, or Product, will be the Sphere's Content in their respective Gallons.

*Example.* Suppose the Axis or Diameter of a Sphere or Globe be 2 Inches, how many Ale Gallons may it hold?

Here 22×22×22 = 10648; and 538,57) 10648 (19,76 *A. G.* Or 10648 × 0,001856 = 19,76 Ale Gal. the Content required. And so for either Wine or Corn Gallons, as Occasion requires.

## PROBLEM XV.

*To find the Content of a* Segment *of a* Sphere *in Gallons.*

In the Scholium, P. 424, there are two Theorems for resolving this Problem according to the Data.

1. If the Diameter of the Segment's Base and its Height are given, the Content may be found by the first of those Theorems, which gives this Rule:

RULE 1. { To the Triple Square of half the Diameter add the Square of the Height; then multiply that Sum into the Height, and divide the Product by 538,57 for *A G*, or by 441,17 for *W G*, &c. as above.

2. But if the Axis of the Sphere and the Height of the Segment are given, the Content may be found by the Second of those Theorems.

RULE 2. { From the triple Product of the Axis into the Height, subtract twice the Square of the Height; then multiply the Remainder into the Height, and divide that Product by 538,57, *&c.* as in the last Problem.

Either of these Rules will produce the Content of the Segment in Gallons.

*Example.* Suppose the Diameter of the Segment's Base be 28 Inches, and its Height be 8 Inches, what may it contain in Ale Gallons?

First 2) 28 (14. Then (by Rule 1.) $14 \times 14 \times 3 = 588$. And $6 \times 6 = 36$. Next $588 + 36 = 624$. Again $624 \times 6 = 3744$. Lastly, 538,57) 3744 (6,95 the Content required.

*Note*, This Problem may be of Use in Gauging the Crowns of Brewers Coppers, *&c.*

Sect. 4. *The Practical Method of Gauging any fix'd Tun or Copper, and making a Table to shew what it will hold at every Inch deep, usually call'd Inching of a Tun,* &c.

First, you must know, that most (if not all) Brewers Tuns are so fix'd as to lean a little for Conveniency of cleansing their Drink, which is usually call'd the Drip or Fall of the Tun. Now this Drip or Fall of any Tun is the Hoof of such a Solid as that Tun is suppos'd to represent, and under that Consideration it may be found, as in Theor. 16, P. 420: But the practical (and indeed the best) Way is, to measure into the Tun (when 'tis dry) so much Liquor as will just cover its Bottom; for by that means you do not only find the true Fall, but also a true horizontal or level Plain over the Bottom of the Tun; from which if the Depth of the Tun (*viz.* the nearest Distance from the Top of the Tun to the Surface of the Liquor) be set off upon every one of its Sides, you will then have a true parallel Plain at the Top of the Tun to that of the Liquor. Then, if the Sides of the Tun are streight from the Top to the Bottom, take as many Dimensions in the aforesaid two Plains as are needful to find the true Area of each; and by those two Area's and the foresaid Depth find so much of the Tun's Content (by the general Rule at Problem X.) as is betwixt those two Plains.

Next, to inch that Tun, divide the Difference between the Top and Bottom Area's by the aforesaid Depth, and the Quotient will be an Addend or fixed Number; which being added to the lesser Area, the Sum will be the Area of the next Inch; and, being added to that Area, their Sum will be the Area of the third Inch; and so on from Inch to Inch, until the Area of every single Inch be found; the Sum of those Area's (if the Work be true) will amount (or be equal) to the Content found, as above. And if the

the Tun's Drip or Fall be added to the Sum of all thofe Area's, that Sum will be the whole or full Content of that Tun.

Now, from hence it muft needs be eafy to conceive, that if 1, 2, 3, or any Number of thofe Area's accounted from the Bottom, be added to the Fall, that Sum will fhew the Quantity of Liquor or Drink that is in the Tun, to fuch a Number of wet Inches from the Bottom as there were Area's added together. Or, if the Sum of any Number of thofe Area's (being accounted from the Top) be fubtracted from the Tun's whole Content, the Remainder will fhew what Quantity of Liquor or Drink is in the Tun, when there is fuch a Number of dry Inches from the Top as there were Area's fubtracted.

This being well confider'd, it will be eafy to make a Table either to every wet or dry Inch of any regular Tun (*viz.* whofe Sides are ftreight from Top to Bottom) what Form foever its Bafes are of, and whether it ftand upon the greater or leffer Bafe.

But if the Sides of the Tun are irregular (*viz.* not ftreight from its Top to the Bottom) then the beft and eafieft Way will be to divide or part the Tun into feveral Fruftums, each of ten Inches deep; and finding the Content of every fingle Fruftum, by taking the Diameters in the Middle of every one of thofe ten Inches (that is, the firft Diameter at 5 Inches from the Top; the fecond Diameter at 15 Inches from the Top, *&c.*) and multiplying their refpective Area's with 10, (which is done by only removing the feparating Comma's one Place forward to the right Hand) if the Sum of all thofe Fruftums be added to the Fall, (as before); that Sum will be the whole Content of the Tun.

*Note*, If you take the Height of the 'forefaid ten Inch Fruftums in the Side of the Tun, you muft allow for the Difference between the flant Height and the Perpendicular Height in every Fruftum.

Laftly, If from the whole Content of the Tun you fubtract the mean Area of the firft Fruftum ten Times, and from the Remainder fubtract the mean Area of the fecond Fruftum ten Times, and from the laft Remainder fubtract the mean Area of the third Fruftum, *&c.* until there remain nothing but the Fall or Hoof of the Tun, you will then by that Means have a Table that will fhew what Quantity of Drink is in the Tun to any Number of dry Inches.

And this is alfo the Method of Gauging and Inching Brewers Coppers, *viz.* by firft meafuring into the Copper fo much Liquor as will juft cover its Crown, and then dividing its perpendicular Height into Fruftums, and its Sides into four equal Parts; that fo crofs Diameters may be taken in the Middle of each Fruftum: But

but if the Copper be much wider at the Top than at the Bottom, and its Sides ſpheroidal or arching, as generally all large Coppers are; then, inſtead of taking thoſe mean Diameters in the Middle of every ten Inches, as above, you muſt take them in the Middle of every ſix Inches, and proceed on as before.

Now the Quantity of Liquor, that would cover the Crown of the Copper, may be found without meaſuring it, as above. In order to that, I do ſuppoſe the Crown to be the Segment of a Sphere, and the lower Part of the Copper wherein the Crown ariſeth, to be the Fruſtum of a parabolick Conoid; then, if the Diameter at the Top of the Crown, and its perpendicular Height are given, the Quantity of Liquor may be found by this following Rule:

RULE. From the Area of the Plain at the Top of the Crown ſubtract $\frac{11}{3}$ of the Area of the Crown's Height; the Remainder, being multiply'd into half the Height of the Crown, will produce the Quantity or Number of Gallons that will cover the Crown.

This *Rule* is deduced from *Scholium*, *Page* 424, and *Theorem* 15. *Page* 430.

*Sect.* 5. To compute the Content of any cloſe Caſk in Gallons, *viz.* of any Butt, Pipe, Hogſhead, Barrel, *&c.*

In order to perform this difficult Part of Gauging, the three following Dimenſions of the propoſed Caſk muſt be truly taken in Inches, and Decimal Parts of an Inch.

*Viz.*
- The Bulge or Bung Diameter within the Caſk.
- Either of the Head Diameters ſuppoſing them both equal.
- And the Length of the Caſk within.

*Note*, In taking of theſe Dimenſions, it muſt be carefully obſerved,

1. That the Bung-hole be in the Middle of the Caſk; alſo that the Bung-ſtaff and the Staff over-againſt the Bung-hole are both regular or even within.

2. That the Heads of the Caſk are equal and truly circular; if ſo, the Diſtance between the Inſide of the Chine to the Outſide of its oppoſite Staff will be the Head Diameter within the Caſk, very near.

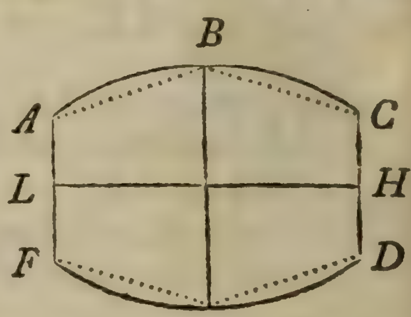
3. With a ſliding pair of Calipers (made on purpoſe for that Uſe) take the ſhorteſt Diſtance at Length between the Outſides of the two Heads; (ſuppoſing them even) from that Length ſubtract 1 $\frac{1}{2}$ Inch (more, or leſs, according to the Largeneſs of the

 Caſk)

Cafk) for the Thicknefs of the two Heads, the Remainder will be the Length of the Cafk within.

Now by thefe Dimenfions, one would fuppofe the Content of the Cafk were perfectly limited ; but it will be eafy to perceive, by the following Figure, that the Diameters (abovefaid) and the Length of one Cafk may be equal to thofe of another, and yet one of thofe Cafks may contain or hold feveral Gallons more than the other.

As for Inftance, fuppofe the annex'd Figure *A B C D G F*, to reprefent a Cafk; then it is plain, that, if the outward and curved Lines *A B C* and *F G D* are the Bounds or Staves of the Cafk, it muft needs hold more than if the inner ftreight or prick'd Lines were its Bounds or Staves; and yet the Bung Diameter *B G*, Head Diameter *C D* and *A F*, and the Length *L H*, are the fame in both thofe Cafks.




Whence it plainly appears, that no one certain or general Rule can be prefcrib'd to find the true Content of all Sorts of Cafks, and therefore Gaugers do ufually fuppofe every Cafk to be in Form of fome one of thefe following Solids.

*Viz.*
- I. The middle Zone or Fruftum of a Spheroid.
- II. The middle Zone or Fruftum of a Parabolick Spindle.
- III. The lower Fruftums of two equal Parabolick Conoids.
- IV. The lower Fruftums of two equal Cones.

Now the Way of Gueffing at the Cafk's Form, and computing its Content, according to its fuppos'd Form, I fhall here fhew in their Order.

I. If the Staves of the Cafk are very much curved or arching (as the outward Lines of the laft Figure) then the Cafk is fuppos'd to be in the Form of the middle Zone or Fruftum of a Spheroid, whofe Content may be computed, by *Theorem* 22. *Page* 427, which gives thefe two *Rules*.

RULE I. To twice the Square of the Bung Diameter add the Square of the Head Diameter; multiply that Sum into the Length, and divide the Product by 1077,15. *Viz.* 3, 8197×282 for Ale Gallons; and by 882,36. *Viz.* 3, 8197×231 for Wine Gallons. Or thus,

RULE

Rule 2. { To twice the Area of the Bung Circle, add the Area of the Head Circle; multiply their Sum into one Third of the Length, and the Product will be the Content in their reſpective Gallons.

*Example* 1. Suppoſe a Caſk in the Form of the middle Zone of a Spheroid, whoſe Bung Diameter is 31,5, Head Diameter 24,5, and its Length 42 Inches.

Firſt 31,5×31,5×2=1984,5. And 24,5×24,5=600,25
Again 1984,5+600,25=2584,75. And 2584,75×42=108559,5
Then 1077,15)108559,5(100,78 the Content in Ale Gallons.
And 882,35) 108559,5 (123,03 the Content in Wine Gallons.

Or thus, by the Second Rule.

Bung Diameter 31,5 twice its Circle's Area is 5,5270
Head Diameter 24,5 its Circle's Area is 1,6718
The Length 42 divided by 3 is 14. 7,1988= their Sum.
Then 7,1988×14=100,78, the Content in A. Gallons as before.
And ſo the Content in Wine Gallons may be found.

II. If the Staves of the Caſk are not quite ſo much curved or arching, as was ſuppos'd before, the Caſk is then taken for the middle Fruſtum of a parabolick Spindle, and its Content is computed, as by *Theorem* 27. *Page* 432. Which gives this *Rule*.

Rule. { To twice the Square of the Bung Diameter add the Square of the Head Diameter; from their Difference ſubtract four Tenths of the Square of the Difference of the Diameters; multiply the Remainder into the Length, and divide the Product by 1077,15, &c. as above.

*Example* 2. Suppoſe the Dimenſions the ſame as before. Then 31,5 × 31,5 × 2 : + 24,5 × 24,5 = 2584,75. And 31,5 — 24,5 = 7. Again 7×7×0,4 = 19,6. And 2584,75 — 19,6 × 42 = 107736,3. Then 1077,15) 107736,3 (100,01 the Cont. in *A. G.* &c. for *W. G.*

III. When the Staves of the Caſk are but very little curved or arching, then it's ſuppos'd to be in the Form of the Fruſtums of two equal parabolick Conoids, abutting or joining together upon one common Baſe at the Bulge, and the Content may be found by *Theorem* 25. *Page* 430. which gives theſe *Rules*.

**Rule 1.** To the Square of the Bung Diameter add the Square of the Head Diameter; multiply their Sum into the Length, and divide the Product by 718,08 (*viz.* 2,5464×282) for Ale Gallons: or by 588,22 (*viz.* 2,5464×231) for Wine Gallons. Or thus,

**Rule 2.** To the Area of the Bung Circle add the Area of the Head Circle; multiply the Sum into half the Length, and the Product will be the Content required.

*Example.* 3. With the same Dimensions as before. Then

315×31,5 + 24,5×24,5 = 1592,5. And 1592,5×42 = 66885
And 718,08) 66885 (93,01 the Content in Ale Gallons.
Or 588,22) 66885 (113,7 the Content in Wine Gallons.

IV. If the Staves of the Cask are streight from the Bulge to the Head, as the inner prick'd Lines in the last Figure (if such a Cask can be made) it is then taken for the lower Frustums of two equal Cones, abutting or joining together upon one common Base at the Bulge. And its Content may be computed as at Problem 13. *Page* 445. or by *Theorem* 15. *Page* 419. Thus,

**Rule.** To the Sum of the Squares of the Head and Bung Diameters add their Product; then multiply that Sum into the Length, and divide the last Product by 1077,15. Or by 882,36. The Quotient will be the Content, &c.

*Example* 4. With the same Dimensions as before.

First 31,5×31,5 + 24,5×24,5 + 31,5×24,5 = 2364,25
And 2364,25×42 = 99298,5. Then 1077,15) 99298,5 (92,18 the Content in Ale Gallons, and so on for Wine Gallons.

Thus you have the Methods of computing the true Contents of the four Solids, in whose Form all Casks are suppos'd to be. And by the Examples it appears, that four such Casks as have their Dimensions all equal, and the same with those above-mention'd, their Contents will be as in the Margin.

| | Ale Gallons. | Differ. |
|---|---|---|
| I. | 100,78 | |
| II. | 100,01 | 0,77 |
| III. | 93,01 | 7,00 |
| IV. | 92,18 | 0,83 |

From the Disproportion or Inequality of these Differences it will be easy to conceive, that there may be several Casks whose Contents cannot be truly found, according to the aforesaid suppos'd Forms; and therefore, in order to rectify the said Inequalities, some Authors (that have written upon this Subject) have laid down *Theorems* of their own Invention; and yet call'd them by by these Names) others have propos'd Tables for the same Purpose. But since it is so, that we can only guess at the Truth, the plainest and easiest Way is to be preferr'd in Practice; and that is, by finding such a mean Diameter as will reduce the propos'd Cask to a Cylinder.

Thus, { Multiply the Difference between the Head and Bung Diameters, with 0,7. or with 0,65. or with 0,6. or with 0,55. according as the Staves of the Cask are more or less arching; add the Product to the Head Diameter, and the Sum will be the mean Diameter required. Then find the Content, as at *Prob.* 11. *Page* 444. }

*Example.* With the same Dimensions as before. Then the Bung Diameter less the Head Diam. is 31,5 — 24,5 = 7. And

| | | *MD.* | | *AG.* | *Cont.* | *Dif.* |
|---|---|---|---|---|---|---|
| 24,5+ | 7×0,7 | = 29,40 | its *Area* | 2,4073×42 = | 101,10 | |
| | 7×0,65 | = 29,05 | ——— | 2,3504×42 = | 98,71 | 2,39 |
| | 7×0,6 | = 28,70 | ——— | 2,2941×42 = | 96,35 | 2,36 |
| | 7×0,55 | = 28,35 | ——— | 2,2385×42 = | 94,02 | 2,32 |

From these it may be observ'd, that the Difference between each Cask's Content is regular, and very near equal; which plainly shews, that there is not so much Room left for Error this Way of computing their Contents, as was by the aforesaid Forms.

Now the first of these four (*viz.* with 0,7) is very commonly used among *Gaugers* for all Sorts of Casks; but I did never gauge any Cask that would contain quite so much as that Rule did make it; and the Reason doth appear very plain from *Theorem* 22. *Page* 427. being compar'd with *Theorem* 19. *Page* 426. and the last Figure; *viz.* that no (Cask being regularly made) can hold more than the middle Fustrum of a Spheroid. But I always found by Experience, that if the second and third of these Rules (*viz.* with 0,65 and 0,6) were duly apply'd, they would answer very near the Truth amongst the common Sort of Casks; and the fourth Rule (*viz.* with 0,55) will come pretty near the Truth in computing the Contents of Casks, whose Staves are almost streight betwixt the Head and Bung, *viz.* such as Wine Pipes, *&c.*

*Sect.* 6. To find what Quantity of Liquor is either drawn forth, or remaining in any spheroidical Cask, usually call'd the Ullage of a Cask; hath two Cases.

*Case* 1. To find what Quantity of Liquor is in the Cask, when its Axis is perpendicular to the Horizon, *viz.* when it stands upright upon one of its Heads.

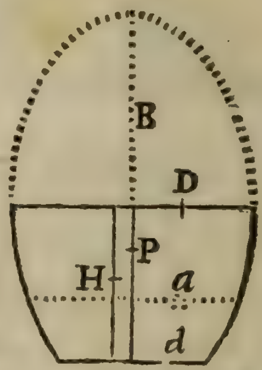
In

In order to perform this the easiest Way, it will be convenient to know how to calculate the Area of any Circle betwixt the Bung and Head, whose Distance from the Bung or middle of the Cask is given. Now that may be done by this Proportion.

*Viz.* { As the Square of half the Length of the Cask: is to the Difference between the Bung and Head Area's : : so is the Square of any Circle's Distance from the Bung: to the Difference between the Bung Area, and the Area of the Circle, *viz.* the Area of the Liquor's Surface.

Demonstration.

Let { $H$ = Half the Length of the Cask
$D$ = Half the Bung Diameter.
$d$ = Half the Head Diameter.




And { $P$ = the Distance of any Circle from the Bung
$a$ = Half the Diameter of that Circle.

Then according to the common Property of the Ellipsis, *Page* 368, it will be,

$BB:DD::BB-HH:dd$. And $BB:DD::BB-PP:aa$.

Ergo $\left\{ \frac{DDHH}{DD-dd} = BB. \right.$ And $\left\{ \frac{DDPP}{DD-aa} = BB. \right.$

Consequently, $\left\{ \frac{DDHH}{DD-dd} = \frac{DD-aa}{DDPP}. \right.$

This Æquation, being brought out of the Fractions, will become $DDHH - aaHH = DDPPddPP$, which gives this *Analogy* $HH:DD-dd::PP:DD-aa$. Then $DD-aa$, being subtracted from $DD$, will leave $aa$. But Circles Area's are in Proportion to the Squares of their Diameters, by *Theorem* 6, *Page* 407. Therefore, &c. Q. E. D. Then, from the Bung Area subtract one third Part of the aforesaid Difference, *viz.* between the Bung Area and the Area of the Liquor's Surface; multiply the Remainder with the Liquor's Distance from the Bung, and the Product will shew what Quantity of Liquor is either above or under half the Content of the Cask.

*Example.* Let us suppose a Cask of the same Dimensions with that in the first *Example*, *Page* 451. and let it be required to find what Quantity of Liquor is in it (of Ale Measure) when there is but 9 Inches wet. Here half the Length of the Cask is 21

 Inches,

Inches, whose Square is 441, and the Liquor's Distance from the Bung is 21 — 9 = 12. Its Square is 144. The Difference between the Bung and Head Area's is 1,0917 (= 2,7635 — 1,6718.) Then 441 : 1,0917 :: 144 : 0,3564.
And 2,7635 — 0,3564 = 2,4071 the Area of the Liquor's Surface.

Again 3) 0,3564 (0,1188. And 2,7635 — 0,1188 = 2,6447 Then 2,6447 × 12 = 31,7364, what the Cask wants of being half full. Consequently 50,39 — 31,73 = 18,66 will be the Quantity of Liquor in the Cask at 9 Inches wet in Ale Gallons.

And if the Cask had wanted but 9 Inches of being full; then 50,39 + 31,73 = 82,12 would have been the Quantity of Liquor in the Cask.

*Note*, because the two first Terms (*viz.* 441 and 1,0917) in the Proportion are fix'd, *viz.* continue the same for any Distance, 'twill be very easy to calculate the Area's of all the Circles betwixt the Bung and Head to every Inch, and by that Means to make a Table that will shew what Quantity of Liquor is either drawn out or remaining in the Cask at any Depth.

*Case* 2. To find what Quantity of Liquor is in any Cask, when its Axis is parallel to the Horizon, *viz.* when it lies along.

There are Variety of Tables to be found in Books of *Gauging* for this Purpose; but I always observed, that the following Method of computing the Ullage, by a Table of the Segments of a Circle, came very near the Truth in all Sorts of Casks, which is thus perform'd:

1. By the Bung and Head Diameters, find such a mean Diameter as you judge will reduce the propos'd Cask to a Cylinder, by the Method laid down in *Page* 453. And then find its full Content, as in those *Examples.*

2. From the Bung Diameter subtract the mean Diameter and half the Difference, (*viz.* divide it by 2.)

3. From the wet Inches of the propos'd Ullage, subtract the said half Difference, and call it *x*; then observe this Proportion.

*Viz.* { As the mean Diameter : is to 100 (the Diameter of the tabular Circle) :: so is the last Difference (*viz.* *x*) : to a versed Sine in the Table. (*Page* 441.)

Then if the tabular Segment, which stands against that versed Sine, be multiply'd into the Content of the Cask, the Product will shew the Ullage, *viz.* what Quantity of Liquor is either in the Cask, or drawn forth.

*Example*

*Example* 1. Let the Cask be that of the second Sort, in *Page* 453. *viz.* whose Bung Diameter is 31,5 Inches, mean Diameter 29,05, and the Content 98,71 Ale Gallons; and suppose there were 10,5 Inches wet in it, it is required to find the wet and dry Gallons?

Here 31,5—29,05=2,45; its half is 1,12. And 10,5—1,22=9,28
Then 29,05 : 100 : : 9,28 : 0,319= V. Sine; its Segm. is 0,2748
And 98,71×0,2748=27,12 the Number of wet Gallons.

Again 31,5—10,5=21 the dry Inches; and 21—1,22=19,78
Then 29,05 : 100 : : 19,78 : 0,68; its Segment is 0,7241
And 98,71×0,7241=71,48 the Number of dry Gallons.
Proof 71,48+27,12=98,6 the Contents of the Cask very near; which plainly shews the Truth of this Method.

Thus far may suffice concerning Gauging of Backs or Coolers, Tuns, Coppers, and Casks, &c. To which I shall only add, that as the Contents of all Brewers Utensils are to be computed by the Ale Gallons, so the Contents of all Distillers Utensils (*viz.* all their Wash-Backs, Stills, and Casks, &c.) must be computed by the Wine Gallon.

And in gauging of Malt (upon which there is now a Duty of four Shillings per Bushel) you must observe, That a Corn or Malt Bushel doth contain 2150,42 cubick Inches; (See *Page* 42.) and therefore in gauging of Malt-Cisterns, or other Vessels, 2150,42 will be a constant or fixed Divisor for finding the Area's of right-lin'd Figures in Bushels at one Inch deep, and 2738 will be a constant or fix'd Divisor for finding the Area's of circular Figures.

I have omitted the Business of gauging Mash-Tuns, and taking an Account of the Goods or Grains, in order to estimate what Quantity of Worts were produc'd from them, &c. because I could never find (by all my Observations) any Certainty therein; nor is it possible there should be any, by Reason of the great Difference that is in Malt (and its Grinding too) for the best Malt (well ground) will yield or produce the most Worts, and least Grains; on the contrary, bad Malt (being ill ground) yields the least Worts and most Grains.

A SUP-

# A SUPPLEMENT

Not in any of the former EDITIONS of this

# BOOK.

Containing the

# HISTORY

OF

# LOGARITHMS,

WITH

Several easy METHODS of Constructing the *Tables* of the LOGARITHMS and SINES, *&c.* Also the Demonstration of the AXIOMS and Doctrine of *Plane*

# TRIGONOMETRY.

Extracted from the

*Philosophical* TRANSACTIONS and the WORKS of Dr. *KEIL*, *RONAYNE*, *WARD*, &c.

---

*Cuncta* Trigonus *habet*, *satagit quæ docta* Mathesis,
Ille *aperit clausum quicquid* Olympus *habet.*

---

THE

# PREFACE.

*THE Mathematicks formerly received considerable Advantages; first, by the Introduction of the* Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematicks. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for other Phænomena of Nature; and lastly, the Usurer computes the Interest of his Money.*

*The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom, taking in the whole Doctrine, have indeed wrote learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouch'd. My Design therefore, in the following Tract, is to supply what seemed still wanting,* viz. *to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.*

*The*

*The wonderful Invention of Logarithms we owe to the Lord* Neper, *who was the first that constructed and published a Canon thereof, at* Edinburgh, *in the Year* 1614. *This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is universally allow'd the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

*The same Lord* Neper *afterwards invented another and more commodious Form of Logarithms, which he afterwards communicated to Mr.* Henry Briggs, *Professor of Geometry at* Oxford, *who was hereby introduced as a Sharer in the completing thereof: But, the Lord* Neper *dying, the whole Business remaining was devolved upon Mr.* Briggs, *who, with prodigious Application, and an uncommon Dexterity, compass'd a LogarithmicCanon, agreeable to that new Form for the first twenty Chiliads of Numbers (or form* 1 *to* 20000) *and for eleven other Chiliads,* viz. *from* 90000 *to* 101000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publish'd at* London *in the Year* 1624.

Adrian Vlacq *published again this Canon at* Goudæ *in* Holland *in the Year* 1628, *with the intermediate Chiliads before omitted, filled up according to* Briggs's *Prescriptions; but these Tables are not so useful as* Briggs's, *because the Logarithms are continued but to* 10 *Places of Figures.*

*Mr.* Briggs *also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to* 15 *Places of Figures, and has subjoined to them the natural Sines, Tangents, and Secants,* *to* 15 *Places of Figures. The Logarithms of the Sines and Tangents are called Artificial Sines and Tangents, but the Sines and Tangents themſelves are called natural. Theſe Tables, together with their Conſtruction and Uſe, were publiſh'd after* Briggs's *Death, at* London, *in the Year* 1633, *by* Henry Gellibrand, *and by him called* Trigonometria Britannica.

*Since then, there have been publiſhed, in ſeveral Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, conſiſt of but ſeven Places of Figures, and wherein are only the Logarithms of the Numbers from* 1 *to* 100000, *which may be ſufficient for moſt Uſes.*

*The beſt Diſpoſition of theſe Tables, in my Opinion, is that, firſt thought of by* Nathaniel Roe, *of* Suffolk; *and, with ſome Alterations for the better, followed by* Sherwin *in his Mathematical Tables publiſhed at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 101000 *conſiſting of* 7 *Places of Figures. To which are ſubjoined the Differences and proportional Parts, by Means of which may be found eaſily the Logarithms of Numbers to* 10000000, *obſerving at the ſame Time that theſe Logarithms conſiſt only of* 7 *Places of Figures. Here are alſo the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with ſome other Tables of Uſe in practical Mathematicks.*

THE

# THE CONSTRUCTION OF LOGARITHMS.

THESE most excellent and useful Numbers were first invented by the famous and never to be forgotten Lord *Neper*, Baron of *Merchiston* in *Scotland*, aforesaid) *Ann.* 1614.) who ingeniously contriv'd to perform Multiplication and Division of Natural Numbers, by only adding or subtracting certain Artificial Numbers, which he called *Logarithms*, and the Extraction of Roots by dividing the Log. by 2 for the Square : by 3 for the Cube : by 4 for the Biquadrate, *&c.*

This Invention of his (no doubt) proceeded from a mature Consideration of the Coherence that is betwixt Numbers in Geometrical Proportion and those in Arithmetical Progression.

As in these following:

*Viz.* { 1 . 2 . 4 . 8 . 16 . 32 . 64 . 128, *&c.* Geometrical.
0 . 1 . 2 . 3 . 4 . 5 . 6 . 7, *&c.* Arithmetical.

It is very perceptible, that, as the Numbers in the Geometrical Proportionals are produced by *Multiplication* or *Division*, those in the Arithmetical Progression are produced by *Addition* or *Subtraction*: As doth appear in this Example:

*Viz.* { $4 \times 32 = 128$ / $2 + 5 = 7$ } or { $128 \div 32 = 4$ Geometr. / $7 - 5 = 2$ Arithmet.

Again, { 1 . 10 . 100 . 1000 . 10000 . 100000, *&c.* Geometr.
0 . 1 . 2 . 3 . 4 . 5, *&c.* Arithmet.

The same Coherence is betwixt these latter, as was between the two first Ranks.

*Viz.* { $1000 \times 10 = 10000$ / $3 + 1 = 4$ } or { $100000 \div 1000 = 100$ Geometr. / $5 - 3 = 2$ Arithmet.

Either of these Examples do sufficiently shew the Reason and very Ground of Logarithms.

And from the latter of these it was, that the prime Logarithms or Characteristics were first assigned.

As

As in this Table:

| *Natural Num.* | *Logarithms.* |
|---|---|
| 1 | 0,0000000 |
| 10 | 1,0000000 |
| 100 | 2,0000000 |
| 1000 | 3,0000000 |
| 10000 | 4,0000000 |
| 100000 | 5,0000000 |

Having laid this Foundation, the next Work was to find out the Logarithms of the intermediate Numbers situated betwixt 1 and 10, *viz.* of 2, 3, 4, 5, 6, 7, *&c.* and of those betwixt 10 and 100, *viz.* of 11, 12, 13, 14, 15, *&c.* and so on for the rest. This was a Work of some Difficulty, and very laborious.

The first Step in order thereunto (as I conceive) was to find out a Rank of continual Means betwixt 10 and 1, so as that the last (and least thereof) might be a mixed Number less than 2, and so near 1, as to have such a Number of Cyphers before the significant Figures thereof, as was intended the Places of Logarithms in the Table should consist of. Which Means are to be found, by extracting the square Root of 10 (having first annexed a competent Number of Cyphers thereunto;) then extracting the Root of that Root, and so by a continued Extraction of Root out of Root, until there be a Root so qualify'd as before-mention'd: Which, to make a Table to seven Places in the Logarithm, will require twenty-five several Extractions, the last of which will produce this Number, 1,0000000686238.

The next Step was to find out a Number betwixt (1) and (0) in Arithmetical Progression, that might truly correspond with the Mean before found (betwixt 10 and 1) such a Number must consequently be its Logarithm. And this may be found by a continual bisecting (or halving) of 1, so often as was the Number of the foregoing Extractions (to wit, twenty-five) the last of which Bisections will produce 0,0000000298023 22, *&c.* the true Logarithm of 1,0000000686238.

For as 1,0000000686238 by twenty-five continued Involutions (*viz.* first into itself, then that Product into itself, and so on successively) will produce 10; so will 0,0000000298023 2, by the like Number of Doublings and Redoublings, produce 1.

This Mean (or Number) and its Logarithm being thus found, it will follow by Proportion, *As the significant Figures of this Mean : are to the significant Figures of its Logarithm : : so are the significant*

*ficant Figures of any Mean, betwixt any given Number and* 1 : (having feven Cyphers before fuch Figures, as this hath) *to the fignificant Figures of its Logarithm.* To which muft be prefixed feven Cyphers to complete it. After which, being doubled, and redoubled according to the Number of Extractions required to produce its correfponding Mean, will at laft difcover the true Logarithm of the given Number. For the clearing of this, take an Example.

Suppofe it were required to find the Logarithm of the Number 2, to feven Places. Firft, by a continued Extraction of Root out of Root, beginning at 2, find fuch a Mean, or Root as before, betwixt 2 and 1, as will have feven Cyphers before its fignificant Figures; which, after twenty-three feveral Extractions, will be this Number 1,00000008262958. Then, according to the foregoing Proportions, it will be 6862238 : 2980232 : : 8262958 : 3588557. To which prefix feven Cyphers, as before directed, then will 1,00000008269958 have for its Logarithm, ,00000003588557; which being doubled and redoubled, as abovefaid, will produce 0,30102997958658, the true Logarithm of 2; which being contracted to feven Places, according to the firft Defign, and agreeable to the feven Places of Cyphers, then it will become 0,3010299: But, in all the Tables that I have feen, the Logarithm of 2 is 0,3010300: I conceive the Reafon is, becaufe the remaining Figures 7958658 come fo near Unity of the laft Place in the retained Figures.

And, by the fome Method that this Logarithm of 2 is made, may the Logarithm of any other Number be found. But when once the Logarithms of a few of the prime Numbers, *viz.* of 3. 7. 11. 13. &c. (that is, of fuch Numbers as cannot be produced by the multiplying of two Integer Factors) are obtained, the reft may be eafily compofed by *Addition* and *Subtraction* only. For as $3 \times 2 = 6$ fo Log. of 3 + Log. of 2 = Log. of 6. And as $10 \div 2 = 5$ fo Log. of 10 — Log. of 2 = Log of 5. The like of all Numbers that have aliquot Parts (that is, fuch Integer Numbers as may be divided by Integers.) And indeed the Logarithms of feveral of the prime Numbers may alfo be obtained by *Addition* or *Subtraction*, as might eafily be fhewed, and is not difficult to conceive by any one, who but duly confiders the Nature and Defign of Logarithms, &c. of which I fhall forbear faying any thing in this Place, and keep to my firft Defign herein, which was to give a brief Account of the ingenious Author's Method, as I conceive it, of making the fame: who undoudtedly found it a very difficult Work, by Reafon there are required fo many feveral Extractions of Roots out of Roots, which muft needs render it both troublefome and laborious. Then to propofe a different Method of raifing the Loga-

2 rithms

rithms of such prime Numbers before-mentioned, which require the Extraction of Roots to obtain their respective Means, with one tenth Part of the Trouble and Time required by the foregoing Method. And not only so, but more exact; for, by our present Method of converging Series, the Root of any Power, how high soever it be, is easily found at one single Extraction: and thereby the Errors which would arise by extracting a *Surd Root* out of a *Surd Root*, especially when often repeated, are avoided; and consequently such a Mean, as may be required betwixt any Number and Unity, is thereby more exactly found.

Now, how this may be performed, I here intend to shew, as briefly as I can. In order thereunto, take this as a Model.

Let $a =$ the Root, or Mean required betwixt any Number and Unity:

$$\text{Then}\begin{cases} a^2 = \square\, a \;.\; a^4 = \square\, a^2 \;.\; a^8 = \square\, a^4 \\ a^{16} = \square\, a^8 \;.\; a^{32} = \square\, a^{16} \;.\; a^{64} = \square\, a^{32} \\ a^{128} = \square\, a^{64} \;.\; a^{256} = \square\, a^{128} \;.\; a^{512} = \square\, a^{256} \end{cases}$$

And so on successively with the Indices in Geometrical Progression, until the Power of $a$ be made equal to such a Term in that Progression, as that the Root, or Value of $a$ may have, betwixt Unity and its significant Figures, so many Cyphers, as are the intended Number of Places in the Logarithms.

For Instance, let it be required to find the Mean between 10 and 1, then the Power of $a$ must be $a^{33554432} = 10$, this Index 33554432 being the 25th Term in Geometrical Pregression, which may be thus determined.

Let 1, the Characteristic or Logarithm of 10, be divided by such a Term in Geometrical Progression, as will cause such a Number of Cyphers to be before the significant Figures in the Quotient, as are required to be before the Figures of the Root $a$; suppose 7, as before. Then 1, ÷ 33554432 = ,000000029802322, &c. which is the true Arithmetical Mean (as before found, by a continual bisecting of 1) correspondent to that signify'd by $a$; and therefore the Value of $a$ found by extracting the respective Root of $10 = a^{33554432}$ will be the Mean required; *viz.* 1,0000000686238 whose Log. is ,000000029802322. These, being found, are the Foundation of the rest, as before.

Then suppose it be required to find the Logarithm of any of the prime Numbers; if you please, that of two. In order thereunto, let $a =$ the Root or Mean sought betwixt 2 and 1, as before; then must $a$ be continually involved, as by the above Model, until its Index be equal to the greatest Term in Geometrical Progression, whose Number of Places of Figures are to be equal to the Number of required Cyphers before $a$, to wit 7. According to which, the Power

Power of $a$ will be $a^{8388608} = 2$ (this 8388608 being the 23d Term in Geometrical Progression) consequently the respective Root of $2 = a^{8388608}$ will be the Mean requir'd.

*Example.*

Let $r + e = a$

Then will $r^{8388608} + 8388608\, r^{8388607} e$
$+ 35184367894528\, r^{8388606} ee = a^{8388606} = 2$

Suppose $r = 1$

Then $1 + 8388608e + 35184367894528ee = 2$

That is $8388608e + 35184367894528\, ee = 1$

Each Part being divided by the Co-efficient found prefixed to $ee$, viz. 351843, &c. then it will become

$,00000023e + ee = ,000000000000284 = D$

$$Consequently \left\{ \frac{D}{,00000023 + e} = e \right.$$

. . . . . . . . .

$,000000000000284 = D$

| | | |
|---|---|---|
| $,00000023$ | 248 | $(,00000008 = e$ |
| $+ e = ,00000008$ | 36 | |
| Divisor $,00000031$ | | |

First $r = 1$,
$+ e = ,00000008$

New $r = 1,00000008$

which being duly involved, in the same Order as the Model denotes, and multiplied into the respective Co-efficients, will then produce these Numbers,

*Viz.* $1,9563638967 + 16411168e + 6883341606628 9ee = 2$

Then $16411168e + 68833416066289ee = ,0436361033$

And $,0000002384e + ee = ,00000000000000063393 = D$

$$Consequently \left\{ \frac{D}{,0000002384 + e} = e \right.$$

. . . . . . . . . . . .

$,00000000000000063393 = D$

| | | |
|---|---|---|
| $,0000002384$ | 480 | $(,0000000263 = e$ |
| $+ e = ,0000000026$ | | |
| | 15393 | |
| Divisor $,000000240$ | 14400 | |
| Divisor $,0000002410$ | 9330 | |
| | 7230 | |
| | 000 | |

Last

Last $r = 1,00000008$
$+ e = ,00000000263$

New $r = 1,00000008263$

I take only $1,0000000268 = r$; the which being involved, and ordered as before, will produce these following Numbers, *viz.*
$1,999503684867 + 16773028e + 70351267454084\,ee = 2$
Then $16773028e + 70351267454084\,ee = ,000496315133$
$,0000002384186e + ee = 000000000000000000705481443 = D$

*Consequently* $\left\{ \dfrac{D}{,0000002384186 + e} = e \right.$

. . . . . . . . . . . . . . . .

$,000000000000000000705481443 = D$

| | | |
|---|---|---|
| ,0000002384186 | 47686 | (,0000000000295 = e |
| + e = ,0000000000295 | | |
| | 2286214 | |
| Divisor ,00000023843 | 2146023 | |
| | | |
| Divisor ,000000238447 | 14019143 | |
| | 11922405 | |
| Divisor ,0000002384481* | | |
| | *20967380 | (879 = e |
| Here I desist forming a new Divisor and make use of the Abridgement. | 19075848 | |
| | 891532 | |
| | 1669136 | |
| | 222396 | |
| Last $r = 1,0000000826$ | 214596 | |
| $+ e = ,0000000000295879$ | | |

$a = 1,0000000826295879$

This Value of $a = 1,0000000826295879$ is the Geometrical Mean betwixt 2 and 1, as was required; (agreeable to that before found, by twenty-three several Extractions.) And by this Method of proceeding, may be found the Mean betwixt 10 and 1, *viz.* 1,0000000686 2238, or betwixt any other of the (before-mentioned) Prime Numbers and Unity, as might easily be shewed. But for Brevity Sake, I shall omit giving more Examples thereof, this one being sufficient (especially to the Ingenious) if well considered, and but once understood, to shew the Nature of, and Manner how to proceed on the like Occasion, of finding any proposed

2 Mean.

Mean. The next Thing will be to find the Logarithm of the Number from whence ſuch Mean was produced, which may be thus performed :

Firſt, find its correſponding Arithmetical Mean, or Logarithm, by Proportion (as in *Page* 462.) Then multiply that correſponding Mean, ſo found, into the Index Number of ſuch Power as the Geometrical Mean was produced from; that Product will be the Logarithm of the given Number (without a continued Doubling and Redoubling, as before.) For the clearing of this, let it be required to complete the Logarithm of 2.

Having firſt found 1,0000000686238, the proper Geometrical Mean betwixt 10 and 1 ; alſo its correſponding Logarithm ,0000002980232 (as before directed) with them and the Mean betwixt 2 and 1, laſt found, *viz.* 1,0000000826295879 ; make uſe of the above-mentioned Proportion (as in *Page* 463.) *viz.*

6862238 : 2980232 : : 826295879 : 358855729

To which prefix ſeven Cyphers to complete it (as before.) Then it will become ,0000000358855729. This Number being multiplied into the Power of *a* (what that is, ſee *Page* 465.) will produce the Logarithm of 2.

*viz.* 0000000358855729 × 8388608 = 0,30103000391352

But according to the firſt Deſign, it is required to have but ſeven Places, *viz.* 0301300; which is the true Logarithm of 2 without any Defect.

Thus I have preſented you with a new and expeditious Method of making Logarithms ; which if they were required to fourteen or fifteen Places (I can modeſtly ſay) they might then be made with one twentieth Part of the Time and Trouble required by the firſt Method.

## METHOD III.

*A New Table of* Logarithms. *Compos'd by Mr.* LONG. *Finding the* Logarithm *by* Division *only, and the Natural Number belonging to a* Logarithm, *by* Multiplication *only.*

| *Log.* | *Nat. Num.* | *Log.* | *Nat. Num.* |
|---|---|---|---|
| 0,9 | 7.943282347 | 0,00009 | 1.000207254 |
| 0,8 | 6.309573445 | 0,00008 | 1.000184224 |
| 0,7 | 5.011872336 | 0,00007 | 1.000161194 |
| 0,6 | 3.981071706 | 0,00006 | 1.000138165 |
| 0,5 | 3.162277660 | 0,00005 | 1.000115136 |
| 0,4 | 2.511886432 | 0,00004 | 1.000092106 |
| 0,3 | 1.995262315 | 0,00003 | 1.000069080 |
| 0,2 | 1.584893193 | 0,00002 | 1.000046053 |
| 0,1 | 1.258925412 | 0,00001 | 1.000023026 |
| 0,09 | 1.230268771 | 0,000009 | 1.000020724 |
| 0,08 | 1.202264435 | 0,000008 | 1.000018421 |
| 0,07 | 1.174897555 | 0,000007 | 1.000016118 |
| 0,06 | 1.148153621 | 0,000006 | 1.000013816 |
| 0,05 | 1.122018454 | 0,000005 | 1.000011513 |
| 0,04 | 1.096478196 | 0,000004 | 1.000009210 |
| 0,03 | 1.071519305 | 0,000003 | 1.000006908 |
| 0,02 | 1.047128548 | 0,000002 | 1.000004605 |
| 0,01 | 1.023292992 | 0,000001 | 1.000002302 |
| 0,009 | 1.020939484 | 0,0000009 | 1.000002072 |
| 0,008 | 1.018591388 | 0,0000008 | 1.000001842 |
| 0,007 | 1.016248694 | 0,0000007 | 1.000001611 |
| 0,006 | 1.013911386 | 0,0000006 | 1.000001381 |
| 0,005 | 1.011579454 | 0,0000005 | 1.000001151 |
| 0,004 | 1.009252886 | 0,0000004 | 1.000000921 |
| 0,003 | 1.006931669 | 0,0000003 | 1.000000690 |
| 0,002 | 1.004615794 | 0,0000002 | 1.000000460 |
| 0,001 | 1.002305238 | 0,0000001 | 1.000000230 |
| 0,0009 | 1.002074475 | 0,00000009 | 1.000000207 |
| 0,0008 | 1.001843766 | 0,00000008 | 1.000000184 |
| 0,0007 | 1.001613109 | 0,00000007 | 1.000000161 |
| 0,0006 | 1.001382506 | 0,00000006 | 1.000000138 |
| 0,0005 | 1.001151956 | 0,00000005 | 1.000000115 |
| 0,0004 | 1.000921459 | 0,00000004 | 1.000000092 |
| 0,0003 | 1.000691015 | 0,00000003 | 1.000000069 |
| 0,0002 | 1.000460623 | 0,00000002 | 1.000000046 |
| 0,0001 | 1.000230285 | 0,00000001 | 1.000000023 |

This

This Table I sometimes make use of for finding the Logarithm of any Number propos'd, and *vice versa*. Suppose I had Occasion to find the Logarithm of 2000. I look in the first Class of my Table (the whole Table consists of 8 Classes) for the next less to 2, which is 1.995262315, and against it is 3, which consequently is the first Figure of the Logarithm sought. Again, dividing the Number propos'd 2, by 1.995262315 the Number found in the Table, the Quotient is 1.002374467; which being look'd for in the second Class of the Table, and finding neither its Equal, nor a Lesser, I add 0 to the Part of the Logarithm before found, and look for the said Quotient, 1.002374467 in the third Class, where the next less is 1.002305238, and against it is 1, to be added to the Part of the Logarithm already found; and dividing the Quotient 1.002374467, by 1.002305238, last found in the Table, the Quotient is 1.000069070; which being sought in the fourth Class gives 0, but being sought in the fifth Class gives 2, to be added to the Part of the Logarithm already found; and dividing the last Quotient by the Number last found in the Table, *viz.* 1.000046053, the Quotient is 1.000023015, which, being sought in the sixth Class, gives 9 to the Part of the Logarithm already found: And dividing the last Quotient by the new Divisor, *viz.* 1.000002072, the Quotient is 1.000000219, which being greater than 1.000000115 shews that the Logarithm already found, *viz.* 3.3010299 is less than the Truth by more than half an Unit; wherefore adding 1, you have *Briggs*'s Logarithm of 2000, *viz.* 3.3010300.

If any Logarithm be given, suppose 3,3010300, throw away the Characteristic, then overagainst these Figures 3 . . . 0 . . 1 . . 0 . . 0, you have in their respective Classes 1,995262315 . . . . . 0 . . . . . 1,002305238 . . . . . 0 . . . . . 1,000069080 . . . . 0 . . . 0 which multiply'd continually into one another, the Product is 2.000000019966, which, by reason the Characteristic is 3, becomes 2,000,000019966, &c. that is, 2000, the Natural Number desired. I shall not mention the Method by which this Table is fram'd, because you will easily see that from the Use of it.

It is obvious to the intelligent Reader, that these Classes of Numbers are no other than so many Scales of mean Proportionals: in the first Class, between 1 and 10; so that the last Number thereof, *viz.* 1,258925412 is the tenth Root of 10, and the rest in order ascending are the Powers thereof. So in the second Class, the last Number 1,023292992 is the hundreth Root of 10, and the rest in the same Manner are Powers thereof. So 1.002305238, in the third Class, is the tenth Root of the last of the second, and the

the reſt its Powers, *&c.* Or, which is all one, each Number, in the preceding Claſs, is the tenth Power of the correſponding Number in the next following Claſs: Whence 'tis plain, that to conſtruct theſe Tables requires no more than one Extraction of the fifth or ſurſolid Root for each Claſs, the reſt of the Work being done by the common Rules of Arithmetick.

## METHOD IV.

Their Conſtruction, according to the common Rules, given by many Extractions of Roots, is tedious; the beſt Way yet known is this which follows.

*To make a Table of* Logarithms.

*Firſt*, Put for the Logarithm of 1 a Cypher for the Index, and a competent Number of Cyphers for the Logarithm, according to the Number of Places you would have your Logarithms conſiſt of; for 10 and Unit, with the ſame Number of Cyphers; for 100, 2, with as many Cyphers; for 1000, 3, with as many Cyphers, *&c.*

*Secondly*, Find the Difference between ſome two Logarithms above 1000, or rather above 10000, that differ by Unity; thus multiply the two Numbers together, and that Product you muſt multiply again by 434294481903251838 96 * which laſt Product divided by the Arithmetical Mean between both Numbers, the Quotient is the Difference ſought.

Suppoſe we would find the Difference between the Log. 10000, and 10001, the Product of theſe two Numbers is 1.00010000. which multiplied by 4343 produced 43434343; this divided by 10000.5, quotes 4343. Now if to the Logarithm of 10000, which is 4.0000000, you add the Difference before found, to wit, 434, the Sum 4.0000434 is the true Logarithm of 10001 to 7 Places.

*Thirdly*, Having thus found the Difference of any two Logarithms differing by Unity, and conſequently ſome of the Logarithms by dividing the Difference found by the Arithmetical Mean, between any two Numbers differing by Unity, you ſhall have the Difference of the Logarithm of thoſe two Numbers.

Thus to find the Difference betwixt the Logarithm of 274, and 275.; divide 4343, the Difference of the Logarithm of 10000, and 10001 by 2745 the Quotient 15821, is the Difference ſought.

*Fourthly*, Having by this Means found a few of the prime Logarithms, the reſt are made by Addition and Subtraction, and having

* *Which is the Subtangent of the Curve expreſſing* Briggs's *Logarithms.* *See* Keil's *Trig.* Pag. 135, 140, *&c.*

ing made the Canon upwards, above 1000 to 10000, by Consequence it is made for all inferior Numbers.

The prime Numbers to which Logarithms must be found, in the first Place are these, 2 . 3 . 7 . 11 . 13 . 17 . 19 . 23 . 29 . 31 . 37 . 41 . 43 . 47 . 53 . 59 . 61 . 67 . 71 . 73 . 79 . 89 . 97, &c. or the same Numbers with Cyphers.

But since it was very tedious and laborious, to find the Logarithms of the prime Numbers, and not easy to compute Logarithms by Interpolation, by first, second, and third, &c. Differences, therefore the great Men, Sir *Isaac Newton*, *Mercator*, *Gregory*, *Wallis*, and lastly, Dr. *Halley*, have published infinite converging Series, by which the Logarithms of Numbers to any Number of Places may be had more expeditiously and truer: Concerning which Series, Dr. *Halley* has written a learned Tract, in the *Philosophical Transactions*, wherein he has demonstrated those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found, easily and expeditiously, the Logarithms of large Numbers.

Let $z$ be an odd Number, whose Logarithm is sought; then shall the Number $z - 1$ and $z + 1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms will be had, which let be called $y$: Therefore, also the Logarithm of a Number which is a Geometrical Mean between $z - 1$ and $z + 1$ will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9}$ &c. shall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z - 1$ and $z + 1$, has to the Arithmetical Mean, *viz.* to the Number $z$.

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But if $z$ be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2, with the Index 4 prefix'd to it; and the Difference of the Logarithms of 20000 and 20002, is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0.0000434272 7687. And if this Difference be divided by $4z$, or 80004, the Quo-

| | |
|---|---|
| Quotient $\frac{y}{4z}$ ſhall be ——————— | 0. 00000 0000542814 |
| And if the Logarithm or the Geometrical Mean be added to the Quotient, the | 4. 30105 1709302416 |
| Sum will be the Logarithm of 20001. | 4. 30105 1709845230 |

Wherefore it is manifeſt, that to have the Logarithm to 14 Places of Figures, there is no Neceſſity of continuing out the Quotient beyond 6 Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq*'s Tables, the two firſt Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will moſtly conſiſt in ſetting down the Numbers. *Note*, This Series is eaſily deduced from that found out by Dr. *Halley*; and thoſe who have a Mind to be inform'd more in this Matter, let them conſult his abovenam'd Treatiſe.

## Mr. *WARD*'s *Eaſy* Method of making the Canon of Sines, Tangents, &c.

FIRST, let me premiſe two Things, that the Periphery of a Circle, whoſe Radius is Unity or 1, is 6.283185, &c. and that the natural Sine of one Minute doth ſo inſenſibly differ from the Length of the Arch of one Minute, that it may be taken for the ſame.

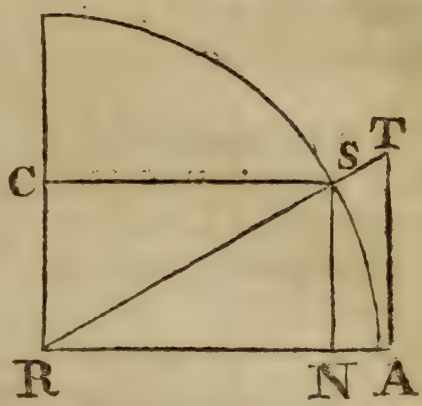
Conſequently, { *As the Periphery in Minutes : is to the Periphery in equal Parts of the Radius : : ſo is one Minute : to the Parts agreeing to that Minute.*

That is, 21600′ : 6,283185 : : 1′ : 0,000290888 = the Natural Sine of one Minute; which agrees with the largeſt Table of Sines I ever ſaw.

Having thus got the Sine of one Minute, its Co-ſine may be thus found:

Suppoſe

Suppose $RA = RS$ the Radius of any Circle, $SN =$ the Sine of the Arch $SA$. Then $RN = CS$ is the Co-Sine of that Arch. But $\square RS - \square SN = \square RN$; consequently $\sqrt{\square RS - \square SN} = RN$.




That is, From the Square of the Radius, subtract the Square of the Sine of 1′, the square Root of the Remainder will be the Co-Sine of 1′, per *Chap.* 9. *Prop.* 1. In Numbers, the Sine of 1′ is 000290885, its Square is 0,000000084612; and 1 — 0,000000084612 = 0,999999915388, the Square Root thereof is ,99999995 = the Co-Sine required.

The Sine and Co-Sine of one Minute being thus obtain'd, all the rest of the Sines in the Quadrant may be gradually calculated by Mr. *Michael Dary*'s Sinical Proportions; which I shall here insert, to the same Effect as they are in his Miscellanies; and then explain and demonstrate the Truth of those Proportions.

If a Rank of Arches be equi-different;

Then { *As the Sine of any Arch in that Rank : is to the Sum of the Sines of any two Arches equally remote from it on each Side : : so is the Sine of any other Arch in the said Rank : to the Sum of the Sines of two Arches next it on each Side; having the same common Distance.*

Immediately after these Proportions, he lays down the following Æquations:

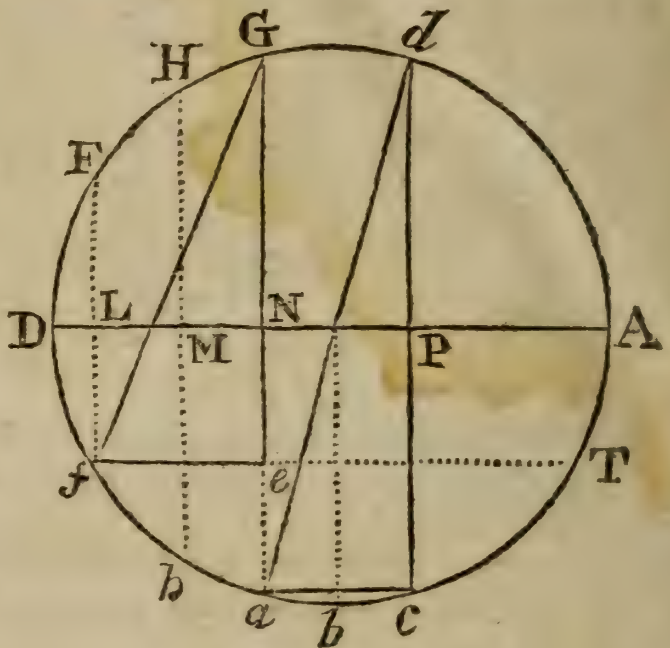
Three Arches equi-different being proposed; if (saith he) you put Z = the Sine of the great Extreme, $y$ = the Sine of the lesser Extreme; $M$ = the Sine of the Mean; $m$ = the Co-Sine thereof; $D$ the Sine of the common Difference; $d$ = the Co-Sine thereof; and $R$ = the Radius.

1. Then $Z + y = \frac{2\,Md}{R}$. Then $Z - y = \frac{2\,mD.}{R}$

3. Then $Zy = MM - DD$. 4. Then $\frac{Z}{y} = \frac{Md + mD.}{Md - Md}$

From the foregoing it is evident (saith he) that if two Thirds, *viz.* either the former or latter 60 Degrees, or the former 30 Degrees, and the latter 30 Degrees of the Quadrant be completed with Sines; the remaining Part of the Quadrant may be completed by Addition, or Subtraction only.

Thus far is from the ingenious Mr. *Dary*, concerning these excellent Proportions; the Truth whereof I shall thus demonstrate.




In the annexed Circle $DA = da$ are Diameters, $fh = ha = ab = bc$, are equal Arches.

Draw $fT$ parallel to $DA$; then will $Ne = Lf$. And the $\triangle\, dac$, like the $\triangle\, Gfe$, being both right-angled at $c$ and $e$, and $\angle\, d = \angle\, G$ because subtended by the equal Arches $ac = fa$.

Therefore $da : dc :: Gf : Ge$.

Consequently $\frac{1}{2} da : dc :: \frac{1}{2} Gf : Ge$. But $Hh = Gf$, whence $HM = \frac{1}{2} Gf$, and $\frac{1}{2} da =$ the Radius, $dp = \frac{1}{2} dc$. Therefore it will be, Radius : $2\,dp = :: HM = \frac{1}{2} Gf : GN + Ne = GN + Lf$. That is, as the Radius : is to twice the Sine $dp$ :: so is the Sine $HM$ : to the Sum of the two Sines $GN$ and $FL = fL$. Q. E. D.

I shall now explain these Proportions, and shew how they may be applied in Practice: Having the Sine of one Minute, and its Co-Sine as before; let the Radius be made the mean or middle Term between those two Extremes; then the Proportions will run

Thus { *As the Radius : is to the double Co-sine of one Minute :: so is the Sine of one Minute : to the Sine of two Minutes, and of* 00′ *: and so is the Sine of* 2′ *:: to the Sum of the Sines of* 3′ *and* 1′ *:: and so is the Sine of* 3′ *: to the Sum of the Sine of* 4′ *and* 2′.

*And so on in a successive Order, from Minute to Minute.*

And then, if from the Sum of the Sines of 3′ and 1′ be taken the Sine of 1′, the Remainder will be the Sine of 3′: And the like, if, from the Sum of the Sines of 4′ and 2′, be taken the Sine of 2′ there will remain the Sine of 4′, &c.

Proceeding on by this Method, all the Natural Sines in the Quadrant may be easily calculated by Addition, and Subtraction only. For the Radius, or First Term in the Proportion, being 1,0000000

or

or Unity, Division is wholly avoided: And because the second Term in the Proportion varies not, if a Tariffa, or small Table be made thereof, to all the nine Digits, then Multiplication is also avoided. For, by the Help of that Tariffa, the whole Work may be perform'd by Addition and Subtraction, until all the Sines are gradually made.

Thus you have an easy Way of making the Canon of Sines; which being once done, the Tangents and Secants may be found by the following

Proportions { *As the Co-sine of any Arch : is to the Sine of that Arch : : so is the Radius : to the Tangent of the same Arch.*

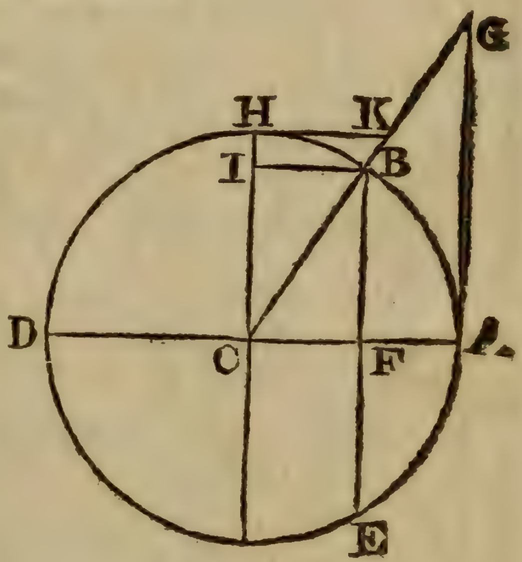
That is, by *the first Scheme of this Problem*,

$RN : SN :: RA : TA$. And $RN : RS :: RA : RT =$ the Secant of that Arch.

# Plane Trigonometry.

## Definitions.

1. A Circle is suppos'd to be divided into 360 equal Parts, called Degrees; and each Degree into 60 equal Parts, called Minutes; and each Minute into 60 equal Parts, called Seconds, &c. Any Portion of whose Circumference is called an Arch, and is measured by the Number of Degrees it contains.




2. A Chord or Subtense is a straight Line, connecting the Extremities of an Arch; as BE is the Chord of the Arches BAE, BDE.

3. A Sine (or Right-sine) is a straight Line drawn from one End of an Arch perpendicular to that Diameter passing thro' the other End; or it is half the Chord of twice the Arch; so BF is the Sine of the Arches BA, BD. And here it is evident, that the Sine of 90 Degrees (which is equal to the Radius or Semi-Diameter of the Circle) is the greatest of all Sines, the Sine of an Arch greater than a Quadrant being less than the Radius.

4. The Difference of an Arch from a Quadrant, whether it be greater or less, is call'd its Complement; so HB is the Complement of the Arches BA, BD; BI is the Sine of that Complement,

 and

and therefore it is called the Co-ſine, or Sine-Complement of the Arches BA, BD.

5. The Secant of an Arch is a ſtraight Line drawn from the Center thro' one End of the Arch till it meet with the Tangent, which is a ſtraight Line touching the Circle at the Extremity of that Diameter which cuts the other End of the Arch; ſo CG is the Secant, and AG the Tangent of the Arches BA, BD: And CK is the Coſecant, and HK the Co-tangent of the ſaid Arches.

6. A Verſed Sine is the Segment of the Diameter intercepted between the Arch and its Sine: Thus FA is the Verſed Sine of the Arch BA, and FD of the Arch BD.

7. Whatever Number of Degrees an Arch wants of a Semi-Circle is called its Supplement.

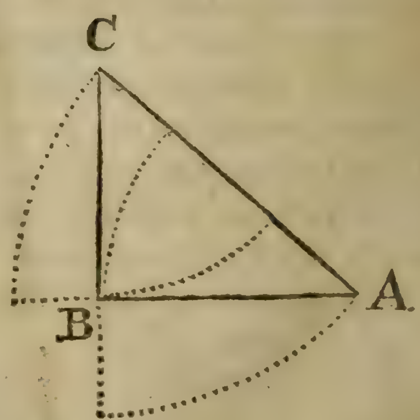
8. That Part of the Radius which is betwixt the Center and Sine is equal to the Co-ſine; thus CF is = IB.

9. If an Arch be greater or leſs than a Quadrant the Sum or Difference of the Radius or Co-ſine is equal to the Verſed Sine.

In a Triangle are ſix Parts, *viz.* three Sides and three Angles: Any three of which being given, except the three Angles of a Plain Triangle, the other three may be found either Mechanically, by the Help of a Scale of equal Parts and Line of Chords, or by an Arithmetical Calculation, if, ſuppoſing the Radius divided into any Number of equal Parts, we know how many of thoſe equal Parts are in the Sine, Tangent, or Secant of any Arch propos'd: The Art of inferring which is called *Trigonometry*, and it is either Plane or Spherical.

*Plane Trigonometry* is ſolv'd by the Help of four fundamental Propoſitions call'd *Axioms*.

*Axiom* I.




In a Right-angled Triangle ABC, if one Leg of the Right-angle, as AB or CB, be made the Radius of a Circle, then ſhall the other Leg CB or AB be the Tangent of the Angle oppoſite to it, and the Hypothenuſe AC (or Side oppoſite to the Right-angle) its Secant (by Definition 5.)

But if the Hypothenuſe AC be made the Radius of a Circle, then will the Legs (or Sides including the Right-angle) to wit, CB and AB be the Sines of the Angles oppoſite (by Definition 3.)

Upon

Upon this *Axiom* depends the Solution of the ſeven Caſes of Right-angled Plane Triangles.

*Note*, That the three Angles of a Plane Triangle make two Right-Angles, or 180 Degrees, by 32. 1 *Eucl.*

For the more eaſy making the Proportions for the Solution of Right-angled Triangles, obſerve, that as different Sides are made Radius, ſo the other Sides require different Names, which Names are either Sines, Tangents, or Secants, and are to be taken out of your Table.

To find a Side, any Side may be made Radius: Then ſay, as the Name of the Side given is to the Name of the Side required; ſo is the Side given to the Side required.

But to find an Angle, one of the given Sides muſt be made Radius; then, as the Side made Radius, is to the other Side; ſo is the Name of the firſt Side (which is Radius) to the Name of the ſecond Side; which fourth Proportional muſt be found among the Sines or Tangents, &c. to be determined by the Side made Radius, and againſt it is the Angle required.

The Proportions for the Solution of ſeven Caſes of Plane Right-angled Triangles. [*See the next foregoing* Fig.]

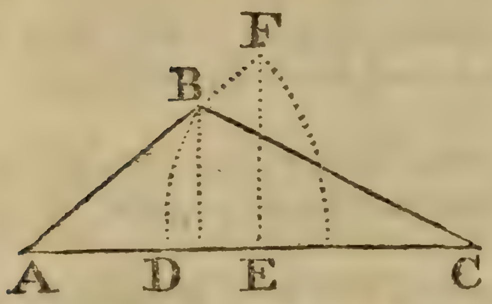
| *Given.* | *Reqd.* | *Proportions.* | *Rad.* | *Caſe.* |
|---|---|---|---|---|
| AB<br>A and<br>C | BC | Coſi. A : Si. A :: AB : BC.<br>R : Tan. A :: AB : BC.<br>Co-t. A : R :: AB : BC. | AC<br>AB<br>BC | 1 |
| AB<br>A and<br>C | AC | Coſi. A : R :: AB : AC.<br>R : Sec. A :: AB : AC.<br>Tan. A : Coſe. A :: AB : AC. | AC<br>AB<br>BC | 2 |
| AB<br><br>BC | A<br>and<br>C | AB : BC :: R : Tan. A.<br>Complement is C.<br>BC : AB :: R : Tan. C.<br>Complement is A. | AB<br><br>BC | 3 |
| AB<br>BC | AC | AB : BC :: R : Tan. A; then<br>Coſi. A : R :: AB : AC, or<br>$\surd : \overline{\square AB + \square BC} : = AC$ (*per* 47. 1.<br>*Eucl.* | AB<br>AC | 4 |
| AB<br>AC | A and<br>C | AC : BC :: R : Coſi. A.<br>AB : AC :: R : Secant A. | AC<br>AB | 5 |
| AB<br>AC | BC | AC : AB :: R : Coſi. A; then<br>R : Tan. A :: AB : BC, or<br>$\surd : \overline{\square AC - \square AB} : = BC$. | AC<br>AB | 6 |
| AC<br>A and<br>C | AB | R : Coſi. A :: AC : AB.<br>Sec. A : R :: AC : AB.<br>Coſ. A : Cot. A :: AC : AB. | AC<br>AB<br>BC | 7 |

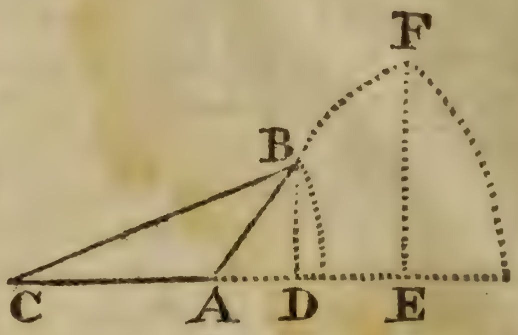
*Axiom*

### *Axiom* II.

In any Triangle the Sides are proportional to the Sines of the oppoſite Angles.

## Demonſtration.



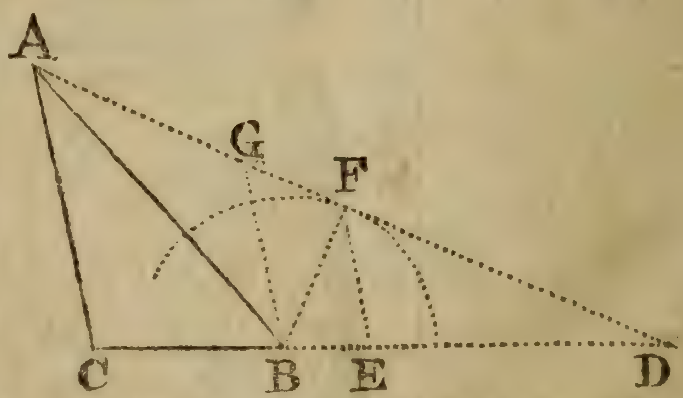

Produce the leſſer Side of the Triangle ABC, to wit AB to F, making AF = BC : Let fall the Perpendiculars BD, FE, upon the Side CA produc'd if Need be; then will FE be the Sine of the Angle A, and BD the Sine of the Angle C, to the Radius BC = AF.

Now the Triangles ABD and AFE, having the $\angle$ A common to them both, and the $\angle$ D = $\angle$ E = to a Right-Angle, are ſimilar; wherefore (by 4. 6 *Eucl. Elem.*) AF (BC) : AB : : FE : BD; *viz.* : : Si. A : Si. C. *Q. E. D.* Otherwiſe thus; by *Ax.* I. AB : R : : BD : Si. A, and BC : R : : BD : Si. C; therefore AB $\times$ Si. A (= R $\times$ BD) = BC $\times$ Si. C; wherefore AB : BC : : Si. C : Si. A. *Q. E. D.*

### *Axiom* III.

The Sum of the Legs of any Angle of a Plane Triangle is to their Difference, as the Tangent of half the Sum of the Angles oppoſite to thoſe Legs is to the Tangent of half their Difference.

## Demonſtration.



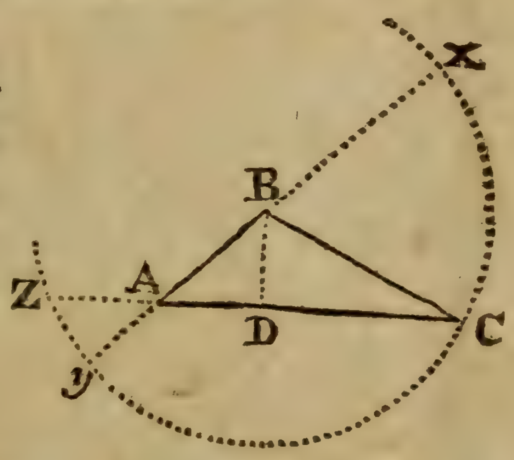

In the Triangle ABC produce CB, the leſſer Leg of the Angle B, till BD becomes = BA, the greater Leg, and then biſect CD in E; join A D and biſect it alſo in F; draw BF, which (by 8. 1 *Eucl. El.*) will be perpen. to AD; and draw EF, which (by 2 6 *Eucl. Elem.*) will be parallel to AC. Then will the Angle ABF = FBD = $\frac{1}{2}$ ABD, which external Angle ABD is (by 32. 1 *Eucl. Elem*) = BAC + C, that is to the Sum of the oppoſite Angles required.

Draw then BG parallel to AC, ſo will the Angle GBA be (by 29. 1 *Eucl. Elem.*) equal to its Alternate one BAC; and if from half the Sum

Sum of the opposite Angles you take the lesser Angle, *i. e.* If from ∠ ABF you take the ∠ GBA, there will remain ∠ GBF = half the Difference of the opposite Angles: And so also, if from CE half the Sum of the Legs, you take CB the lesser Leg, there will remain BE equal to half the Difference of the Legs. And then since the △ABF is Right-angled, if BF be made Radius, AF will be the Tangent of ∠ ABF (*i. e.* the Tangent of half the Sum of the opposite Angles); and in the little △ GBF, FG will be the Tangent of the ∠GBF (*i. e.* the Tangent of half the Difference of the opposite Angles): But the Segments of the Legs of any Triangle cut by Lines parallel to the Base, being (by *Schol.* to 2. 6 *Eucl. El.*) proportional; EC : EB : : FA : FG; that is in Words, half the Sum of the Legs is to half their Difference, as the Tangent of half the Sum of the opposite Angles is to the Tangent of half their Difference: But Wholes are as their Halves; wherefore the Sum of the Legs is to their Difference, as the Tangent of half the Sum of the Angles opposite is to the Tangent of half their Difference. *Q. E. D.*

*Axiom* IV.

The Base or greatest Side of any Plane Triangle is to the Sum of the Legs, as the Difference of the Legs is to the Difference of the Segments of the Base made by a Perpendicular let fall from the Angle opposite to the Base.




### Demonstration.

From the ∠ B, on the Base AC, of the △ ABC, let fall the Perpendicular BD; on B, as a Center, with the greater Leg BC, as a Radius, describe the Circle B*x*C*y* Z; and produce AB to *x* and *y*, and CA to Z. Then by the 35. 3 *Eucl. Elem.* A*y* × A*x* is = AC ×AZ; *viz.* : BC—BA : × : BC + BA : = AC × : DC — DA; therefore AC : BC + BA : : BC — BA : DC — DA. *Q. E. D.* Otherwise, let the Difference of the Squares of the Sides BC and AB be taken and divided by the Base AC, the Quotient shall be the Difference of the Segments of the Base aforesaid: Or, square all the 3 Sides, and deduct the Square of one of the less Sides out of the Sum of the other two Squares, divide half the Remainder by the longest Side, the Quotient is the Alternate Segment of the Base.

The Proportions for the Solution of the six Cases of Plane oblique Triangles. [*See the last* Fig.]

*Given.*

2

| *Given.* | *Reqd.* | *Proportions.* | *Ax.* | *Case.* |
|---|---|---|---|---|
| AB<br>BC<br>and C | A | AB : BC : : Si. C : Si. A. | 2 | 1 |

*N. B. 1st,* If the given Angle be Obtuse, the other two Angles then are each Acute. *2dly,* If the Side opposite to the given Angle is longer than the Side opposite to the Angle sought, then is the Angle sought Acute; but if shorter, then is the said Angle doubtful, and may be either Acute or Obtuse, because both the Sine and its Complement to two Right-Angles are the same: Wherefore to be certain of what Quality the Angle opposite to the greatest Side is: Take the Sum and Difference of the greatest Side and Middle (or least) and their Logarithms, if the Half of them be equal to the Logarithm of the third Side, the Angle opposite to the greatest Side is a Right-Angle; but if the Logarithm of the third Side be greater than the Half, it is Acute, if less, it is Obtuse: Or, without Logarithms, multiply the said Sum by the Difference abovesaid, and extract the Square Root,

which if { Equal to / Greater than / Less than } the third Side, then is the greatest Angle { Right / Obtuse / Acute }

| *Given.* | *Reqd.* | *Proportions.* | *Ax.* | *Case.* |
|---|---|---|---|---|
| AB<br>BC<br>and C | AC | AB : BC : : Si. C : Si. A.<br>Hence, by Subtraction, the $\angle$ B will be known.<br>Si. A : Si. B : : BC : AC. | 2 | 2 |
| A, C<br>and BC | AB | Si. A : Si. C : : BC : AB. | 2 | 3 |
| B<br>AB<br>BC | A and C | BC + AB : BC — AB : : Tan. $\frac{1}{2}$ Sum of the $\angle$s opposite : Tan. $\frac{1}{2}$ Difference of the $\angle$s opposite. Then $\frac{1}{2}$ Sum + $\frac{1}{2}$ Difference = greater $\angle$ A; and $\frac{1}{2}$ Sum — $\frac{1}{2}$ Difference = lesser $\angle$ C. | 3 | 4 |
| B<br>AB<br>BC | AC | First, find the Angles by the last; then<br>Si. C : Si. B : : AB : AC. | 3<br>2 | 5 |
| AB<br>BC<br>AC | A<br>B<br>C | AC : BC+BA : : BC—BA : DC—DA :<br>Then $\frac{1}{2}$ AC + $\frac{1}{2}$ DC — $\frac{1}{2}$ DA = DC.<br>And $\frac{1}{2}$ AC — : $\frac{1}{2}$ DC — $\frac{1}{2}$ DA : = DA.<br>Then AB : AD : : R : Cosi. A.<br>And CB : DC : : R : Cosi. C.<br>And 180° — $\angle$ A — $\angle$ C = $\angle$ B. | 4<br><br><br>1<br>1 | 6 |

Or more readily at one Operation.

From half the Sum of the Sides subduct each particular Side, and let the Sum of the Logarithm of the half Sum and Difference of the Side subtending the enquired Angle be subducted from the Sum of the Log. of the other Difference and the double Radius, half the Remainder shall be the Log. of the Tangent of half the enquired Angle.

*Agreeable to this* Axiom *in* Geilibrand's Trig. Britannica, p. 46.

*As the Rectangle of half the Sum of the Sides and the Difference between that half Sum and the Side opposite to the Angle required, is to the Rectangle of the other two Remainders; so is the Square of Radius to the Square of the Tangent of half the Angle sought.*

*Ex Angulis latera, vel ex lateribus Angulos & mixtim in Triangulis tam planis quam Sphæricis assequi, Summa Gloria Mathematici est: Sic enim Cœlum & Terras & Maria felici & admirando calculo mensurat.* Fran. Vieta.

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